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OPEN Matching relations for optimal entanglement concentration and purification

Fan-Zhen Kong^{1,2}, Hui-Zhi Xia¹, Ming Yang¹, Qing Yang¹ & Zhuo-Liang Cao³

The bilateral controlled NOT (CNOT) operation plays a key role in standard entanglement purification process, but the CNOT operation may not be the optimal joint operation in the sense that the output entanglement is maximized. In this paper, the CNOT operations in both the Schmidt-projection based entanglement concentration and the entanglement purification schemes are replaced with a general joint unitary operation, and the optimal matching relations between the entangling power of the joint unitary operation and the non-maximal entangled channel are found for optimizing the entanglement in- crement or the output entanglement. The result is somewhat counter-intuitive for entanglement concentration. The output entanglement is maximized when the entangling power of the joint unitary operation and the quantum channel satisfy certain relation. There exist a variety of joint operations with non-maximal entangling power that can induce a maximal output entanglement, which will greatly broaden the set of the potential joint operations in entanglement concentration. In addition, the entanglement increment in purification process is maximized only by the joint unitary operations (including CNOT) with maximal entangling power.

Entanglement plays an important role in quantum information processing. It is a prerequisite in quantum teleportation^{1,2}, entanglement swapping³, quantum cryptography⁴ and so on. Generating or enhancing entanglement between separated physical systems is of paramount importance in quantum information processing. Joint unitary operations can transform product states into maximal entangled states locally. In the nonlocal case (remote users), local operations can enhance the entanglement of non-maximally entangled states in a probabilistic way, such as entanglement concentration⁵ and entanglement purification⁶.

The idea of the standard entanglement concentration and purification processes is to extract a smaller number of more entangled pairs from a larger number of less entangled pairs by local operations and classical communications(LOCCs)^{5,6}. The core part of the concentration and purification processes is the local CNOT operation on the two representatives from two different less entangled pairs. In experiment, the implementation of the CNOT operation is not a easy task, so efforts have been made to find easier ways to realize or to avoid the CNOT operation, such as in linear optical system⁷⁻¹⁰, in ionic or atomic system^{11,12} etc. In the purification of higher-dimensional entanglement, joint unitary operation also plays an very important role¹³⁻¹⁶. Pan et al. replaced the CNOT operation with a polarization beam splitter to purify polarization entangled mixed states^{7,8}, and Gisin et al. replaced the CNOT operation with a general beam splitter to purify phase-error single-photon entangled states^{9,10}. These results show that the complicated CNOT operation is not a necessity in entanglement concentration and purification, and the bilateral coupling between the two copies of the states to be concentrated or purified is the core part of an entanglement concentration or purification process. The role of this kind of coupling has been demonstrated^{9,10}, and the optimal output fidelity is achieved with a general coupling rather than a CNOT-like coupling. But this result only applies to the phase-error single-photon entangled mixed states, and whether there exist the similar results for the purification of general mixed entangled states(such as Werner state) or the concentration of unknown non-maximally entangled pure states is not clear. In addition, the focus of entanglement concentration and purification is entanglement, so the entangling property of the coupling operation maybe more important than the coupling itself. So, in this paper, we will study the role of the joint unitary operation in a general

¹School of Physics & Material Science, Anhui University, Hefei 230601, China. ²Department of Computer Science, Jining University, Qufu, Shandong 273155, China. ³Institute for Quantum Control and Quantum Information, School of Electronic and Information Engineering, Hefei Normal University, Hefei 230601, China. Correspondence and requests for materials should be addressed to M.Y. (email: mingyang@ahu.edu.cn) or Z.-L.C. (email: zlcao@ahu. edu.cn)





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entanglement concentration or purification process. Because it is not possible to get a pure maximally entangled state from purifying a finite ensemble of general mixed states, we will focus on the entanglement increment in purification process. Meanwhile, the maximal output entanglement is the most important target of entanglement concentration. Here, we are going to focus on the entangling power of the joint unitary operations at both sides, the parameter of the quantum channel and the entanglement of the output state (entanglement increment) so as to find the matching relation between the entanglement property of the joint unitary operations at both sides and the quantum channel to maximize the output entanglement (entanglement increment) in entanglement concentration (purification).

The entanglement property of a unitary operator has been studied from different aspects, such as entangling power^{17–22}, entanglement measure^{18,23,24}, entangling capacity^{25–30}, and entanglement-changing power³¹. Entangling power of a joint unitary operation is defined as the mean entanglement (linear entropy) produced by applying the joint unitary operation on a given distribution of pure product states¹⁷. In this paper, we use the approach introduced in¹⁷ for measuring the entangling power of joint unitary operations, and concurrence is used to measure the entanglement of the output states³².

In this paper, we revisited the entanglement purification and the Schmidt-projection based entanglement concentration schemes with the CNOT operation being replaced with a general joint unitary operation, and the optimal matching relations between the entangling power of the joint unitary operation and the non-maximal entangled channel are found for optimizing the entanglement increment and output entanglement in entanglement purification and concentration.

Results

Matching relation in entanglement concentration. The bilateral CNOT operation in standard entanglement concentration process is replaced with a general bilateral bipartite operation, which is elaborated in the **Methods** section. The entanglement of the output state can be measured in terms of concurrence³². For the sake of brevity, the entanglement of the output state is called the output entanglement, and it is denoted by C_{out} . The entanglement increment of the concentration process is denoted by ΔC . It is obvious that the joint control operation induced by the component A_z only does not have a positive contribution to the entanglement concentration. Hence, we only study the case in which \vec{A} only has A_x or A_y component, and, for the sake of simplicity, the corresponding bilateral operations are referred to as controlled-x (U_{A_x}) and controlled-y (U_{A_y}) operations, respectively.

For measurement result $|00\rangle_{34}$, C_{out} and ΔC are plotted as functions of the parameter θ of input state and the entangling power of U_{A_v} in Fig. 1, and C_{out} and ΔC are plotted in Fig. 2 for the case with U_{A_v} .

Figure 1 (Fig. 2) shows that there exists an optimal matching relation between the entangling power of the U_{A_x} (U_{A_y}) and the parameter θ of the input state so that C_{out} can reach the maximum value 1. The explicit expressions of the optimal matching relations for the cases with U_{A_x} and U_{A_y} can be expressed respectively as:

$$C_{out}^{A_x} = \left| \frac{2\sin\theta\cos^2\theta[\cos\theta - \sin^2A_x(\sin\theta + \cos\theta)]}{\cos^4\theta + \sin^2\theta[\cos\theta - \sin^2A_x(\sin\theta + \cos\theta)]^2} \right| = 1,$$
(1)

$$C_{out}^{A_{y}} = \left| \frac{2\sin\theta\cos^{2}\theta \left[\cos\theta + \sin^{2}A_{y}(\sin\theta - \cos\theta)\right]}{\cos^{4}\theta + \sin^{2}\theta \left[\cos\theta + \sin^{2}A_{y}(\sin\theta - \cos\theta)\right]^{2}} \right| = 1,$$
(2)

where $C_{out}^{A_x}(C_{out}^{A_y})$ is the C_{out} for the case with $U_{A_y}(U_A)$.



Figure 2. C_{out} and ΔC are plotted as functions of the parameter θ of input state and the entangling power of the joint unitary operation U_{A_v} . Here the measurement result is $|00\rangle_{34}$.



Figure 3. C_{out} and ΔC are plotted as functions of the parameter θ of input state and the entangling power of the joint unitary operation U_{A_v} . Here the measurement result is $|11\rangle_{34}$.



Figure 4. C_{out} and ΔC are plotted as functions of the parameter θ of input state and the entangling power of the joint unitary operation U_{A_y} . Here the measurement result is $|11\rangle_{34}$.

Figures 1 and 2 also show that the considerable ΔC is presented around $\theta = \frac{\pi}{2}$ with a small entangling power of the joint unitary operation. From Figs 1 and 2, the following conclusion can be made: if the parameter θ of the input state is given a fixed value θ_0 within the range $\frac{\pi}{4} < \theta_0 < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta_0 < \frac{3\pi}{4}$, the solutions of Eq. (1) or (2) will fix some joint control operations which can not only produce the maximal C_{out} but also induce a considerable ΔC .

Similarly, for the measurement result $|11\rangle_{34}$, C_{out} and ΔC have been plotted as functions of the parameter θ of the input state and the entangling power of the joint operation $U_{A_x}(U_{A_y})$ in Figs 3 and 4. Figs 3 and 4 show that, for the input states within the range $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$, the optimal matching relation between the entan-



Figure 5. C_{out} (for different input purities p) are plotted as functions of the parameter θ of input state and the entangling power of the joint unitary operation U_{A_x} or U_{A_y} . Here the measurement result is $|00\rangle_{34}$.

gling power of the joint operation and the parameter θ of the input state always can be found for driving C_{out} to reach the maximum 1. Here the explicit expressions of the optimal matching relations can be expressed respectively as:

$$C_{out}^{\tilde{A}_x} = \left| \frac{2\cos\theta [\sin\theta - \sin^2 A_x (\sin\theta + \cos\theta)]}{\cos^2\theta + [\sin\theta - \sin^2 A_x (\sin\theta + \cos\theta)]^2} \right| = 1,$$
(3)

$$C_{out}^{\tilde{A}_{y}} = \left| \frac{2\cos\theta \left[\sin\theta + \sin^{2}A_{y}(\cos\theta - \sin\theta)\right]}{\cos^{2}\theta + \left[\sin\theta + \sin^{2}A_{y}(\cos\theta - \sin\theta)\right]^{2}} \right| = 1,$$
(4)

for the cases with U_{A_x} and U_{A_y} . In addition, when the entangling power of the joint operation reaches the maximum value $\frac{2}{9}$, C_{out} is always the maximum 1 except for the product initial state case. This exception results from the fact that zero input entanglement leads to zero C_{out} in entanglement concentration. When the entangling power reaches the maximum value $\frac{2}{9}$, ΔC can be obtained as:

$$\Delta C = 1 - |2 \sin \theta \cos \theta|. \tag{5}$$

So when the input state is very close to a product state and the entangling power of the joint unitary operation reaches the maximum $\frac{2}{\alpha}$, both C_{out} and ΔC are maximized for the measurement result $|11\rangle_{34}$ case.

We can conclude that the set of joint unitary operations with maximal entangling power (including CNOT) is not the only set of joint operations which can maximize the C_{out} in entanglement concentration, and a variety of joint operations with lower entangling power can also induce a maximal C_{out} , which will greatly broaden the set of the potential joint operations in entanglement concentration.

Matching relation in entanglement purification. If the concentration process studied above is applied to mixed initial entangled states, it is called entanglement purification. Similarly, the joint control operation induced by the component A_z only does not have a positive contribution to the entanglement purification, so we only study the cases with U_{A_x} and U_{A_y} , respectively.

Figures 5 and 6 show that the maximal C_{out} is increasing along with the purity p of the quantum channel. For a fixed purity of the quantum channel, the maximal C_{out} can be attained only when the entangling power of the



Figure 6. C_{out} (for different input purities p) are plotted as functions of the parameter θ of input state and the entangling power of the joint unitary operation U_{A_x} or U_{A_y} . Here the measurement result is $|11\rangle_{34}$.



Figure 7. C_{out} and ΔC are plotted as functions of the purity p of the input state and the entangling power of the joint unitary operation U_{A_x} . Here the measurement result is $|00\rangle_{34}$, and $\theta = \frac{\pi}{4}$.

joint unitary operation reaches the maximum $\frac{2}{9}$ and the input entanglement reaches the maximum, i.e. the parameter θ is equal to $\frac{\pi}{4}$ or $\frac{3\pi}{4}$. So, we can study the optimal matching relation between the channel and the entangling power of the joint operation for maximizing the ΔC by fixing $\theta = \frac{\pi}{4}$. Because the output state for the measurement result $|11\rangle_{34}$ case is a conjugate of that for the measurement result $|00\rangle_{34}$ case when θ is equal to $\frac{\pi}{4}$, we only discuss the measurement result $|00\rangle_{34}$ case.

Figures 7 and 8 show that for different initial purities p within the range $(\frac{1}{3},1)$, the maximal C_{out} can be achieved only when the entangling power of the joint unitary operation approaches the maximum value $\frac{2}{9}$. With the entangling power of the joint unitary operation being the maximum $\frac{2}{9}$ and the purity p of quantum channel being 0.6944, the only maximal ΔC (0.0775) can be reached. That is to say, the set of the joint unitary operations with the maximal entangling power (including CNOT) is required for achieving the maximal ΔC in entanglement purification. Although the CNOT operation in the standard entanglement purification is the optimal joint operation, our results still broaden the set of the potential joint operations in entanglement purification, because the CNOT operation is not the only joint operation whose entangling power is the maximum value $\frac{2}{9}$.



Figure 8. C_{out} and ΔC are plotted as functions of the purity p of the input state and the entangling power of the joint unitary operation U_{A_y} . Here the measurement result is $|00\rangle_{34}$, and $\theta = \frac{\pi}{4}$.

Conclusion

In this paper, by replacing the CNOT operation with a general joint operation, we studied the role of entangling power of the joint unitary operation in entanglement concentration and entanglement purification and found the matching relations between the entangling power of the joint unitary operation and the parameter of the quantum channel so that the entanglement of the output state or the entanglement increment is maximized. The results show that the set of joint unitary operations with maximal entangling power $\frac{2}{2}$ are needed for maximizing the entanglement increment of entanglement purification. But, for entanglement concentration, the result is somewhat counter-intuitive. Besides the set of joint unitary operations with maximal entangling power (including CNOT), there are a variety of joint operations with non-maximal entangling power that can induce a maximal output entanglement in entanglement concentration too, which will greatly broaden the set of the potential joint operations in entanglement concentration. In addition, because the CNOT operation is not the only joint operation whose entangling power is the maximum value $\frac{2}{2}$, the CNOT operation in the standard entanglement purification is just a typical one of the optimal joint operations which can maximize the entanglement increment in entanglement purification. So the results presented here can greatly broaden the set of the potential joint unitary operations, which can maximize the output entanglement for entanglement concentration and the entanglement increment for entanglement purification, and may help the experimentalists to find a joint unitary operation which is both optimal and simple.

Methods

Entanglement concentration. Two pairs of particles (1, 2 and 3, 4) are prepared in non-maximally entangled pure states initially, and then distributed to two users Alice (1, 3) and Bob (2, 4). Each user will carry out a general control operation instead of CNOT operation on the two particles he (or she) possesses and then measure the target particle. If the target particles are in the same state, the source pair is retained. Next, we will analyze the relationship between the non-maximally entangled channel, entangling power of the general joint operation and the entanglement of the output state.

Suppose that particles (1, 2 and 3, 4) are in the following non-maximally entangled pure states:

$$|\psi\rangle_{12} = \cos\theta|00\rangle + \sin\theta|11\rangle, \tag{6}$$

$$|\psi\rangle_{34} = \cos\theta|00\rangle + \sin\theta|11\rangle, \tag{7}$$

where $\theta \in (0,\pi)$. The bilateral control operation is in the following general form: $U = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes e^{i\delta}e^{i\vec{A}\cdot\vec{\sigma}}$, where $e^{i\delta}$ is a relative phase factor, $e^{i\vec{A}\cdot\vec{\sigma}} = \cos |\vec{A}| + i\frac{\sin |\vec{A}|}{|\vec{A}|}\vec{A}\cdot\vec{\sigma}$, $\vec{A}\cdot\vec{\sigma} = A_x\sigma_x + A_y\sigma_y + A_z\sigma_z$, and $\sigma_{x(y,z)}$ is the Pauli operator. Here the first pair (1, 2) is the source pair, and the second pair (3, 4) is the target one. According to the definitions^{17,20}, the entangling power of the general control operation U can be expressed as follows:

$$e_p(U) = \frac{2}{9}\sin^2|\vec{A}|.$$
 (8)

If the measurement after the control operation gives same results for both of the target particles 3, 4, then the source pair 1, 2 is retained and its state is regarded as the output state of the concentration process.

Entanglement purification. If the concentration process studied above is applied to mixed initial entangled states, it is called entanglement purification. Without loss of generality, the initial mixed state to be purified can be expressed as the mixture of a pure non-maximally entangled state and an identity operator:

$$\rho_{12} = p|\psi\rangle_{12} \langle \psi| + (1-p)\frac{I}{4}, \tag{9}$$

$$\rho_{34} = p|\psi\rangle_{34} \langle \psi| + (1-p)\frac{I}{4}, \tag{10}$$

where $0 , and <math>|\psi\rangle_{12} = |\psi\rangle_{34} = \cos\theta|00\rangle + \sin\theta|11\rangle$. As discussed in the entanglement concentration process, if the target particles (3, 4) are measured in same states after the general joint controlled operation, the quantum state of the source particles 1, 2 is the output state, whose entanglement is calculated in terms of concurrence.

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Author Contributions

F.-Z.K. and H.-Z.X. carried out the calculations. M.Y., Q.Y. and Z.-L.C. conceived the idea. All authors contributed to the interpretation of the results and the writing of the manuscript. All authors reviewed the manuscript.

Additional Information

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