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Class of gauge-invariant models of quantum electrodynamics with nonlocal interaction

Tao Mei

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We present a class of gauge-invariant models of quantum electrodynamics with nonlocal interaction. The models have translation, Lorentz and gauge invariance and reduce to the conventional local quantum electrodynamics under the appropriate limit conditions, both the equations of motion of the charged particle and electromagnetic field obtained by the action principle lead to the normal form of current conservation. Quantization of the models is realized by taking advantage of the formalism based on the Yang-Feldman equations and the Lehmann-Symanzik-Zimmermann reduction formulas. Finally, we employ a special choice of the models to calculate the vacuum polarization as an example to demonstrate the possibility of establishing a theory of quantum electrodynamics without divergence.

There is a large literature on nonlocal field theory (for a review and references, see refs 1, 2). One of the motivations for investigating nonlocal field theory is to establish a theory without divergence; since the renormalized theory of local gauge fields leads to finite predictions and to an unprecedented agreement between theory and experiment, if divergences still appear in the corresponding nonlocal theory and must still be eliminated by a renormalization procedure, then such a nonlocal field theory is not interesting.

Of course, according to the contemporary point of view, any field theory must be renormalized; if a nonlocal field theory involves finite renormalization constants, then such a theory accords with the contemporary point of view and standard mathematical theory at the same time, since standard mathematical theory allows ignoring small or finite quantities (but ignoring an infinitely large quantity does not accord with standard mathematical theory)³.

Some models of quantum electrodynamics (QED) with nonlocal interaction have been proposed^{4–6}. In this paper, we present a class of models of QED with nonlocal interaction.

After looking back on some known nonlocal models of QED, we present the action of a class of models and the equations of motion of a charged particle and electromagnetic field obtained by the action principle. Translation, Lorentz and gauge invariance of the models are proved, and we prove that both the equations of motion of a charged particle and electromagnetic field lead to the normal form of current conservation. The models are reduced to the conventional local QED under the appropriate limit condition.

Similar to some known nonlocal modes of QED, for guaranteeing the gauge invariance of the theory, the form of the models presented in this paper is far from that of conventional field theory.

Next, based on the fact that a free charged particle and free electromagnetic field still obey the local Dirac equation and the local Maxwell equation of free fields, respectively, quantization of the model is realized by taking advantage of the formalism based on the Yang-Feldman equations and the Lehmann-Symanzik-Zimmermann reduction formulas. Some properties, e.g., Lorentz and gauge invariance, unitarity and causality of the elements of the S -matrix, are discussed.

The models presented in this paper provide a wide range of choices; we employ a special choice of the models to calculate the vacuum polarization as an example to show the possibility of establishing a theory of QED without divergence.

All symbols and conventions of this paper follow ref. 7, for example, $g^{\alpha\beta} = g_{\alpha\beta} = \text{diag}(+1, -1, -1, -1)$ and $A \cdot B = A^\lambda B_\lambda = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}$.

Looking back on models of quantum electrodynamics with nonlocal interaction

As is well known, the action of the conventional local QED is

Department of Journal, Central China Normal University, Wuhan, Hubei PRO, People's Republic of China. Correspondence and requests for materials should be addressed to T.M. (email: meitao@mail.ccnu.edu.cn)

$$S_{EM} + S_e + S_1^{(0)},$$

where

$$S_{EM} = -\frac{1}{4} \int d^4x (A_{\mu,\nu}(x) - A_{\nu,\mu}(x))(A^{\mu,\nu}(x) - A^{\nu,\mu}(x)), \tag{1}$$

$$S_e = \int d^4x \bar{\psi}(x) \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \psi(x), \tag{2}$$

$$S_1^{(0)} = -e \int d^4x j^\mu(x) A_\mu(x). \tag{3}$$

In (3), $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$. The theory is invariant under the gauge transformation

$$\psi'(x) = e^{-ie\chi(x)}\psi(x), A'_\mu(x) = A_\mu(x) + \frac{\partial\theta(x)}{\partial x^\mu}, \tag{4}$$

where $\theta(x)$ is an arbitrary scalar function. The relation between the two functions $\chi(x)$ and $\theta(x)$ in (4) reads

$$\chi(x) = \theta(x); \tag{5}$$

and both the equations of motion of the charged particle and electromagnetic field obtained by the action principle lead to the current-conservation equation

$$\frac{\partial j^\mu(x)}{\partial x^\mu} = 0. \tag{6}$$

Since $S_1^{(0)}$ given by (3) can be written in the form $S_1^{(0)} = -e \int d^4x d^4y j^\lambda(x) \delta^4(x-y) A_\lambda(y)$, by replacing $\delta^4(x-y)$ by a scalar function $f(x-y)$, which is independent of $\psi(x)$ and $A^\mu(x)$, $S_1^{(0)}$ becomes

$$\tilde{S}_1^{(0)} = -e \int d^4x d^4y j^\mu(x) f(x-y) A_\mu(y). \tag{7}$$

Thus, H. McManus in ref. 4 establishes a model of QED with nonlocal interaction, of which the action is $S_{EM} + S_e + \tilde{S}_1^{(0)}$, where S_{EM} , S_e and $\tilde{S}_1^{(0)}$ are given by (1), (2) and (7), respectively. H. McManus in ref. 4 proves that the theory is invariant under the gauge transformation (4), but now, the relation between the two functions $\chi(x)$ and $\theta(x)$ in (4) reads

$$\chi(x) = \int d^4y f(x-y)\theta(y). \tag{8}$$

Specifically, by choosing $f(z) = \frac{1}{(2\pi)^4} \frac{1}{1+r_0^4(k^2)^2} e^{ik\cdot z}$, where r_0 is a constant, a theory of classical electrodynamics without singularities is obtained in ref. 4. However, M. Chretien and R. E. Peierls in ref. 5 point out that this theory is incapable of removing the divergence of the vacuum polarization in the corresponding quantum theory.

On the other hand, $S_e + S_1^{(0)}$, in which S_e and $S_1^{(0)}$ are given by (2) and (3), respectively, can be written in the form⁸

$$S_e + S_1^{(0)} = \int d^4x \int d^4y \bar{\psi}(x) \exp\left[-ie \int_{y^\rho}^{x^\rho} A_\sigma(z) dz^\sigma\right] \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \delta^4(x-y) \psi(y), \tag{9}$$

where the integral $\int_{y^\rho}^{x^\rho} A_\sigma(z) dz^\sigma$ is taken over the straight line in space-time joining x^ρ and y^ρ . By replacing $\delta^4(x-y)$ in (9) by a scalar function $f(x-y)$, which is independent of $\psi(x)$ and $A^\mu(x)$, M. Chretien and R. E. Peierls in ref. 5 establish a different model of nonlocal QED, of which the action is $S_{EM} + S_{e-1}^{(0)}$, where S_{EM} is still given by (1), and

$$S_{e-1}^{(0)} = \int d^4x \int d^4y \bar{\psi}(x) \exp\left[-ie \int_{y^\rho}^{x^\rho} A_\sigma(z) dz^\sigma\right] \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) f(x-y) \psi(y). \tag{10}$$

Although the form of this theory is far from that of conventional field theory, the theory is still invariant under the gauge transformation (4) and (5). For this theory, current conservation is no longer the normal form (6), and there still exist divergences in the theory⁵.

K. Scharnhorst in ref. 6 investigates a generalization of the above theory, of which the action is $S_{EM} + \tilde{S}_{e-1}^{(0)}$, where S_{EM} is still given by (1), and

$$\begin{aligned} \tilde{S}_{e-1}^{(0)} &= \int d^4x \int d^4y \bar{\psi}(x) \exp\left[-ie \int_{y^\rho}^{x^\rho} A_\sigma(z) dz^\sigma\right] \\ &\times \left[a(x-y) \gamma^\mu \left(i \frac{\partial}{\partial x^\mu} - e A_\mu(x) \right) - mb(x-y) \right] \psi(y). \end{aligned} \tag{11}$$

Class of models of quantum electrodynamics with nonlocal interaction

A generalization of (7) is to replace x^λ and y^λ in the function $f(x^\lambda - y^\lambda)$ in (7) by two four-vector functions $U^\lambda(x)$ and $V^\lambda(y)$, respectively, $\tilde{S}_1^{(0)}$ thus becomes

$$\tilde{S}_1 = -e \int d^4x d^4y j^\mu(x) f(U^\lambda(x) - V^\lambda(y)) A_\mu(y). \tag{12}$$

However, we can prove that the theory whose action is $S_{EM} + S_e + \tilde{S}_1$, where S_{EM} , S_e and \tilde{S}_1 are given by (1), (2) and (12), respectively, breaks gauge invariance, no matter how the relation between the two functions $\chi(x)$ and $\theta(x)$ in (4) is chosen.

In order to restore gauge invariance, for the two functions $U^\mu(x)$ and $V^\mu(y)$, we introduce

$$U_\nu^\mu(x) \equiv \frac{\partial U^\mu(x)}{\partial x^\nu}, \quad V_\nu^\mu(y) \equiv \frac{\partial V^\mu(y)}{\partial y^\nu}. \tag{13}$$

If we regard $U_\nu^\mu(x)$ and $V_\nu^\mu(y)$ as two 4×4 matrices, then we ask that both the corresponding determinants $U(x)$ and $V(y)$ do not vanish:

$$U(x) \equiv \det[U_\nu^\mu(x)] \neq 0, \quad V(y) \equiv \det[V_\nu^\mu(y)] \neq 0. \tag{14}$$

Hence, the corresponding inverse matrices $\tilde{U}_\nu^\mu(x) \equiv \frac{\partial x^\mu}{\partial U^\nu(x)}$ and $\tilde{V}_\nu^\mu(y) \equiv \frac{\partial y^\mu}{\partial V^\nu(y)}$ exist and satisfy

$$U_\lambda^\mu(x) \tilde{U}_\nu^\lambda(x) = \delta_\nu^\mu, \quad U_\nu^\lambda(x) \tilde{U}_\lambda^\mu(x) = \delta_\nu^\mu; \quad V_\lambda^\mu(y) \tilde{V}_\nu^\lambda(y) = \delta_\nu^\mu, \quad V_\nu^\lambda(y) \tilde{V}_\lambda^\mu(y) = \delta_\nu^\mu. \tag{15}$$

Taking advantage of the functions $U_\nu^\mu(x)$, $V(y)$ and $\tilde{V}_\nu^\mu(y)$ corresponding to $U^\mu(x)$ and $V^\mu(y)$, respectively, we construct an interaction action

$$S_I = -e \int d^4x d^4y j^\alpha(x) U_\alpha^\beta(x) f(U^\lambda(x) - V^\lambda(y)) V(y) \tilde{V}_\beta^\gamma(y) A_\gamma(y). \tag{16}$$

We ask that $f(z)$ be independent of $\psi(x)$ and $A^\mu(x)$; $U^\mu(x)$ and $V^\mu(y)$ can be dependent on $\psi(x)$ and $A^\mu(x)$, but both of them are invariant under the transformation (4). Under these conditions, we now prove that the action $S_{EM} + S_e + S_I$, where S_{EM} , S_e and S_I are given by (1), (2) and (16), respectively, is invariant under the gauge transformation (4), but now, the relation between the two functions $\chi(x)$ and $\theta(x)$ in (4) reads

$$\chi(x) = \int d^4y f(U^\lambda(x) - V^\lambda(y)) V(y) \theta(y). \tag{17}$$

At first, it is easy to obtain that under the transformation (4), the sum of S'_{EM} and S'_e , in which the field quantities are $A'_\mu(y)$ and $\psi'(x)$, becomes

$$S'_{EM} + S'_e = S_{EM} + S_e + e \int d^4x j^\mu(x) \frac{\partial \chi(x)}{\partial x^\mu}. \tag{18}$$

On the other hand, under the transformation (4), $j'^\alpha(x) = j^\alpha(x)$ and $U_\alpha^\beta(x) f(U^\lambda(x) - V^\lambda(y)) V(y) \tilde{V}_\beta^\gamma(y)$ are invariant, because $f(z)$ is independent of $\psi(x)$ and $A^\mu(x)$, and both $U^\mu(x)$ and $V^\mu(y)$ are invariant; therefore, under the transformation (4), S'_I , in which the field quantities are $A'_\mu(y)$ and $\psi'(x)$, becomes

$$\begin{aligned} S'_I &= -e \int d^4x d^4y j'^\alpha(x) U_\alpha^\beta(x) f(U^\lambda(x) - V^\lambda(y)) V(y) \tilde{V}_\beta^\gamma(y) A'_\gamma(y) \\ &= -e \int d^4x d^4y j^\alpha(x) U_\alpha^\beta(x) f(U^\lambda(x) - V^\lambda(y)) V(y) \tilde{V}_\beta^\gamma(y) \left(A_\gamma(y) + \frac{\partial \theta(y)}{\partial y^\gamma} \right) \\ &= S_I + \Delta S_I, \end{aligned} \tag{19}$$

where S_I is expressed by (16), and

$$\begin{aligned} \Delta S_I &= -e \int d^4x d^4y j^\alpha(x) U_\alpha^\beta(x) f(U^\lambda(x) - V^\lambda(y)) V(y) \tilde{V}_\beta^\gamma(y) \frac{\partial \theta(y)}{\partial y^\gamma} \\ &= -e \int d^4x d^4y j^\alpha(x) U_\alpha^\beta(x) f(U^\lambda(x) - V^\lambda(y)) \\ &\quad \times \left[\frac{\partial(V(y) \tilde{V}_\beta^\gamma(y) \theta(y))}{\partial y^\gamma} - \frac{\partial(V(y) \tilde{V}_\beta^\gamma(y))}{\partial y^\gamma} \theta(y) \right]. \end{aligned} \tag{20}$$

In the last section of this paper, we shall prove

$$\frac{\partial(V(y) \tilde{V}_\sigma^\rho(y))}{\partial y^\rho} = 0. \tag{21}$$

Thus, the last term in (20) vanishes; by an integration by parts with respect to y^λ , ΔS_I becomes

$$\Delta S_I = e \int d^4x d^4y j^\alpha(x) U_\alpha^\beta(x) \frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial y^\gamma} V(y) \tilde{V}_\beta^\gamma(y) \theta(y). \tag{22}$$

In the last section of this paper, we shall prove

$$\frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial y^\alpha} = -\tilde{U}_\beta^\gamma(x) \frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial y^\gamma} V_\alpha^\beta(y). \tag{23}$$

Substituting (23) into (22) and using some of the formulas in (15), ΔS_I becomes

$$\begin{aligned} \Delta S_I &= -e \int d^4x d^4y j^\alpha(x) U_\alpha^\beta(x) \tilde{U}_\rho^\sigma(x) \frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial x^\sigma} V_\gamma^\rho(y) V(y) \tilde{V}_\beta^\gamma(y) \theta(y) \\ &= -e \int d^4x d^4y j^\alpha(x) \frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial x^\alpha} V(y) \theta(y). \end{aligned} \tag{24}$$

Combining (18) with (19), and using (24), we obtain

$$\begin{aligned} S'_{EM} + S'_e + S'_I &= S_{EM} + S_e + S_I + e \int d^4x j^\mu(x) \frac{\partial \chi(x)}{\partial x^\mu} - e \int d^4x j^\alpha(x) \frac{\partial}{\partial x^\alpha} \\ &\quad \times \int d^4y f(U^\lambda(x) - V^\lambda(y)) V(y) \theta(y), \end{aligned}$$

thus, if the function $\chi(x)$ is as introduced by (17), then we obtain $S'_{EM} + S'_e + S'_I = S_{EM} + S_e + S_I$; this means that the action $S_{EM} + S_e + S_I$ is invariant under the gauge transformation (4) and (17).

It is obvious that the action $S_{EM} + S_e + S_I$ has Lorentz invariance; therefore, $S_{EM} + S_e + S_I$ is Lorentz and gauge invariant. However, if both $U^\mu(x)$ and $V^\mu(y)$ are independent of $\psi(x)$ and $A^\mu(x)$, then it is difficult to find two functions $U^\mu(x)$ and $V^\mu(y)$ such that $S_{EM} + S_e + S_I$ has translation invariance. Contrarily, if $U^\mu(x)$ and $V^\mu(y)$ are dependent on $\psi(x)$ or $A^\mu(x)$, then it is easy to choose $U^\mu(x)$ and $V^\mu(y)$ such that $S_{EM} + S_e + S_I$ has translation invariance.

For example, if we take

$$\begin{aligned} U^\mu(x) &= x^\mu + a' u^\mu(x), \quad V^\mu(y) = y^\mu + a v^\mu(y), \\ u^\lambda(x) &= b_1 \int d^4x' \frac{W'_1(x-x')}{\partial x_\lambda} \bar{\psi}(x') \psi(x') + b_2 \int d^4x' W'_3(x-x') \bar{\psi}(x') \gamma^5 \gamma^\lambda \psi(x'), \\ v^\lambda(y) &= b_3 \int d^4y' W_2(y-y') \bar{\psi}(y') \gamma^\lambda \psi(y') \\ &\quad + b_4 \int d^4y' W_4(y-y') (A^{\lambda,\tau}(y') - A^{\tau,\lambda}(y')) \bar{\psi}(y') \gamma_\tau \psi(y'), \end{aligned} \tag{25}$$

where a' , a and b_k are all constants (some constants can be zero), and $W'_k(x-x')$ and $W_k(y-y')$ are independent of $\psi(x)$ and $A^\mu(x)$, then it is obvious that $u^\lambda(x)$ and $v^\lambda(y)$, as well as $U^\mu(x)$ and $V^\mu(y)$, are invariant under the transformation (4), and both $U^\mu(x)$ and $V^\mu(y)$ satisfy (14). Besides these properties, according to translation invariance of the field quantities $\psi(x)$ and $A^\mu(x)$, and using

$$\begin{aligned} f(U^\lambda(x) - V^\lambda(y)) &= f(x^\lambda + a' u^\lambda(x) - y^\lambda - a v^\lambda(y)) \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} \left(a' u^\rho(x) \frac{\partial}{\partial x^\rho} \right)^l \sum_{m=0}^{\infty} \frac{1}{m!} \left(-a v^\sigma(y) \frac{\partial}{\partial y^\sigma} \right)^m f(x^\lambda - y^\lambda), \end{aligned}$$

we can prove that $S_{EM} + S_e + S_I$ has translation invariance.

Of course, besides (25), there are many choices for $U^\mu(x)$ and $V^\mu(y)$ such that $S_{EM} + S_e + S_I$ has translation invariance.

On the other hand, in the appropriate limit, $S_{EM} + S_e + S_I$ should reduce to the conventional local QED. This condition can be realized very easily; for example, $S_I \rightarrow S_I^{(0)}$ in the limit $a' \rightarrow 0$, $a \rightarrow 0$ in (25) and $f(z) \rightarrow \delta^4(z)$.

Although the function $f(z)$ in (7), (10) and (16) seems to be arbitrary, it must satisfy $f(z) \rightarrow \delta^4(z)$; hence, all different forms of $f(z)$ have the same feature. On the other hand, the characteristics of $S_{EM} + S_e + S_I$ vary with the forms of the two functions $U^\mu(x)$ and $V^\mu(y)$ in (16); therefore, $S_{EM} + S_e + S_I$ represents a class of models rather than a single model.

We now take

$$u^\lambda(x) = 0, \quad v^\lambda(y) = \int d^4y' W(y - y') j^\lambda(y') \tag{26}$$

in (25) and $f(z) = \delta^4(z)$ as an example to show some properties of the corresponding equations of motion of a charged particle and electromagnetic field. For this choice, (16) becomes

$$S_{I(1)} = -e \int d^4x d^4y j^\alpha(x) \delta^4(x^\lambda - V^\lambda(y)) V(y) \tilde{V}_\alpha^\beta(y) A_\beta(y), \tag{27}$$

where $V^\lambda(y)$ is given by (25) and (26), i.e.,

$$V^\lambda(y) = y^\lambda + a v^\lambda(y) = y^\lambda + a \int d^4y' W(y - y') j^\lambda(y');$$

$V(y)$ and $\tilde{V}_\alpha^\beta(y)$ are given by (58) and (59), respectively, in the last section of this paper, and $v^\lambda(y)$ in (58) and (59) is given by (26).

From the action $S_{EM} + S_e + S_{I(1)}$ and the action principle, we obtain the equations of motion of the electromagnetic field and charged particle:

$$\frac{\partial}{\partial y^\nu} (A^{\mu,\nu}(y) - A^{\nu,\mu}(y)) = e J^\mu(y), \quad J^\mu(y) = V(y) \tilde{V}_\nu^\mu(y) \int d^4x j^\nu(x) \delta^4(x^\lambda - V^\lambda(y)); \tag{28}$$

$$\left(i \gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \psi(x) = e \gamma^\alpha \Phi_\alpha(x) \psi(x), \quad \bar{\psi}(x) \left(i \gamma^\mu \frac{\partial}{\partial x^\mu} + m \right) = -e \bar{\psi}(x) \gamma^\alpha \Phi_\alpha(x); \tag{29}$$

$$\begin{aligned} \Phi_\alpha(x) = & \int d^4y \delta^4(x^\lambda - V^\lambda(y)) V(y) \tilde{V}_\alpha^\beta(y) A_\beta(y) - a \int d^4z d^4y j^\beta(z) \\ & \times \delta^4(z^\lambda - V^\lambda(y)) V(y) \tilde{V}_\alpha^\rho(y) \tilde{V}_\beta^\sigma(y) (A_{\rho,\sigma}(y) - A_{\sigma,\rho}(y)) W(z - x). \end{aligned} \tag{30}$$

It is obvious that (28) is invariant under the transformation given in (4) and (17). By the same method as in the proof that the action $S_{EM} + S_e + S_I$ is invariant under the gauge transformation (4) and (17), we can prove that (29) has the same gauge invariance.

It is easy to prove that the equation of motion (29) of a charged particle leads to the current conservation (6); on the other hand, as is well known, a very important and basic property of the equation of motion of the electromagnetic field $\frac{\partial}{\partial y^\nu} (A^{\mu,\nu}(y) - A^{\nu,\mu}(y)) = e j^\mu(y)$ of the conventional local QED is that which leads to the current conservation (6) by the approach $0 = \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu} (A^{\mu,\nu}(y) - A^{\nu,\mu}(y)) = e \frac{\partial j^\mu(y)}{\partial y^\mu}$. We prove that (28) has the same property.

From (28), we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu} (A^{\mu,\nu}(y) - A^{\nu,\mu}(y)) = e \frac{\partial J^\mu(y)}{\partial y^\mu} \\ &= e \frac{\partial}{\partial y^\mu} (V(y) \tilde{V}_\nu^\mu(y) \int d^4x j^\nu(x) \delta^4(x^\lambda - V^\lambda(y))) \\ &= e \frac{\partial (V(y) \tilde{V}_\nu^\mu(y))}{\partial y^\mu} \int d^4x j^\nu(x) \delta^4(x^\lambda - V^\lambda(y)) \\ &\quad + e V(y) \tilde{V}_\nu^\mu(y) \int d^4x j^\nu(x) \frac{\partial \delta^4(x^\lambda - V^\lambda(y))}{\partial y^\mu}, \end{aligned} \tag{31}$$

according to (21), the first term in the last expression of (31) vanishes; according to (23), we have $\frac{\partial \delta^4(x^\lambda - V^\lambda(y))}{\partial y^\mu} = -\frac{\partial \delta^4(x^\lambda - V^\lambda(y))}{\partial x^\rho} V_\mu^\rho(y)$. Thus (31) becomes

$$\begin{aligned}
 0 &= -eV(y)\tilde{V}_\nu^\mu(y)V_\mu^\rho(y)\int d^4xj^\nu(x)\frac{\partial\delta^4(x^\lambda-V^\lambda(y))}{\partial x^\rho} \\
 &= eV(y)\delta_\nu^\rho(y)\int d^4x\frac{\partial j^\nu(x)}{\partial x^\rho}\delta^4(x^\lambda-V^\lambda(y)) \\
 &= eV(y)\int d^4x\frac{\partial j^\nu(x)}{\partial x^\nu}\delta^4(x^\lambda-V^\lambda(y)),
 \end{aligned}$$

where we have used some of the formulas in (15) and performed integration by parts with respect to x^λ . From the above result, we obtain the current conservation (6).

In the above discussions, $U^\mu(x)$ and $V^\mu(y)$ are mainly dependent on ψ and $\bar{\psi}$; on the other hand, $U^\mu(x)$ and $V^\mu(y)$ can be dependent on A^λ alone. For example, if we take $u^\lambda(x) = 0$ and $v^\lambda(y) = A_\perp^\lambda(y)$ in (25) and $f(z) = \delta^4(z)$, where A_\perp^λ is the transverse four-vector corresponding to electromagnetic four-potential A^μ , whose formal definition can be found in ref. 9,10, then (16) becomes

$$S_{I(2)} = -e\int d^4x d^4y j^\alpha(x)\delta^4(x^\lambda - y^\lambda - aA_\perp^\lambda(y))V(y)\tilde{V}_\alpha^\beta(y)A_\beta(y),$$

where $V(y)$ and $\tilde{V}_\alpha^\beta(y)$ are still given by (58) and (59), respectively, in the last section of this paper, but now in (58) and (59), $v^\lambda(y) = A_\perp^\lambda(y)$.

For this case, from the action $S_{EM} + S_e + S_{I(2)}$, we obtain the equations of motion of the charged particle and electromagnetic field:

$$\left(i\gamma^\mu\frac{\partial}{\partial x^\mu} - m\right)\psi(x) = e\gamma^\alpha\int d^4y\delta^4(x^\lambda - V^\lambda(y))V(y)\tilde{V}_\alpha^\beta(y)A_\beta(y)\psi(x), \tag{32}$$

$$\frac{\partial}{\partial y^\nu}(A^{\nu,\mu}(y) - A^{\nu,\mu}(y)) = eJ^\mu(y) + aeK_\perp^\mu(y), \tag{33}$$

where $J^\mu(y)$ is given by (28), $K_\perp^\lambda(y)$ is the transverse four-vector corresponding to $K^\mu(y)$, and

$$\begin{aligned}
 K^\mu(y) &= \eta^{\mu\alpha}((v_{\alpha,\beta}(z) - v_{\beta,\alpha}(z)) - a(v_{\alpha,\lambda}(z)v_{\beta,\lambda}^\lambda(z) - v_{\beta,\lambda}(z)v_{\alpha,\lambda}^\lambda(z)) \\
 &\quad + a^2(v_{\alpha,\rho}(z)v_{,\sigma}^\rho(z)v_{,\beta}^\sigma(z) - v_{\beta,\rho}(z)v_{,\sigma}^\rho(z)v_{,\alpha}^\sigma(z))) \\
 &\quad \times \int d^4x\delta^4(x^\lambda - V^\lambda(z))j^\beta(x).
 \end{aligned} \tag{34}$$

The action $S_{EM} + S_e + S_{I(2)}$ and two equations of motion (32) and (33) are invariant under the gauge transformation (4) and (17), since the transverse four-vector A_\perp^λ is invariant under the gauge transformation (4); besides, it is easy to prove that the equation of motion (32) of the charged particle leads to the current conservation (6), and, using the above method that proves that (28) leads to (6), we can prove that the equation of motion of the electromagnetic field (33) leads to (6).

Similar to (9), it is easy to prove that $S_e + S_I$ can be written in the form

$$\begin{aligned}
 S_e + S_I &= \int d^4x \int d^4y \bar{\psi}(x)\exp\left[-ie\int_{y^\rho}^{x^\rho} dz_1^\sigma \int d^4z_2 U_\sigma^\alpha(z_1)f(U^\lambda(z_1) - V^\lambda(z_2))\right. \\
 &\quad \left.\times V(z_2)\tilde{V}_\alpha^\beta(z_2)A_\beta(z_2)\right]\left(i\gamma^\mu\frac{\partial}{\partial x^\mu} - m\right)\delta^4(x - y)\psi(y),
 \end{aligned}$$

hence, similar to (10), by replacing $\delta^4(x - y)$ in the above formula by a scalar function $g(x - y)$ that is independent of $\psi(x)$ and $A^\mu(x)$, we can establish a class of models of nonlocal QED, of which the action is $S_{EM} + S_{e-I}$, where S_{EM} is still given by (1), and

$$\begin{aligned}
 S_{e-I} &= \int d^4x \int d^4y \bar{\psi}(x)\exp\left[-ie\int_{y^\rho}^{x^\rho} dz_1^\sigma \int d^4z_2 U_\sigma^\alpha(z_1)f(U^\lambda(z_1) - V^\lambda(z_2))\right. \\
 &\quad \left.\times V(z_2)\tilde{V}_\alpha^\beta(z_2)A_\beta(z_2)\right]\left(i\gamma^\mu\frac{\partial}{\partial x^\mu} - m\right)g(x - y)\psi(y).
 \end{aligned}$$

We can further extend the above action in terms of (11).

Quantization of the models

From (28) and (29), or (32) and (33), we see that the free charged particle and free electromagnetic field still obey the local Dirac equation and the local Maxwell equation of free fields, respectively. The models thus belong to the so-called “theory with nonlocal interaction”. For this theory, so far, the unique method of quantization is to employ the formalism based on the Yang-Feldman equations^{11,12}. Here, we list only the main steps of quantization of the models as follows. In the last step, we include brief discussions, since we shall employ the Lehmann-Symanzik-Zimmermann reduction formulas instead of the S-matrix.

The method of quantization works in the Heisenberg Picture; for simplicity, we employ the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ for electromagnetic four-potential A^μ .

I “in” and “out” fields and the Fock space.

The field operators $\psi_{\text{in(out)}}(x)$, $\bar{\psi}_{\text{in(out)}}(x)$ and $A_{\text{in(out)}}^\mu(y)$ of the “in” and “out” fields satisfy the equations $(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m)\psi_{\text{in(out)}}(x) = 0$, $\mathbf{A}_{\text{in(out)}}(y) = 0$, and $\nabla \cdot \mathbf{A}_{\text{in(out)}}(y) = 0$, respectively, for which the concrete expressions are given by (13.50) and (14.33) in ref. 7. All states of the corresponding Fock space are generated by repeatedly acting with the creation operators $b_{\text{rin}}^+(\mathbf{p})$, $d_{\text{rin}}^+(\mathbf{p})$ and $a_{\text{rin}(\lambda)}^+(\mathbf{k})$ in the expressions of $\psi_{\text{in(out)}}(x)$, $\bar{\psi}_{\text{in(out)}}(x)$ and $A_{\text{in(out)}}^\mu(y)$ on the vacuum state $|0\rangle$.

II The asymptotic conditions and the Yang-Feldman equations.

Using the idea of “weak convergence”^{7,13,14}, the asymptotic conditions for the field operators $\psi(x)$, $\bar{\psi}(x)$ and $\mathbf{A}(y)$ read:

$$\psi(x) \rightarrow \psi_{\text{in(out)}}(x), \bar{\psi}(x) \rightarrow \bar{\psi}_{\text{in(out)}}(x), \mathbf{A}(y) \rightarrow \mathbf{A}_{\text{in(out)}}(x), \text{ as } x^0 \rightarrow - (+)\infty. \tag{35}$$

And, furthermore, considering the asymptotic conditions (35), from the equations of motion (30) and (28) we obtain the Yang-Feldman equations⁷:

$$\psi(x) = \psi_{\text{in(out)}}(x) + e \int d^4x' S_{\text{ret(adv)}}(x - x') \gamma^\alpha \Phi_\alpha(x') \psi(x'), \tag{36}$$

$$\bar{\psi}(x) = \bar{\psi}_{\text{in(out)}}(x) + e \int d^4x' \bar{\psi}(x') \Phi_\alpha(x') \gamma^\alpha S_{\text{adv(ret)}}(x' - x), \tag{37}$$

$$\mathbf{A}(y) = \mathbf{A}_{\text{in(out)}}(y) + e \int d^4y' D_{\text{ret(adv)}}(y - y') \mathbf{J}^{\text{tr}}(y'). \tag{38}$$

In (38), $\mathbf{J}^{\text{tr}}(y)$ is the transverse three-vector corresponding to $J^\mu(y)$ introduced by (28). If the equations of motion of the charged particle and electromagnetic field are (32) and (33), respectively, then it is easy to write out the corresponding Yang-Feldman equations, similar to (36)~(38); besides, under the Coulomb gauge and the asymptotic conditions (35), the transverse four-vector A_\perp^μ corresponding to A^μ reads

$$A_\perp^\mu(y) = (0, \mathbf{A}_{\text{in}}(y)) + e \int d^4y' D_{\text{ret}}(y - y') (J^\mu(y') + aK_\perp^\mu(y')).$$

III The Lehmann-Symanzik-Zimmermann (LSZ) reduction formulas

The last step of the method of quantization employing the formalism based on the Yang-Feldman equations is to introduce the S-matrix by

$$\psi_{\text{out}}(x) = S^{-1} \psi_{\text{in}}(x) S, \bar{\psi}_{\text{out}}(x) = S^{-1} \bar{\psi}_{\text{in}}(x) S, \mathbf{A}_{\text{out}}(x) = S^{-1} \mathbf{A}_{\text{in}}(x) S; \tag{39}$$

some examples of calculating the S-matrix in terms of (39) can be found in refs 15, 16.

However, generally speaking, for infinite-dimensional matrices $\psi_{\text{in(out)}}(x)$, $\bar{\psi}_{\text{in(out)}}(x)$ and $\mathbf{A}_{\text{in(out)}}(y)$, perhaps such a matrix S does not exist^{7,13}; even if it exists, it may not be unitary¹¹.

Conversely, for the theory with nonlocal interaction, we can prove that the LSZ reduction formulas^{7,13,14} still hold. Hence, we employ the LSZ reduction formulas instead of the S-matrix to calculate the transition amplitude. For example, for the case that the initial and final states are an electron, the corresponding transition amplitude is

$$\begin{aligned} \langle (\mathbf{p}', r')_{\text{out}} | (\mathbf{p}, r)_{\text{in}} \rangle &= \langle 0 | b_{r' \text{out}}(\mathbf{p}') b_{\text{rin}}^+(\mathbf{p}) | 0 \rangle = \delta_{r'r} \delta^3(\mathbf{p}' - \mathbf{p}) \\ &\quad - \int d^4z_1 \int d^4z_2 \bar{\psi}_{r'r}^{(+)}(z_1, \mathbf{p}') \left(i\gamma^\mu \frac{\partial}{\partial z_1^\mu} - m \right) \\ &\quad \times \langle 0 | T(\psi(z_1) \bar{\psi}(z_2)) | 0 \rangle \left(-i\gamma^\nu \frac{\partial}{\partial z_2^\nu} - m \right) \psi_r^{(-)}(z_2, \mathbf{p}). \end{aligned} \tag{40}$$

Substituting the field operators $\psi(x)$ and $\bar{\psi}(x)$ obtained from (36)~(38) by an iterative method into (40), we can calculate the transition amplitude $\langle (\mathbf{p}', r')_{\text{out}} | (\mathbf{p}, r)_{\text{in}} \rangle$.

A question naturally arises from the above discussions: now that the transition amplitude is only an element of the S-matrix, if the S-matrix defined by (39) does not exist, or exists but is not unitary, then is the transition amplitude determined by the LSZ reduction formula correct?

Here, we only address this question briefly. Notice that there are in fact two types of S-matrices: one is defined based on operators, e.g., introduced by (39); the other is defined based on a state vector^{7,13,14} whose element, i.e., transition amplitude, can be determined by the LSZ reduction formula. The differences between the two types of

S-matrices are subtle. For example, the asymptotic conditions (35) based on operators are in fact incorrect; the correct forms of the asymptotic conditions are based on the state vector^{7,13,14}:

$$\begin{aligned}\langle\beta|\psi(x)|\alpha\rangle &\rightarrow\langle\beta|\psi_{\text{in(out)}}(x)|\alpha\rangle,\langle\beta|\bar{\psi}(x)|\alpha\rangle\rightarrow\langle\beta|\bar{\psi}_{\text{in(out)}}(x)|\alpha\rangle, \\ \langle\beta|\mathbf{A}(x)|\alpha\rangle &\rightarrow\langle\beta|\mathbf{A}_{\text{in(out)}}(x)|\alpha\rangle,\quad\text{as }x^0\rightarrow-(+)\infty.\end{aligned}$$

(The exact expressions of the asymptotic conditions can be found in ref. 14, e.g., the formulas 7(106a) and 7(106b) in ref. 14). And, thus, instead of (36)–(38), the correct forms of the Yang–Feldman equations should be¹⁴

$$\langle\beta|\psi(x)|\alpha\rangle=\langle\beta|\psi_{\text{in(out)}}(x)|\alpha\rangle+e\int d^4x'S_{\text{ret(adv)}}(x-x')\bar{\gamma}^\alpha\langle\beta|\Phi_\alpha(x')\psi(x')|\alpha\rangle,$$

etc. Hence, regardless of whether the S-matrix based on operators exists, we can introduce the type of S-matrix based on the state vector, and, in addition, take advantage of the LSZ reduction formula to determine the element of this type of S-matrix, i.e., the transition amplitude.

We therefore obtain a complete quantum theory of the models of QED with nonlocal interaction, as presented in the above section.

However, although we employ the LSZ reduction formulas instead of the S-matrix, basic principles such as Lorentz and gauge invariance, unitarity and causality of the S-matrix must be satisfied, of course. We now ask that the elements of S-matrix, i.e., transition amplitude, satisfy these principles. And, furthermore, according to the LSZ reduction formulas, the question of whether the elements of the S-matrix satisfy these principles is reduced to investigating the properties of such Green functions as $\langle 0|T(\dots\psi(x)\dots\bar{\psi}(y)\dots\mathbf{A}(z)\dots)|0\rangle$.

For example, it is obvious that if such Green functions as $\langle 0|T(\dots\psi(x)\dots\bar{\psi}(y)\dots\mathbf{A}(z)\dots)|0\rangle$ are Lorentz and gauge invariant, then the elements of the S-matrix are as well. And, the condition of unitarity corresponding to that of the S-matrix is now $\sum_\gamma\langle\beta|S^+|\gamma\rangle\langle\gamma|S|\alpha\rangle=\langle\beta|\alpha\rangle$, which also leads to the investigation of the properties of such Green functions as $\langle 0|T(\dots\psi(x)\dots\bar{\psi}(y)\dots\mathbf{A}(z)\dots)|0\rangle$.

As for the condition of causality of the S-matrix, following N. N. Bogoliubov and D. V. Shirkov¹⁷, we write the interaction action (16) in the form

$$S_I=-e\int d^4x d^4y g(x)j^\alpha(x)U_\alpha^\beta(x)f(U^\lambda(x)-V^\lambda(y))V(y)\tilde{V}_\beta^\gamma(y)A_\gamma(y),$$

where $0\leq g(x)\leq 1$. When $g(x)=0$, the interaction does not exist; when $0<g(x)<1$, the interaction is joined partly; when $g(x)=1$, the interaction is joined fully. Because the function $g(x)$ appears in S_I , it will appear in the equations of motion of the charged particle and electromagnetic field, and, additionally, in such Green functions as $\langle 0|T(\dots\psi(x)\dots\bar{\psi}(y)\dots\mathbf{A}(z)\dots g(w)\dots)|0\rangle$ and the elements of the S-matrix $\langle\beta|S(g)|\alpha\rangle$. Therefore, determining whether the elements of the S-matrix satisfy the condition of causality

$$\sum_\gamma\frac{\delta}{\delta g(x)}\left(\frac{\delta\langle\beta|S(g)|\gamma\rangle}{\delta g(y)}\langle\gamma|S^+(g)|\alpha\rangle\right)=0\quad(x\leq y),$$

which corresponds to the condition of causality of the S-matrix given by ref. 17 (see the formula (17.30) in ref. 17), is still reduced to investigating the properties of such Green functions as $\langle 0|T(\dots\psi(x)\dots\bar{\psi}(y)\dots\mathbf{A}(z)\dots g(w)\dots)|0\rangle$.

On the other hand, experiences with some known theories show that the properties mentioned above of Green functions can be investigated only after the theories are researched deeply. For example, if we employ the Coulomb gauge in the conventional local QED, then Lorentz and gauge invariance of the elements of the S-matrix can be proved only after the rules of the Feynman diagrams of the theory are obtained (see ref. 7, Chapter 17). A different example is unitarity of the S-matrix of the theory of the non-Abelian gauge field: although as early as 1963, R. P. Feynman pointed out that one must add some additive terms in the theory of the non-Abelian gauge field to guarantee unitarity of the S-matrix under Feynman gauge¹⁸, and even though a general approach of adding additive terms was given by L. Fadeev and V. N. Popov¹⁹, the unitarity of the S-matrix of the theory of the non-Abelian gauge field can be proved only after Slavnov–Taylor identities are established.

What should we do when the models presented in this paper do not satisfy unitarity or (and) causality? A revelation from the theory of the non-Abelian gauge field is that, if the elements of the S-matrix of the models presented in this paper do not satisfy unitarity or (and) causality, then maybe we can add some appropriate additive terms to restore unitarity and causality, similarly to the addition of ghost fields to restore the unitarity of the S-matrix in the theory of the non-Abelian gauge field. These questions will be studied further.

Calculation result of the vacuum polarization

Concrete calculation of the transition amplitude based on the LSZ reduction formula is lengthy and complex even for the conventional local QED. For example, for the conventional local QED, (correspondingly, $\Phi_\alpha=A_\alpha$ and $J^\mu=j^\mu$ in (36)–(38)), in order to obtain the self energy of the electron and the vacuum polarization in terms of (40), we must calculate the field operators $\psi(x)$ and $\bar{\psi}(x)$ in terms of (36)–(38) by an iterative method up to second-order and third-order approximations, respectively. Of course, the vacuum polarization can be obtained from the self energy of the photon, but for this, we still must calculate the field operator $\mathbf{A}(x)$ in terms of (36)–(38) by an iterative method up to second-order approximation. On the other hand, G. Källén in ref. 20 presents an approach to obtain the vacuum polarization for the conventional local QED: the approach is to calculate the linear response of the vacuum expectation value of current to an external electromagnetic field without using the

LSZ reduction formula; one only needs to calculate the field operators $\psi(x)$ and $\bar{\psi}(x)$ in terms of (36) and (37) by an iterative method up to first-order approximation. Here we follow this approach to calculate the vacuum polarization; of course, in (36) and (37), $\Phi_\alpha(x)$ is now given by (30).

The concrete calculation process is lengthy and complex; here we only present the main steps and results. According to (36) and (37), up to first-order approximation, we have

$$\begin{aligned} \psi(x) &= \psi_{\text{in}}(x) + e \int d^4x' S_{\text{ret}}(x-x') \gamma^\alpha \Phi_{\alpha(\text{in})}(x') \psi_{\text{in}}(x'); \\ \bar{\psi}(x) &= \bar{\psi}_{\text{in}}(x) + e \int d^4x' \bar{\psi}_{\text{in}}(x') \Phi_{\alpha(\text{in})}(x') \gamma^\alpha S_{\text{adv}}(x'-x). \end{aligned} \tag{41}$$

In (41), $\Phi_{\alpha(\text{in})}(x)$ is given by (30), in which all field quantities are zero-order approximations, namely,

$$j^\mu(x) = j_{\text{in}}^\mu(x), \quad A^\mu(x) = A_{\text{in}}^\mu(x) + A_{\text{ext}}^\mu(x), \quad A_{\text{in}}^\mu(x) = (0, \mathbf{A}_{\text{in}}(x)), \tag{42}$$

where A_{ext}^μ is an external electromagnetic field.

If we use

$$\begin{aligned} \delta^4(x^\lambda - V^\lambda(y)) &= \delta^4(x^\lambda - y^\lambda - av^\lambda(y)) \\ &= \delta^4\left(x^\lambda - y^\lambda - a \int d^4y' W(y-y') j_{(n-1)}^\lambda(y')\right) \\ &= \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot (x-y)} \exp\left(-iak_\lambda \int d^4y' W(y-y') j_{(n-1)}^\lambda(y')\right) \\ &= \delta^4(x-y) - a \frac{\partial}{\partial x^\lambda} \delta^4(x-y) \int d^4y' W(y-y') j_{(n-1)}^\lambda(y') \\ &\quad + \frac{a^2}{2!} \frac{\partial^2}{\partial x^{\lambda_1} \partial x^{\lambda_2}} \delta^4(x-y) \int d^4y_1 W(y-y_1) \int d^4y_2 W \\ &\quad \times (y-y_2) j_{(n-1)}^{\lambda_1}(y_1) j_{(n-1)}^{\lambda_2}(y_2) + \dots \end{aligned} \tag{43}$$

to deal with the factor $\delta^4(x^\lambda - V^\lambda(y))$ in (30), then we should in principle calculate all terms in (43); in other words, we should calculate all orders of the parameter a in (43).

Based on (43), we obtain

$$\begin{aligned} \Phi_{\alpha(\text{in})}(x) &\approx \int d^4y \delta^4(x^\lambda - y^\lambda - av^\lambda(y)) V(y) \tilde{V}_\alpha^\beta(y) A_\beta(y) \\ &\approx A_{\alpha(\text{in})}(x) + A_{\alpha(\text{ext})}(x) - av_{,\alpha}^\beta(x) (A_{\beta(\text{in})}(x) + A_{\beta(\text{ext})}(x)) \\ &\quad + a^2 (v_{,\lambda}^\beta(x) v_{,\alpha}^\lambda(x) + v^\lambda(x) v_{,\alpha,\lambda}^\beta(x)) (A_{\beta(\text{in})}(x) + A_{\beta(\text{ext})}(x)) + \dots, \end{aligned} \tag{44}$$

where we have used the expressions (58) and (59) of $V(y)$ and $\tilde{V}_\alpha^\beta(y)$ in the last section of this paper, respectively, and ignored all derivative terms of A_{ext}^μ , since what we calculate is the linear response of the vacuum expectation value of the current to A_{ext}^μ .

Since the function $W(x-y)$ in (26) is a (real) function but not a matrix, we can choose

$$W(x-y) = \int \frac{d^4k}{(2\pi)^4} \sum_{n=1}^N b_n (k^2)^n e^{-b_n(k^2)^2} e^{-ik \cdot (x-y)}, \tag{45}$$

or by taking advantage of²⁰

$$\bar{\Delta}(x; m) = -\mathbf{PP} \cdot \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip \cdot x}}{p^2 - m^2} = \frac{1}{4\pi} \delta(x^2) - \frac{m}{8\pi} \theta(x^2) \frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}}, \tag{46}$$

where $J_1(z)$ is a Bessel function, we choose

$$W(x-y) = \sum_{n=1}^N b_n \frac{\partial^n \bar{\Delta}(x; M_n)}{\partial (M_n^2)^n}, \tag{47}$$

where all b_n and M_n are real constants.

Substituting (44) into (41), a problem of operator ordering of ψ_{in} and $\bar{\psi}_{\text{in}}$ arises; we can prove that all operator orderings of ψ_{in} and $\bar{\psi}_{\text{in}}$ in (41) can be determined by the current conservation (6).

Based on the above discussions, we obtain

$$\begin{aligned}
 \langle 0|ej^\mu(x)|0\rangle &= e^2 \langle 0|j_{(1)}^\mu(x)|0\rangle = 4e^2 \int d^4x' A_{\nu \text{ ext}}(x') \Omega_{(0)}^{\mu\nu}(x-x') \\
 &+ 4e^2 \int d^4x' A_{\nu \text{ ext}}(x') \left(\sum_{n=2}^{\infty} a^n \Theta_n^{(0)} \right) \Omega_{(0)}^{\mu\nu}(x-x') \\
 &+ \left(\sum_{n=2}^{\infty} a^n \Theta_n^{(1)} \right) \Omega_{(1)}^{\mu\nu}(x-x') \\
 &+ e^2 \int d^4x' A_{\nu \text{ ext}}(x') \sum_{n=3}^{\infty} a^n \Omega_n^{\prime\mu\nu}(x-x'),
 \end{aligned}
 \tag{48}$$

where all $\Theta_n^{(0)}$ and $\Theta_n^{(1)}$ ($n \geq 2$) are constants arising from (43) and (47),

$$\begin{aligned}
 \Omega_{(0)}^{\mu\nu}(x-x') &= -\int \frac{d^4p_1}{(2\pi)^4} \frac{e^{-ip_1 \cdot (x-x')}}{(p_1^0 + i\varepsilon)^2 - E_{p_1}^2} \int \frac{d^4p_2}{(2\pi)^3} \delta(p_2^2 - m^2) e^{-ip_2 \cdot (x'-x)} \\
 &\times (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu} (p_1 \cdot p_2 - m^2)),
 \end{aligned}
 \tag{49}$$

$$\Omega_{(1)}^{\mu\nu}(x-x') = -\int \frac{d^4p_1}{(2\pi)^4} \frac{e^{-ip_1 \cdot (x-x')}}{(p_1^0 + i\varepsilon)^2 - E_{p_1}^2} \int \frac{d^4p_2}{(2\pi)^3} \delta(p_2^2 - m^2) e^{-ip_2 \cdot (x'-x)} (p_1^\mu + p_2^\mu) p_2^\nu,
 \tag{50}$$

and $\Omega_n^{\prime\mu\nu}(x-x')$ ($n \geq 3$) in (48) are additional terms that cannot be written in the form of the second term in (48). Notice that no term is proportional to $a^1(=a)$ in (48), since the traces of the γ -matrices in this term vanish; this result is obtained by the software FeynCalc.

We can prove that the first term in (48) is exactly the same as the results (2.18) and (2.20) in ref. 21. J. Schwinger in ref. 21 writes $\Omega_{(0)}^{\mu\nu}(x)$, given by (49), in the form

$$\Omega_{(0)}^{\mu\nu}(x) = \Omega_{(0)1}^{\mu\nu}(x) + \Omega_{(0)2}^{\mu\nu}(x),
 \tag{51}$$

$$\begin{aligned}
 \Omega_{(0)1}^{\mu\nu}(x) &= \frac{i}{2} \int \frac{d^4p_1}{(2\pi)^3} \delta(p_1^2 - m^2) \varepsilon(p_1^0) e^{-ip_1 \cdot x} \int \frac{d^4p_2}{(2\pi)^3} \delta(p_2^2 - m^2) e^{ip_2 \cdot x} \\
 &\times (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu} p_1 \cdot p_2 + m^2 g^{\mu\nu}),
 \end{aligned}
 \tag{52}$$

$$\begin{aligned}
 \Omega_{(0)2}^{\mu\nu}(x) &= -\text{PP} \cdot \int \frac{d^4p_1}{(2\pi)^4} \frac{e^{-ip_1 \cdot x}}{p_1^2 - m^2} \int \frac{d^4p_2}{(2\pi)^3} \delta(p_2^2 - m^2) e^{ip_2 \cdot x} \\
 &\times (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu} p_1 \cdot p_2 + m^2 g^{\mu\nu}).
 \end{aligned}
 \tag{53}$$

J. Schwinger in ref. 21 proves that if no real pair creation has occurred, then $\Omega_{(0)1}^{\mu\nu}(x)$ introduced by (52) vanishes; this conclusion is proved in the Interaction Picture in ref. 21, but here we work in the Heisenberg Picture, and, thus, have to use a different method to deal with $\Omega_{(0)1}^{\mu\nu}(x)$.

The method we use is to calculate $\Omega_{(0)1}^{\mu\nu}(x)$ straightforwardly. This method can not only prove J. Schwinger's conclusion in ref. 21, but also obtain a concrete result for $\Omega_{(0)1}^{\mu\nu}(x)$ for the case that real pair creation has occurred.

The calculation result of $\Omega_{(0)2}^{\mu\nu}(x)$ given by J. Schwinger in ref. 22 is

$$\begin{aligned}
 \Omega_{(0)2}^{\mu\nu}(x) &= -\frac{K}{24\pi^2} \int \frac{d^4p}{(2\pi)^4} (p^\mu p^\nu - g^{\mu\nu} p^2) e^{-ip \cdot x} \\
 &- \frac{1}{64\pi^2} \frac{\partial^2}{\partial x^\lambda \partial x_\lambda} \int_0^1 w^2 \left(1 - \frac{w^2}{3} \right) dw \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot x} \frac{p^\mu p^\nu - g^{\mu\nu} p^2}{\frac{1-w^2}{4} p^2 - m^2},
 \end{aligned}$$

where

$$K = \lim_{k \rightarrow \infty} \left(\ln \frac{\sqrt{k^2 + m^2} + k}{m} - 1 \right)
 \tag{54}$$

is logarithmically divergent.

We follow the above approach to deal with $\Omega_{(1)}^{\mu\nu}(x)$ given by (50); for the case that no real pair creation has occurred, we finally obtain

$$\begin{aligned}
 \langle 0|ej^\mu(x)|0\rangle &= e^2\langle 0|j_{(1)}^\mu(x)|0\rangle = K\frac{e^2}{6\pi^2}\left(1 + \sum_{n=2}^\infty a^n\Theta_n^{(0)} + \frac{1}{4}\sum_{n=2}^\infty a^n\Theta_n^{(1)}\right) \\
 &\times\left(\frac{\partial^2}{\partial x_\mu\partial x_\nu} - g^{\mu\nu}\frac{\partial^2}{\partial x^\lambda\partial x_\lambda}\right)A_{\nu\text{ ext}}(x) \\
 &+ \frac{e^2}{(4\pi)^2}\left(1 + \sum_{n=2}^\infty a^n\Theta_n^{(0)} - \frac{1}{2}\sum_{n=2}^\infty a^n\Theta_n^{(1)}\right) \\
 &\times\int d^4x'\left(\frac{\partial^2}{\partial x'_\mu\partial x'_\nu} - g^{\mu\nu}\frac{\partial^2}{\partial x'^\lambda\partial x'_\lambda}\right)A_{\nu\text{ ext}}(x') \\
 &\times\frac{\partial^2}{\partial x^\lambda\partial x_\lambda}\int_0^1 w^2\left(1 - \frac{w^2}{3}\right)dw\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-x')}\frac{1}{\frac{1-w^2}{4}p^2 - m^2} \\
 &- \Lambda^2\frac{e^2}{\pi^2}\left(\sum_{n=2}^\infty a^n\Theta_n^{(1)}\right)\int d^4x'\frac{\partial A_{\text{ext}}^\lambda(x')}{\partial x'^\lambda} \\
 &\times\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-x')}\frac{\theta(-k^2)}{\sqrt{-k^2}}\left(\frac{k^0}{k_3 + \sqrt{-k^2}}\left(\frac{k^\mu}{\sqrt{-k^2}} - \delta_3^\mu\right) + \delta_0^\mu\right) \\
 &+ e^2\int d^4x'A_{\nu\text{ ext}}(x')\sum_{n=3}^\infty a^n\Omega_n'^{\mu\nu}(x-x'), \tag{55}
 \end{aligned}$$

where K is given by (54), $\Lambda = \lim_{p \rightarrow \infty} p$, and the third term including Λ is a new divergent term arising from the new theory.

From (55), we see that

- (1) By choosing constants $\Theta_n^{(0)}$ and $\Theta_n^{(1)}$ satisfying

$$1 + \sum_{n=2}^\infty a^n\Theta_n^{(0)} + \frac{1}{4}\sum_{n=2}^\infty a^n\Theta_n^{(1)} = 0, \quad \sum_{n=2}^\infty a^n\Theta_n^{(0)} - \frac{1}{2}\sum_{n=2}^\infty a^n\Theta_n^{(1)} = 0,$$

the first (divergent) term in (55) is eliminated and the second term in (55) presents the lowest nontrivial order of the vacuum polarization given by the formula (A.39) in ref. 21.

- (2) The third term (new divergent term) in (55) arising from the new theory can be eliminated by imposing Lorenz gauge $\frac{\partial A_{\text{ext}}^\lambda(x)}{\partial x^\lambda} = 0$ on external electromagnetic field $A_{\text{ext}}^\lambda(x)$.
- (3) We have not dealt with $\Omega_n'^{\mu\nu}(x-x')$ ($n \geq 3$) in the last term in (55). It is possible that some terms arising from $\Omega_n'^{\mu\nu}(x-x')$ are similar to the first three terms in (55), and, thus, contribute to the first three terms in (55).
- (4) Generally speaking, since the new theory brings many new terms, it can be predicted that new divergent terms in the new theory presented in this paper will appear. Just as the method of higher covariant derivative regularization of gauge theories²³⁻²⁵ removes divergences in the original theory while at the same time bringing many new divergent terms, of course, all divergences are removed finally. We predict that the divergences brought by new terms arising from the new equations of motion (28) and (30) can be removed finally. For example, even if we don't impose the Lorenz gauge on the external electromagnetic field, the third term (new divergent term) in (55) cannot be eliminated; however, it is possible that it can be eliminated by some terms arising from the last term in (55).
- (5) Our purpose is to establish a theory of QED without divergence, but there are still two divergent constants K and Λ in (55). For this result we emphasize that (I) both K and Λ arise from the calculation approach from (43) to (48); and (II) it is because of the structure and properties of the theory, but not renormalization procedure, that both K and Λ are eliminated. As for the calculation approach from (43) to (48), which not only leads to a complex computation process, but also causes some divergent integrals, e.g., (49) and (50), strictly speaking, the divergent integrals bring uncertainty, especially for the case of changing variables in the integral. Hence, seeking a different calculation approach for the models presented in this paper, especially for the treatment method of the δ -function given by (43), is necessary. Can we find a calculation approach such that there is no divergence in the calculation result? This question will be studied further.

Finally, we emphasize that the theory we employ in this section to calculate the vacuum polarization is only a special choice of the models presented in the section "A class of models of quantum electrodynamics with non-local interaction" of this paper. Concretely, the theory employed in this section is the result of defining the two functions $U^\mu(x)$ and $V^\mu(y)$ in (16) by (25) and (26), and, also of defining the function $W(y-y')$ in (26) by (45) or (47). On the other hand, the models presented in the section "A class of models of quantum electrodynamics with nonlocal interaction" of this paper provide a wide range of choices. Hence, maybe it is possible to establish a theory of QED without divergence by choosing appropriate functions $f(z)$, $U^\mu(x)$ and $V^\mu(y)$ in the action (16).

Proofs of (21) and (23)

Both (21) and (23) play important roles in the proofs of the gauge invariance of the model and the conclusion that the equation of motion of the electromagnetic field (28) leads to the current conservation (6). Eqs (21) and (23) can be proved by a basic function algorithm. Here we prove them by straightforward calculation. Some expressions obtained in the proof process are also used in the calculation of the vacuum polarization.

In order to prove (21), without loss generality, we assume

$$V^\mu(y) = y^\mu + av^\mu(y), \tag{56}$$

since the arbitrary vector function $V^\mu(y)$ can always be written in such a form. According to (13),

$$V_\beta^\alpha(y) = \frac{\partial(y^\alpha + av^\alpha(y))}{\partial y^\beta} = \delta_\beta^\alpha + av_{,\beta}^\alpha(y). \tag{57}$$

By a straight forward calculation, we obtain the determinant $V(y)$ and the inverse matrix $\tilde{V}_\nu^\mu(y)$ of $V_\beta^\alpha(y)$ as follows.

$$V(y) = 1 + \sum_{n=1}^4 \frac{a^n}{n!} V_{(n)}(y); \tag{58}$$

$$\begin{aligned} V_{(1)}(y) &= v_{,\lambda}^\lambda(y), \\ V_{(2)}(y) &= v_{,\mu}^\mu(y)v_{,\nu}^\nu(y) - v_{,\nu}^\mu(y)v_{,\mu}^\nu(y), \\ V_{(3)}(y) &= v_{,\alpha}^\alpha(y)v_{,\beta}^\beta(y)v_{,\gamma}^\gamma(y) + 2v_{,\beta}^\alpha(y)v_{,\gamma}^\beta(y)v_{,\alpha}^\gamma(y) - 3v_{,\lambda}^\lambda(y)v_{,\nu}^\mu(y)v_{,\mu}^\nu(y), \\ V_{(4)}(y) &= v_{,\mu}^\mu(y)v_{,\nu}^\nu(y)v_{,\rho}^\rho(y)v_{,\sigma}^\sigma(y) + 3v_{,\nu}^\mu(y)v_{,\mu}^\nu(y)v_{,\rho}^\sigma(y)v_{,\sigma}^\rho(y) \\ &\quad + 8v_{,\lambda}^\lambda(y)v_{,\beta}^\alpha(y)v_{,\gamma}^\beta(y)v_{,\alpha}^\gamma(y) - 6v_{,\mu}^\mu(y)v_{,\nu}^\nu(y)v_{,\sigma}^\rho(y)v_{,\rho}^\sigma(y) \\ &\quad - 6v_{,\nu}^\mu(y)v_{,\rho}^\nu(y)v_{,\sigma}^\rho(y)v_{,\sigma}^\mu(y). \end{aligned}$$

$$\tilde{V}_\nu^\mu(y) = \frac{1}{V(y)} \left[\delta_\nu^\mu + \sum_{n=1}^3 \frac{a^n}{n!} \tilde{V}_{\nu(n)}^\mu(y) \right]; \tag{59}$$

$$\begin{aligned} \tilde{V}_{\nu(1)}^\mu(y) &= \delta_\nu^\mu V_{(1)}(y) - v_{,\nu}^\mu(y), \\ \tilde{V}_{\nu(2)}^\mu(y) &= \delta_\nu^\mu V_{(2)}(y) - 2v_{,\nu}^\mu(y)V_{(1)}(y) + 2v_{,\lambda}^\mu(y)v_{,\nu}^\lambda(y), \\ \tilde{V}_{\nu(3)}^\mu(y) &= \delta_\nu^\mu V_{(3)}(y) - 3v_{,\nu}^\mu(y)V_{(2)}(y) + 6v_{,\lambda}^\mu(y)v_{,\nu}^\lambda(y)V_{(1)}(y) - 6v_{,\alpha}^\mu(y)v_{,\beta}^\alpha(y)v_{,\nu}^\beta(y), \end{aligned}$$

where $V_{(1)}(y)$, $V_{(2)}(y)$ and $V_{(3)}(y)$ are introduced by (58).

Therefore, $\frac{\partial(V(y)\tilde{V}_\nu^\mu(y))}{\partial y^\mu} = \sum_{n=1}^3 \frac{a^n}{n!} \frac{\partial \tilde{V}_{\nu(n)}^\mu(y)}{\partial y^\mu}$, and according to the above expressions of $\tilde{V}_{\nu(n)}^\mu(y)$, we can verify that $\frac{\partial \tilde{V}_{\nu(n)}^\mu(y)}{\partial y^\mu} = 0$ ($n = 1, 2, 3$). Thus, (21) is proved.

In order to prove (23), based on the Fourier transform $f(x) = \int \frac{d^4k}{(2\pi)^4} \hat{f}(k) e^{ik \cdot x}$ of $f(x)$, we have

$$\begin{aligned} \frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial x^\alpha} &= \frac{\partial}{\partial x^\alpha} \int \frac{d^4k}{(2\pi)^4} \hat{f}(k) e^{ik_\lambda(U^\lambda(x) - V^\lambda(y))} \\ &= \int \frac{d^4k}{(2\pi)^4} ik_\beta \frac{\partial U^\beta(x)}{\partial x^\alpha} \hat{f}(k) e^{ik_\lambda(U^\lambda(x) - V^\lambda(y))} \\ &= U_\alpha^\beta(x) \int \frac{d^4k}{(2\pi)^4} ik_{\beta\lambda} \hat{f}(k) e^{ik_\lambda(U^\lambda(x) - V^\lambda(y))}; \end{aligned}$$

similarly,

$$\frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial y^\alpha} = -V_\alpha^\beta(y) \int \frac{d^4k}{(2\pi)^4} ik_{\beta\lambda} \hat{f}(k) e^{ik_\lambda(U^\lambda(x) - V^\lambda(y))}.$$

From the above two expressions and using some of the formulas in (15), we obtain

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4} ik_\mu \hat{f}(k) e^{ik_\lambda(U^\lambda(x) - V^\lambda(y))} &= \tilde{U}_\mu^\nu(x) \frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial x^\nu} \\ &= -\tilde{V}_\mu^\nu(y) \frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial y^\nu}. \end{aligned} \tag{60}$$

According to (60), each of $\frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial y^\alpha}$ and $\frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial x^\beta}$ can be transformed to the other; for example, from (60) we have

$$\begin{aligned}\tilde{U}_\mu^\nu(x) \frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial x^\nu} V_\tau^\mu(y) &= -V_\tau^\mu(y) \tilde{V}_\mu^\nu(y) \frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial y^\nu} \\ &= -\delta_\tau^\nu \frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial y^\nu} \\ &= -\frac{\partial f(U^\lambda(x) - V^\lambda(y))}{\partial y^\tau},\end{aligned}$$

which is just (23).

When we use a basic function algorithm to prove (60), we do not need to take advantage of the form of the Fourier transform of $f(x)$.

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Additional Information

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