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Multipartite entanglement indicators based on monogamy relations of *n*-qubit symmetric states

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Constructed from Bai-Xu-Wang-class monogamy relations, multipartite entanglement indicators can detect the entanglement not stored in pairs of the focus particle and the other subset of particles. We investigate the k-partite entanglement indicators related to the αth power of entanglement of formation (αEoF) for $k \le n$, $\alpha \in [\sqrt{2}, 2]$ and n-qubit symmetric states. We then show that (1) The indicator based on αEoF is a monotonically increasing function of k. (2) When n is large enough, the indicator based on αEoF is a monotonically decreasing function of α, and then the n-partite indicator based on $\sqrt{2}$EoF works best. However, the indicator based on 2 EoF works better when n is small enough.

Quantum correlations that comprise and go beyond entanglement are not monogamous. Only entanglement can be strictly monogamous[1], that is, they obey strong constraints on how they can be shared among multipartite systems. This is one of the most important properties for multipartite quantum systems[2]. So these monogamy relations can be used to characterize the entanglement structure in multipartite systems[3], and concretely *the difference between the left- and right-hand side of them can be defined as indicators* to detect multipartite entanglement not stored in pairs of the focus particle (e.g., the first particle) and the other subset of particles[4].

For the squared concurrence, the indicator named three-tangle[3] can be used to detect *genuine multipartite entanglement* (*which are entangled states being not decomposable into convex combinations of states separable across any partition*) in three-qubit pure states. However, for three-qubit mixed states, there exist some entangled states that have neither two-qubit concurrence nor three-tangle[5]. To reveal this critical entanglement structure, some multipartite entanglement indicators based on Bai-Xu-Wang-class monogamy relations for the entanglement of formation (EoF) have been proposed[4,6,7]. In this paper, we will study which multipartite entanglement indicator for EoF works better. By "work better" we mean that is larger than the other[8].

We resolve the above problem in the following ways. Firstly, we prove that the αth power of EoF (αEoF, $\alpha \ge \sqrt{2}$) obeys a set of hierarchy k-partite ($k \in [3, n]$) monogamy relations of Eq. (10) in an arbitrary n-qubit state $\rho_{A_1A_2\cdots A_n}$. Here, the k-partition means the partition $A_1, \cdots, A_{k-1}$ and $A_k \cdots A_n$. Based on these monogamy relations, a set of new multipartite entanglement indicators are presented correspondingly, which can work better than the 2 EoF-based indicators in n-qubit symmetric states. However, we find that the 2 EoF-based indicator can work better than the αEoF-based indicators for $\alpha \in [\sqrt{2}, 2)$ when n is small enough (e.g., $n \le 9$).

Results

This section is organized as follows. In the first subsection, we review the monogamy relations for 2 EoF in n-qubit systems. We then prove in the second subsection that the αEoF obeys hierarchy k-partite monogamy relations for $k \in [3,n]$ and any n-qubit states. In the third subsection, we construct the entanglement indicators on n-qubit symmetric states, and show their monotonic properties. Two examples are given in the forth subsection to verify these results.

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Review of monogamy relations for EoF. Coffman, Kundu, and Wootters[3] proved the first monogamy relation for the squared concurrence in three-qubit states. Then, Osborne and Verstraete[9] proved a set of hierarchy k-partite monogamy relations for the squared concurrence in n-qubit states $\rho_{A_1A_2\cdots A_n}$, which have the form

$$C^2(\rho_{A_1|A_2\cdots A_n}) \ge \sum_{i=2}^{k-1} C^2(\rho_{A_1A_i}) + C^2(\rho_{A_1|A_k\cdots A_n}), \tag{1}$$

where A_1 is the focus qubit, $\rho_{A_1A_i} = Tr_{A_2\cdots A_{i-1}A_{i+1}\cdots A_n}(\rho_{A_1A_2\cdots A_n})$, $C(\rho_{A_1|A_2\cdots A_n})$ is the concurrence of $\rho_{A_1A_2\cdots A_n}$ in the bipartition $A_1|A_2\cdots A_n$, and $\rho_{A_1A_2\cdots A_{k-1}(A_k\cdots A_n)}$ is a k-partite n-qubit state.

Based on these Osborne-Verstraete-class hierarchical monogamy relations in Eq. (1), a set of multipartite entanglement indicators can be constructed as follows

$$\tau_{k,C}^{A_1,2}(\rho_{A_1A_2\cdots A_n}) = C^2(\rho_{A_1|A_2\cdots A_n}) - \sum_{i=2}^{k-1} C^2(\rho_{A_1A_i}) - C^2(\rho_{A_1|A_k\cdots A_n}), \tag{2}$$

where the entanglement measure is the squared concurrence. These indicators can detect the entanglement not stored in pairs of A_1 and any other $k-1$ party (i.e., $A_2, \cdots, A_{k-1}$ and $A_k\cdots A_n$)[4]. However, there exists a special kind of entangled state[10] which has zero entanglement indicator. Moreover, the calculation of multiqubit concurrence is extremely hard due to the convex roof extension. Therefore, it is natural to ask whether other monogamy relations beyond the squared concurrence exist.

Recently, Bai *et al.*[4] and Oliveira *et al.*[11] respectively proved that 2 EoF is monogamous in n-qubit states, as follows

$$E_F^2(\rho_{A_1|A_2\cdots A_n}) \ge E_F^2(\rho_{A_1A_2}) + \cdots + E_F^2(\rho_{A_1A_n}). \tag{3}$$

Moreover, Bai *et al.*[6] exactly showed that there are a set of hierarchy k-partite monogamy relations for 2 EoF in an arbitrary n-qubit states, which obey the relation

$$E_F^2(\rho_{A_1|A_2\cdots A_n}) \ge \sum_{i=2}^{k-1} E_F^2(\rho_{A_1A_i}) + E_F^2(\rho_{A_1|A_k\cdots A_n}). \tag{4}$$

Generally, Zhu and Fei[7] proved that αEoF obeys the following monogamy relation in n-qubit states,

$$E_F^\alpha(\rho_{A_1|A_2\cdots A_n}) \ge E_F^\alpha(\rho_{A_1A_2}) + \cdots + E_F^\alpha(\rho_{A_1A_n}), \tag{5}$$

where $\alpha \in [\sqrt{2}, 2)$. (In fact, Eq. (5) obviously satisfies for $\alpha > 2$ which can be obtained from Eq. (4) and ref. 12.)

Because some bipartite multiqubit EoF of $E_F(\rho_{A_1|A_2\cdots A_n})$ can be calculated via quantum discord[13,14], the entanglement indicator $\tau_{k,E_F}^{A_1,\alpha}(\rho_{A_1A_2\cdots A_n})$ from Eqs (3–5) can be obtained and can characterize multipartite entangled states in some n-qubit states[4,6,7]. In these entanglement indicators, how to choose a better indicator to detect that there exists multipartite entanglement is a problem. In the following subsections, we will try to resolve the problem.

Hierarchy *k*-partite monogamy relations for αEoF. In this subsection, we firstly summary of some existing conclusions, and then get the hierarchy k-partite monogamy relations for αEoF.

As we know, EoF is a well defined measure of entanglement for bipartite states. For any two-qubit state ρ_{AB}, an analytical formula was given by Wootters[15] as follows

$$E_F(\rho_{AB}) = f[C^2(\rho_{AB})] = h\left(\frac{1+\sqrt{1-C^2(\rho_{AB})}}{2}\right), \tag{6}$$

where $C(\rho_{AB}) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$ is the concurrence with the decreasing nonnegative λ_i being the eigenvalues of the matrix $\rho_{AB}(\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y)$. Here, $f(x) = h(\frac{1+\sqrt{1-x}}{2})$, and $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary Shannon entropy. Recently, Bai *et al.*[6] proved that $f(x)$ is a monotonic and concave function of x. Moreover, Zhu and Fei[7] proved that $f(x)$ satisfies the following relation

$$f^\alpha(x^2 + y^2) \ge f^\alpha(x^2) + f^\alpha(y^2), \tag{7}$$

where $\alpha \ge \sqrt{2}$, x and $y \in [0,1]$. They also proved that EoF obeys the following relation

$$E_F(\rho_{A_1|A_2\cdots A_n}) \ge f\left[C^2(\rho_{A_1|A_2\cdots A_n})\right]. \tag{8}$$

for the bipartite quantum state $\rho_{A_1|A_2\cdots A_n}$ in $2 \otimes 2^{n-1}$ systems. Because a $2 \otimes 2^{n-1}$ pure state $|\psi\rangle_{A_1|A_2\cdots A_n}$ is equivalent to a two-qubit state under the Schmidt decomposition[16], we have

$$E_F\left(|\psi\rangle_{A_1|A_2\cdots A_n}\right) = f\left[C^2\left(|\psi\rangle_{A_1|A_2\cdots A_n}\right)\right]. \tag{9}$$

From Eqs (1) and (6–9) for n-qubit systems, we can easily obtain that the following hierarchy k-partite monogamy relation holds.

Theorem 1 For any n-qubit state $\rho_{A_1A_2\cdots A_n}$, EoF satisfies the following monogamy relation

$$E_F^{\alpha}(\rho_{A_1|A_2\cdots A_n}) \geq \sum_{i=2}^{k-1} E_F^{\alpha}(\rho_{A_1A_i}) + E_F^{\alpha}(\rho_{A_1|A_k\cdots A_n}), \tag{10}$$

where $k = \{3,4,\cdots, n\}$ and $\alpha \geq \sqrt{2}$.

The αEoF satisfies the hierarchy monogamy inequality (10) for any $\alpha \geq \sqrt{2}$, while the αth power of concurrence satisfies hierarchy monogamy inequalities for any $\alpha \geq 2$[9,12]. This phenomenon shows a difference between the two kinds of entanglement measures. On the other hand, the inequality (10) is a generalization of Eq. (5) in ref. 6 and Eq. (19) in ref. 7. More specifically, Eq. (10) equals to Eq. (4) when $\alpha = 2$, and is the same as Eq. (5) when $k = n$.

Properties of hierarchy entanglement indicators.

For any n-qubit state $\rho_{A_1A_2\cdots A_n}$ and αEoF ($\alpha \in [\sqrt{2},2]$), we can define a hierarchy entanglement indicator based on the corresponding monogamy relation in Eq. (10) as follows

$$\tau_k^{\alpha}(\rho_{A_1A_2\cdots A_n}) = \min\left\{\tau_{k,E_F}^{A_1,\alpha}, \tau_{k,E_F}^{A_2,\alpha}, \cdots, \tau_{k,E_F}^{A_n,\alpha}\right\}, \tag{11}$$

where

$$\tau_{k,E_F}^{A_1,\alpha}(\rho_{A_1A_2\cdots A_n}) = E_F^{\alpha}(\rho_{A_1|A_2\cdots A_n}) - \sum_{i=2}^{k-1} E_F^{\alpha}(\rho_{A_1A_i}) - E_F^{\alpha}(\rho_{A_1|A_k\cdots A_n}). \tag{12}$$

It can be used to detect the entanglement for the k-partite case of an n-qubit system[6] not stored in pairs of A_1 and any other $k-1$ party.

Here it should be noted that, different from the hierarchy entanglement indicator of the concurrence, the indicator of EoF depends on which qubit is chosen to be the focus qubit. Fortunately, the indicators of the concurrence and EoF are all focus-independent in symmetric quantum systems. In the following, we give some properties about the indicators of EoF only for n-qubit symmetric states.

Theorem 2 For any n-qubit symmetric state $\rho_{A_1A_2\cdots A_n}$, the hierarchy entanglement indicator satisfies

$$\tau_k^{\alpha}(\rho_{A_1A_2\cdots A_n}) = \tau_{k,E_F}^{A_1,\alpha}(\rho_{A_1A_2\cdots A_n}), \tag{13}$$

and it is a monotonically increasing function of k, where $k = \{3,4,\cdots, n\}$ and $\alpha \in [\sqrt{2},2]$.

Proof. When $\rho_{A_1A_2\cdots A_n}$ is a symmetric state, it is permutation invariant. Then, $\forall\ i, j \in \{1,2,\cdots, n\}$ and $i \neq j$, we have $E_F(\rho_{A_iA_j}) = E_F(\rho_{A_1A_2})$ and

$$E_F(\rho_{A_i|A_1A_2\cdots A_{i-1}A_{i+1}\cdots A_n}) = E_F(\rho_{A_1|A_2\cdots A_n}). \tag{14}$$

Combining with Eq. (11), we have

$$\tau_k^{\alpha}(\rho_{A_1A_2\cdots A_n}) = \tau_{k,E_F}^{A_1,\alpha}(\rho_{A_1A_2\cdots A_n}). \tag{15}$$

Moreover, according to Eq. (5), we have

$$E_F^{\alpha}(\rho_{A_1|A_k\cdots A_n}) \geq E_F^{\alpha}(\rho_{A_1A_k}) + E_F^{\alpha}(\rho_{A_1|A_{k+1}\cdots A_n}). \tag{16}$$

Then we can derive

$$\begin{aligned} \tau_{k+1}^{\alpha}(\rho_{A_1A_2\cdots A_n}) &= E_F^{\alpha}(\rho_{A_1|A_2\cdots A_n}) - \sum_{i=2}^{k} E_F^{\alpha}(\rho_{A_1A_i}) - E_F^{\alpha}(\rho_{A_1|A_{k+1}\cdots A_n}) \\ &\geq E_F^{\alpha}(\rho_{A_1|A_2\cdots A_n}) - \sum_{i=2}^{k-1} E_F^{\alpha}(\rho_{A_1A_i}) - E_F^{\alpha}(\rho_{A_1|A_k\cdots A_n}) \\ &= \tau_k^{\alpha}(\rho_{A_1A_2\cdots A_n}), \end{aligned} \tag{17}$$

where the inequality holds because of Eq. (16). Therefore, the entanglement indicator $\tau_k^{\alpha}(\rho_{A_1A_2\cdots A_n})$ is a monotonically increasing function of k.

In symmetrical quantum systems, the k-partite n-qubit monogamy relations of αEoF in Eq. (10) can be a monogamy equality (e.g., the corresponding results in the next subsection), and thus the corresponding entanglement indicator $\tau_k^{\alpha}(\rho_{A_1A_2\cdots A_n})$ can not work. However, we can choose an appropriate indicator

$$g(\alpha, n) = \tau_n^{\alpha}(\rho_{A_1A_2\cdots A_n}) \tag{18}$$

to represent a better entanglement indicator which comes from the following result.

Theorem 3 For any n-qubit symmetric state $\rho_{A_1A_2\cdots A_n}$, the entanglement indicator obeys the following relation

$$g(\alpha, n) = b^{\alpha} - (n-1)c^{\alpha}, \tag{19}$$

where $\alpha \in [\sqrt{2}, 2]$, $b = E_F(\rho_{A_1|A_2\cdots A_n})$ and $c = E_F(\rho_{A_1A_2})$. For any n, we have the following results

(1) When $c = 0$, $g(\alpha, n)$ is a monotonically decreasing function of α.
(2) When $c > 0$ and $b < 1$, $g(\alpha, n)$ is a monotonically decreasing function of α if and only if

$$\alpha \geq \frac{\ln\left[(n-1)\frac{\ln c}{\ln b}\right]}{\ln \frac{b}{c}}, \tag{20}$$

and $g(\alpha, n)$ is a monotonically increasing function of α if and only if

$$\alpha \leq \frac{\ln\left[(n-1)\frac{\ln c}{\ln b}\right]}{\ln \frac{b}{c}}. \tag{21}$$

When $c > 0$ and $b = 1$, $g(\alpha, n)$ is also a monotonically increasing function of α.

Proof. From Eqs (10), (12) and (15), we have

$$g(\alpha, n) = E_F^{\alpha}(\rho_{A_1|A_2\cdots A_n}) - \sum_{i=2}^{n} E_F^{\alpha}(\rho_{A_1A_i}) = b^{\alpha} - (n-1)c^{\alpha}. \tag{22}$$

According to the definition of b and c and the monogamy inequality (5), we get $0 \leq c < b \leq 1$.

For any n, we will analytically prove the two necessary and sufficient conditions.

(1) When $c = 0$, we have $g(\alpha, n) = b^{\alpha}$. Because $0 < b \leq 1$, $g(\alpha, n)$ is a monotonically decreasing function of α.
(2) When $c \in (0, b)$, we have

$$\frac{\partial g(\alpha, n)}{\partial \alpha} = b^{\alpha} \ln b - (n-1)c^{\alpha} \ln c. \tag{23}$$

The monotonically decreasing property of $g(\alpha, n)$ is satisfied if and only if the first-order partial derivative $\partial g(\alpha, n)/\partial \alpha \leq 0$, which is equivalent to Eq. (20).

Furthermore, the monotonically increasing property of $g(\alpha, n)$ is satisfied if and only if the first-order partial derivative $\partial g(\alpha, n)/\partial \alpha \geq 0$, which is equivalent to Eq. (21).

From Theorem 3, we can obtain that the necessary and sufficient condition for the unit indicator is $E_F(\rho_{A_1|A_2\cdots A_n}) = 1$ and $E_F(\rho_{A_1A_2}) = 0$. For any n-qubit symmetrical state, we can numerically compute the corresponding bounds to determine which is better, $\sqrt{2}$ EoF indicator or the 2 EoF, as follows:

After some deduction, we numerically obtain two bounds N_1 and N_2 with Eqs (20) and (21). When $n \geq N_1$, the $\sqrt{2}$ EoF indicator is better than the 2 EoF indicator which comes from Eq. (20). The 2 EoF indicator is better than the $\sqrt{2}$ EoF indicator when $n \leq N_2$, which comes from Eq. (21).

These results can be verified via two n-qubit symmetrical states in the next subsection.

Analytical examples. We will investigate the above results on permutationally invariant states, which are the W state, the superposition of the W state and the Greenberger-Horne-Zeilinger (GHZ) state of n qubits respectively.

For the W state. In this part, we analyze the n-qubit W state which has the form

$$|W\rangle_{A_1A_2\cdots A_n} = \frac{1}{\sqrt{n}}(|00\cdots01\rangle + |00\cdots10\rangle + \cdots + |01\cdots00\rangle + |10\cdots00\rangle). \tag{24}$$

For this quantum state, the n-partite n-qubit monogamy relations of αth power of concurrence as shown in ref. 7 are saturated, and thus these concurrence-based entanglement indicators can not work. However, we will show that the αEoF-based indicator can be used to represent the entanglement in the n-partite n-qubit systems.

Using the symmetry of qubit permutations in the W state, $C^2(|W\rangle_{A_1|A_2\cdots A_n}) = 4(n-1)/n^2$, and $C^2(\rho_{A_1A_2}) = 4/n^2$ [17], we have

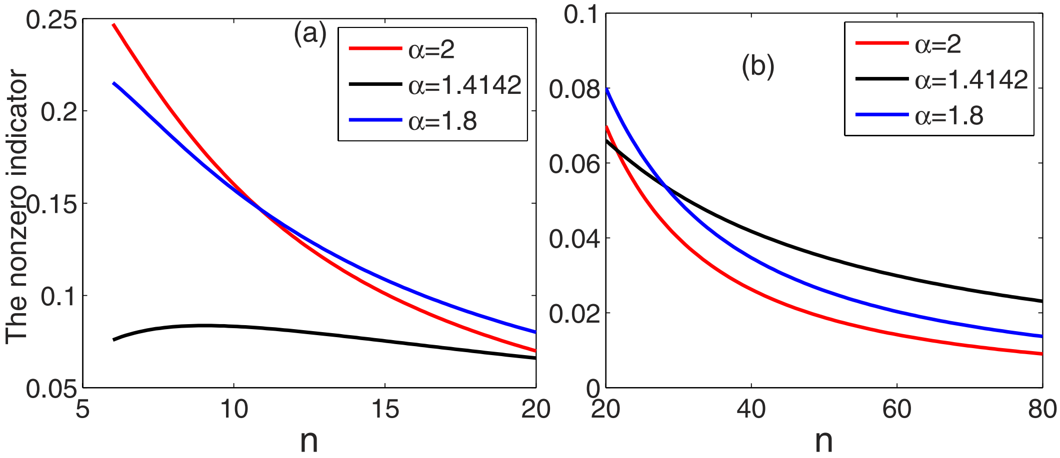

Figure 1. The multipartite entanglement indicators for the W state as functions of n, where $n \in [6, 20]$ in (a) and $n \in [20, 80]$ in (b).

$$\begin{aligned}\tau_n^{\alpha}\left(|W\rangle_{A_1A_2\cdots A_n}\right) &= E_F^{\alpha}\left[C^2\left(|W\rangle_{A_1|A_2\cdots A_n}\right)\right] - (n-1)E_F^{\alpha}\left[C^2(\rho_{A_1A_2})\right]\\ &= f^{\alpha}\left[\frac{4(n-1)}{n^2}\right] - (n-1)f^{\alpha}\left(\frac{4}{n^2}\right)\\ &= b^{\alpha}[p(n)] - (n-1)c^{\alpha}[q(n)],\end{aligned} \tag{25}$$

where $p(n) = 4(n-1)/n^2$ and $q(n) = 4/n^2$. This set of $\tau_n^{\alpha}(|W\rangle_{A_1A_2\cdots A_n})$ are positive since the αEoF is monogamous as shown in Eqs (5) and (10).

In order to study the properties of $g(\alpha, n)$, we firstly prove the function $M(n)$, with

$$M(n) = \frac{\ln\left[(n-1)\frac{\ln c[q(n)]}{\ln b[p(n)]}\right]}{ln\frac{b[p(n)]}{c[q(n)]}} \tag{26}$$

in Eqs (20) and (21), is a monotonically decreasing function of n. The details for illustrating the monotonic property are presented in Methods.

Let

$$g(\alpha, n) = \tau_n^{\alpha}\left(|W\rangle_{A_1A_2\cdots A_n}\right). \tag{27}$$

After some deduction, we can derive

$$M(77) \approx 1.4134 < \sqrt{2} < M(76) \approx 1.4149, \tag{28}$$

when $\alpha = \sqrt{2}$. Thus, combining with the monotonically decreasing property of $M(n)$, we prove that $\alpha \geq M(n)$ when $n \geq 77$, while $\alpha \leq M(n)$ when $n \leq 76$. When $\alpha = 2$, we get

$$M(10) \approx 1.9394 < 2 < M(9) \approx 2.0055, \tag{29}$$

which means $\alpha \geq M(n)$ when $n \geq 10$, while $\alpha \leq M(n)$ when $n \leq 9$. Combining the above two inequations with Eqs (20) and (21), we obtain the two bounds $N_1 = \max\{77,10\} = 77$ and $N_2 = \min\{76,9\} = 9$. And, we know that $\tau_n^{\sqrt{2}}(|W\rangle_{A_1A_2\cdots A_n}) > \tau_n^{2}(|W\rangle_{A_1A_2\cdots A_n})$ when $n \geq N_1$, and $\tau_n^{\sqrt{2}}(|W\rangle_{A_1A_2\cdots A_n}) < \tau_n^{2}(|W\rangle_{A_1A_2\cdots A_n})$ when $n \leq N_2$. Then we complete the proof that $g(\alpha, n)$ obeys these properties.

In Fig. 1, we plot these indicators as functions of n, and then these properties can be verified from the figure. From the Fig. 1, we numerically find that $g(\alpha, n)$ is a monotonically decreasing function of n when $\alpha \in [\sqrt{2}, 2]$ and $n \geq 10$. How to exactly prove the result is an open problem.

These results still hold for symmetric n-qubit mixed states as shown in the next part.

For the superpositions of the GHZ state and the W state. When an n-qubit mixed state is a superpositions of the *GHZ* state and the *W* state, it has the form

$$\rho_{A_1A_2\cdots A_n} = p|GHZ\rangle\langle GHZ| + (1-p)|W\rangle\langle W|, \tag{30}$$

where $|GHZ\rangle = (|00\cdots00\rangle + |11\cdots11\rangle)/\sqrt{2}$ and $p \in (0,1)$. For $n=3$, Lohmayer *et al.*[5] found that, when $p \in (0.292,0.627)$, it is entangled but without two-qubit concurrence and three-tangle. It is still an unsolved problem[4] of how to characterize the entanglement structure in this kind of states for large n.

In Eq. (18), the n-partite entanglement indicators have the forms

$$\tau_n^{\alpha}(\rho_{A_1A_2\cdots A_n}) = E_F^{\alpha}(\rho_{A_1|A_2\cdots A_n}) - (n-1)E_F^{\alpha}(\rho_{A_1A_2}), \tag{31}$$

Then, the calculations of $E_F(\rho_{A_1|A_2\cdots A_n})$ and $E_F(\rho_{A_1A_2})$ are key steps.

Any reduced two-qubit states of $\rho_{A_1A_2\cdots A_n}$ has the same form

$$\begin{aligned}\rho_{A_1A_2} &= \left[\frac{p}{2} + \frac{(n-2)(1-p)}{n}\right]|00\rangle\langle 00| \\ &\quad + \frac{1-p}{n}(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|) + \frac{p}{2}|11\rangle\langle 11|.\end{aligned} \tag{32}$$

Using the effective method for calculating concurrence in ref. 15 and after some calculations, we have

$$C(\rho_{A_1A_2}) \equiv 0, \quad \forall\, p \in (p_L, p_R) \tag{33}$$

where $n \geq 6$ and $p_{L,R} = \frac{(2n^2+8n-9) \mp \sqrt{(2n^2+8n-9)^2 - 16(n-1)(3n^2+4n-5)}}{2(3n^2+4n-5)}$. Then, according to Eq. (6), we obtain $E_F^{\alpha}(\rho_{A_1A_2}) \equiv 0$.

In the following, we will calculate $E_F(\rho_{A_1|A_2\cdots A_n})$. Through introducing a system B which has the same state space as the composite system $A_1A_2\cdots A_n$, $\rho_{A_1A_2\cdots A_n}$ can be purified as

$$|\Psi\rangle_{A_1A_2\cdots A_nB} = \sqrt{p}\,|GHZ\rangle_{A_1A_2\cdots A_n}|0\rangle_B + \sqrt{1-p}\,|W\rangle_{A_1A_2\cdots A_n}|1\rangle_B. \tag{34}$$

According to the Koashi-Winter formula[4,18], the bipartite multiqubit EoF can be calculated by the purified state $|\Psi\rangle_{A_1A_2\cdots A_nB}$, with $\rho_{A_1A_2\cdots A_n} = tr_B|\Psi\rangle\langle\Psi|$,

$$E_F(\rho_{A_1|A_2\cdots A_n}) = D_B(\rho_{A_1B}) + S(A_1|B), \tag{35}$$

where $S(A_1|B)$ is the quantum conditional von Neumann entropy, and the quantum discord $D_B(\rho_{A_1B})$ is defined as[13]

$$D_B(\rho_{A_1B}) = \min_{\{\Pi_B^r\}} \sum_r p_r S(A_1|\Pi_B^r) - S(A_1|B) \tag{36}$$

with the minimum running over all the positive operator-valued measures on the subsystem B. The details for proving Eq. (35) are presented in Methods. Chen *et al.*[19] presented an effective method for choosing an optimal measurement over B and then calculating the quantum discord of two-qubit X states, which can be used to quantify the multipartite entanglement indicator in Eq. (19). After some analysis, we can obtain the optimal measurement for the quantum discord $D_B(\rho_{A_1B})$ is σ_z when $n \geq 6$ and $p \in (p_L, p_R)$. Then, after some deduction, we get

$$E_F(\rho_{A_1|A_2\cdots A_n}) = p + (1-p)h\left(\frac{1}{n}\right). \tag{37}$$

From Eqs (19), (31) and (33), the indicator has the form

$$g(\alpha, n) = \tau_n^{\alpha}(\rho_{A_1A_2\cdots A_n}) = \left[p + (1-p)h\left(\frac{1}{n}\right)\right]^{\alpha}. \tag{38}$$

The distribution of $\tau_n^{\alpha}(\rho_{A_1A_2\cdots A_n})$ has been shown in Fig. 2 for $\alpha = \sqrt{2}$ and $\alpha = 2$ respectively. Furthermore, $\tau_n^{2}(\rho_{A_1A_2\cdots A_n})$ and $\tau_n^{\sqrt{2}}(\rho_{A_1A_2\cdots A_n})$ have some properties as follows.

(1) For any α, $g(\alpha, n)$ is a monotonically decreasing function of n. The monotonically decreasing property of $g(\alpha, n)$ holds because the first-order partial derivative satisfies

$$\frac{\partial g(\alpha, n)}{\partial n} = \alpha\left[p + (1-p)h\left(\frac{1}{n}\right)\right]^{\alpha-1} \frac{1}{\sqrt{1-\frac{1}{n}}\ \ln\ 16} \cdot \ln\left(\frac{1+\sqrt{1-\frac{1}{n}}}{1-\sqrt{1-\frac{1}{n}}}\right)\left(-\frac{1}{n^2}\right) < 0. \tag{39}$$

(2) Combining with Theorem 3 and Eqs (33) and (38), we have $\tau_n^{\sqrt{2}}(\rho_{A_1A_2\cdots A_n}) > \tau_n^{2}(\rho_{A_1A_2\cdots A_n})$.

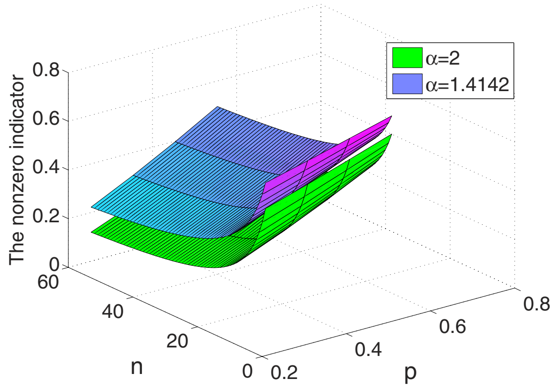

Figure 2. The multipartite entanglement indicators for the superposition state as functions of *n* and *p*, where $n \in [6, 60]$, $\alpha = 2$ and $\sqrt{2}$ respectively.

From the above two properties, we know that the nonzero $\tau_n^{\sqrt{2}}(\rho_{A_1A_2\cdots A_n})$ can indicate the existence of the n-qubit entanglement. These results can also be understood as the fact that $\tau_n^{\sqrt{2}}(\rho_{A_1A_2\cdots A_n})$ can detect as many as possible n-qubit entangled states for large n.

Conclusion

Entanglement monogamy is a fundamental property of multipartite entangled states. Based on our established monogamy relations Eq. (10), we obtain a set of useful tools for characterizing the multipartite entanglement not stored in pairs of the focus particle and the other subset of particles, which overcome some flaws of the concurrence. For any n-qubit symmetric state, we prove that the $\sqrt{2}$ EoF indicator work best when n is large enough, while the 2 EoF indicator works better than the $\sqrt{2}$ EoF indicator for smaller n.

Methods

The monotonic property of the function in Eqs (20) and (21). In order to determine the monotonic property of $M(n)$, with

$$M(n) = \frac{\ln\left[(n-1)\frac{\ln c}{\ln b}\right]}{\ln \frac{b}{c}} \tag{40}$$

in Eqs (20) and (21), we analyze the sign of the first-order derivative $dM(n)/dn$.

After some deduction, we can obtain

$$\begin{aligned}\frac{dM(n)}{dn} = & \frac{1}{\left(\ln \frac{b}{c}\right)^2}\left[\left(\frac{1}{n-1} + \frac{1}{c\ \ln\ c}\frac{dc}{dn} - \frac{1}{b\ \ln\ b}\frac{db}{dn}\right)\ln\ \frac{b}{c}\right.\\ & \left.-\left(\frac{1}{b\ \ln\ b}\frac{db}{dn}\ \ln\ b - \frac{1}{c\ \ln\ c}\frac{dc}{dn}\ \ln\ c\right)\ln\left((n-1)\frac{\ln\ c}{\ln\ b}\right)\right].\end{aligned} \tag{41}$$

Then, $dM(n)/dn < 0$ when

$$\ln\left[(n-1)\frac{\ln\ c}{\ln\ b}\right] > \ln\ \frac{b}{c}, \tag{42}$$

and

$$\frac{\ln\ b}{b\ \ln\ b}\frac{db}{dn} - \frac{\ln\ c}{c\ \ln\ c}\frac{dc}{dn} > \frac{1}{n-1} + \frac{1}{c\ \ln\ c}\frac{dc}{dn} - \frac{1}{b\ \ln\ b}\frac{db}{dn}. \tag{43}$$

Eq. (42) holds if and only if

$$\ln\left[\frac{(n-1)c\ \ln\ c}{b\ \ln\ b}\right] > 0, \tag{44}$$

i.e.,

$$H(b) = b\ \ln\ b > (n-1)c\ \ln\ c = (n-1)H(c). \tag{45}$$

The inequality (45) holds because $H[x(n)] = x(n)\ln\ x(n)$ is a concave function of n with $x(n) \in \{b(n),\ c(n)\}$.

Similarly, we have Eq. (43) holds when

$$F(\ln\ b) > F(n-1) + F(\ln\ c). \tag{46}$$

where

$$F[t(n)] = (1 + t(n)) \cdot \frac{d\ \ln|t(n)|}{dn} \tag{47}$$

and then $F(n-1) > 1/(n-1)$. From ref. 9, we easily get that $dt(n)/dn < 0$ where $t(n) \in \{\ln\ b,\ \ln\ c\}$.

In the following, we will prove Eq. (46). Let $K[t(n)] = t(n) + \ln|t(n)|$, where $t(n) \in \{\ln\ b,\ \ln\ c\}$. Using the definition of the partial derivative, it is not different to verify that $\frac{\partial K[t(n)]}{\partial n}$, $\frac{\partial K[t(n)]}{\partial t}$, $\frac{\partial^2 K[t(n)]}{\partial n \partial t}$ and $\frac{\partial^2 K[t(n)]}{\partial t \partial n}$ are all continuous functions. Combining with the exchange order theorem of two second-order mixed partial derivative, we have

$$\begin{aligned} \frac{dF(t)}{dt} &= \frac{\partial^2 K[t(n)]}{\partial n \partial t} = \frac{\partial^2 K[t(n)]}{\partial t \partial n} = -\frac{1}{t^2}\frac{dt}{dn} > 0, \\ \frac{d^2F(t)}{dt} &= \frac{\partial^3 K[t(n)]}{\partial n \partial t^2} = \frac{\partial^2 K[t(n)]}{\partial t^2 \partial n} = \frac{2}{t^3}\frac{dt}{dn} < 0. \end{aligned} \tag{48}$$

According to Eq. (47), we get that $F(t)$ is monotonic and concave as a function of t.

Combining with Eq. (19), we have

$$F(\ln\ b) \geq F(\ln[(n-1)c]) \geq F((n-1)\ln\ c) \geq F(n-1) + F(\ln\ c). \tag{49}$$

Here, the first inequality holds because f is a concave function of n, and the monotonically increasing property of $F(t)$ in Eq. (48). The second inequality is satisfied because $F(t)$ is a monotonically increasing function in Eq. (48) and $\ln x$ is a concave function of x. And the last inequality holds because $F(t)$ is a concave function as proved in Eq. (48).

Then, we complete the proof that $M(n)$ is a monotonically decreasing function of n.

Proof of the Eq. (35) in the Main Text. Purification can be done for any state $\rho_{A_1A_2\cdots A_n}$, because we can introduce a system B which has the same state space as system $A_1A_2\cdots A_n$ and define a pure state[20] for the combined system

$$|\Psi\rangle_{A_1A_2\cdots A_nB} = \sqrt{p}|GHZ\rangle_{A_1A_2\cdots A_n}|0\rangle_B + \sqrt{1-p}|W\rangle_{A_1A_2\cdots A_n}|1\rangle_B. \tag{50}$$

From ref. 21, we know

$$I^{\longleftarrow}(\rho_{A_1B}) + D_B(\rho_{A_1B}) = I(\rho_{A_1B}). \tag{51}$$

Combining with $I(\rho_{A_1B}) = S(\rho_{A_1}) - S(A_1|B)$, we can find that Eq. (35) is just Eq. (2) in ref. 17. More specifically,

$$\begin{aligned} E_F(\rho_{A_1|A_2\cdots A_n}) &= S(\rho_{A_1}) - I^{\longleftarrow}(\rho_{A_1B}) \\ &= S(\rho_{A_1}) - \left[I(\rho_{A_1B}) - D_B(\rho_{A_1B})\right] \\ &= D_B(\rho_{A_1B}) + S(A_1|B). \end{aligned} \tag{52}$$

Then, we complete the proof of the Eq. (35) in the Main Text.

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Author Contributions
F.L. and F.G. contributed the idea. F.L. performed the calculations and wrote the main manuscript. S.-J.Q. checked the calculations. S.-C.X. and Q.-Y.W. made an improvement of the manuscript. All authors contributed to discussion and reviewed the manuscript.

Additional Information
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