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Transition and Damping of Collective Modes in a Trapped Fermi Gas between BCS and Unitary Limits near the Phase Transition

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We investigate the transition and damping of low-energy collective modes in a trapped unitary Fermi gas by solving the Boltzmann-Vlasov kinetic equation in a scaled form, which is combined with both the *T*-matrix fluctuation theory in normal phase and the mean-field theory in order phase. In order to connect the microscopic and kinetic descriptions of many-body Feshbach scattering, we adopt a phenomenological two-fluid physical approach, and derive the coupling constants in the order phase. By solving the Boltzmann-Vlasov steady-state equation in a variational form, we calculate two viscous relaxation rates with the collision probabilities of fermion's scattering including fermions in the normal fluid and fermion pairs in the superfluid. Additionally, by considering the pairing and depairing of fermions, we get results of the frequency and damping of collective modes versus temperature and *s*-wave scattering length. Our theoretical results are in a remarkable agreement with the experimental data, particularly for the sharp transition between collisionless and hydrodynamic behaviour and strong damping between BCS and unitary limits near the phase transition. The sharp transition originates from the maximum of viscous relaxation rate caused by fermion-fermion pair collision at the phase transition point when the fermion depair, while the strong damping due to the fast varying of the frequency of collective modes from BCS limit to unitary limit.

Strongly-interacting two-component Fermi gases provide a unique testing ground for the many-body theories of exotic systems, such as unconventional superconductors, nuclear matter, neutron stars and dilute atomic Fermi gases, which at first sight have tunable and strong interactions by using a Feshbach resonance (FR)[1,2]. Near the resonance [$\eta \sim 0$, as usual, we conveniently measure the interaction strength in terms of the inverse scattering length $\eta = (k_F a_{sc})^{-1}$, where k_F is the Fermi momentum and a_{sc} the *s*-wave scatting length], the interparticle interactions are unitary limited and universal[1–4].

The study of collective excitations in these systems has attracted much attention in the past decades. The collective excitations are one of the main sources to prob the dynamics of the many-body systems. The high accuracy of frequency measurements and the sensitivity of collective phenomena to interaction effects make them good candidates to unravel the dynamical correlations. Experimental results have been obtained for low-lying collective modes of a two-component Fermi gas [6]Li in wide temperature and interaction regimes, including the radial compression modes[5–10], the axial compressions modes[8,11,12], the radial quadrupole modes[10,13,14], and the scissors mode[10,13]. The thermodynamic quantities like energy and entropy in trapped Fermi gases at unitarity are measured without invoking any specific theoretical model[15,16]. These experiments have in turn stimulated a considerable amount of theoretical works[11,12,17–22].

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However, a theoretical description of this unitary regime is still challenging, particularly at nonzero temperature for sharp transition and strong damping of the collective modes.

There are different strong-coupling theories to study the collective excitations of superfluid Fermi gases in the BCS-BEC crossover. One of them is microscopic theory based on a model Hamiltonian either with a one-channel model for a broad (or weak) FR or with a two-channel model for a narrow FR. The link of these two models is well described in[23]. There are numerous efforts to develop the strong-coupling perturbation theories of interacting fermions. For example, the thermodynamic potential (or action) approach[24–28], the diagrammatic method[29–39], and the many-body T-matrix fluctuation theories[29–44]. Leggett's[45] mean-field theory and then Randeria *et al.*[46] by adding fluctuations get some qualitative correct results at zero temperature. The Quantum Monte Carlo (QMC) simulations[47,48] and the pseudogap approach[43,49–51] have better results in the BCS-BEC crossover. Because the strong coupling atomic Fermi gases are trapped in a finite space at finite temperatures, the inhomogeneous feature of the system, strong pairing fluctuations, and finite temperatures are important keys in considering real cold Fermi gases, which makes the pure microscopic approach difficult to deal with, especially in studying the collective excitations.

Another viewpoint is the quantum hydrodynamical theory based both on the Boltzmann-Vlasov kinetic equation in the normal state[17–22] and on the generalized Gross-Pitaevskii(GP) equation in the superfluid state[52–62] with a phenomenological equation of state. The study of the viscosity of strongly interacting systems is also a topic of great interest both in experimental works[5–14,63,64] and in theoretical works based on the Boltzmann-Vlasov kinetic equation[17–22,65,66]. The radial compression mode reveals a surprising behavior[8]: An abrupt change of the radial collective frequency in a strongly attractive Fermi gas. The radial quadrupole mode has confirmed the transition from collisionless to hydrodynamic behavior at $\eta \simeq -0.8$[14]. The transition is accompanied by very strong damping. The corresponding features cannot be explained on the basis of available theoretical models and new physics is in great need in this regime. How to explain this feature is an open question by now. We still lack a full discussion on the transition and damping of collective modes, especially compared with the experimental results[5–14,63,64]. This is the major motivation for our present study of different collective modes under similar experimental conditions where the system is trapped around the critical temperature T_c.

In this paper, we determine the sharp transition and strong damping of the collective modes at $-2 < \eta < 0$ around T_c after solving the Boltzmann-Vlasov kinetic equation in a scaled form, in which we have combined both the many-body T-matrix fluctuation theory in the normal phase at $T \geq T_c$ and the mean-field theory in the order phase at T, T_c. We first need to get the expressions of the viscous relaxation time, τ_{nn} and τ_{ns}. Here τ_{nn} is related to the scattering between fermions in the normal states, and τ_{ns} is related to the scattering between normal fermions and superfluid fermions. In order to do so, the two-fluid approach is a link to the microscopic and kinetic descriptions of many-body Feshbach scattering. We then calculate two viscous relaxation rates by solving the Boltzmann-Vlasov steady state equation in a variational form. We next calculate the collective-mode frequencies and their damping as a function of the temperature and interaction strength for a trapped gas. We finally compare our results to experiments.

Results

Model and ingredients. For the broad resonance regime of the interacting fermions, a one-channel microscopic model of the strong Feshbach resonance is the most appropriate. For N fermions of two species $\sigma = \uparrow$ and $\sigma = \downarrow$ with fermionic field $\psi_\sigma(\mathbf{r})$, its Hamilton density is

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + g\psi_\sigma^* \psi_{\bar{\sigma}}^* \psi_{\bar{\sigma}} \psi_\sigma. \tag{1}$$

Here $\hat{\mathcal{H}}_0 = \sum_\sigma \psi_\sigma^* (\hat{K} + \hat{V}) \psi_\sigma$ is the single-particle Hamilton density, $\hat{K} = -\hbar^2 \nabla^2 / 2m - \mu$, $\hat{V}(\mathbf{r}) = \frac{1}{2} m\omega_\perp^2 (x^2 + y^2 + \lambda^2 z^2)$ is the external potential with λ being the anisotropic parameter, and $g = 4\pi\hbar a_{sc}/m$ is the interaction coupling constant. In the normal phase at $T \gg T_c$, the system can be treat as a normal fluid. But near above T_c, fermions in the system may exist pre-pairing. While in the order phase at $T < T_c$, parts of fermions become either the Cooper pairs due to the many-body effect at the BCS side or the bosonic molecule due to two-body bound state at the BEC side, and the pair fluctuations become important to the system near the unitary. However, it is hard to find anact solution of the model to describe this picture, also hard to find the expressions for the viscous relaxation rates of the all scattering processes.

Firstly, we start from the Boltzmann-Vlasov kinetic equation and find its scaling solutions. The phenomenological two-fluid physical picture is about N_n normal fermions and $N_s = (N - N_n)/2$ superfluid fermion pairs (Cooper pairs or molecules). Generally, we should write GP equation to describe the dynamic of the superfluid parts. However, our interest focus on the system's behavior and novel phenomenon near T_c, where the effects that form and break pairs play a more important role. So we freeze all superfluid fermion pairs at zero center-mass momentum and omit the dynamics of the superfluid fermions. This approximation is only satisfied at the BCS side. Therefore we focus on two kinds of scattering processes of fermions from BCS limit to unitary limit. One is scattering with the normal fermions in the normal phase by joining the T-matrix approximation at $T \geq T_c$, the other is with the fermion

pairs in the superfluid phase by adopting the mean-field approximation at $T < T_c$. Secondly, in order to consider the pairing and depairing effects of fermions, and to get the pair-pair and fermion-pair coupling constants in the order phase, we need to connect the microscopic and kinetic descriptions of many-body Feshbach scattering. So we solve the one-channel microscopic model in the mean-field approximation. We write the two-liquid phenomenological model and determine the coupling constants by equalizing all the physical quantities obtained from the phenomenological model with those obtained from the microscopic model. Thirdly, we derive the expression of the two viscous relaxation time of fermions, and calculate their relaxation time, τ_{nn} and τ_{ns}, respectively. Finally, we calculate the collective mode frequencies and their damping as a function of the temperature and interaction strength for a trapped gas. We compare our theoretical results to experiments, and conclude that the transition from collisionless to hydrodynamic behavior occurs at the maximum of viscous relaxation rate of normal fermion-superfluid pair collision at phase transition point.

Boltzmann-Vlasov kinetic equation. We first briefly review the Boltzmann-Vlasov kinetic equation. We consider a two-component gas of Fermi atoms with mass m and different spin σ near its normal phase. We assume that the dynamics is described by a semiclassical distribution function $f(\mathbf{r}, \mathbf{p}, t)$ for each component. We omit the index of spin because of the system symmetry. $f(\mathbf{r}, \mathbf{p}, t)$ satisfies the Boltzmann-Vlasov kinetic equation[18,67,68]

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f}{\partial \mathbf{r}} - \frac{\partial(\hat{V} + \varepsilon)}{\partial \mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{p}} = -I[f, \rho_s] = C_{nn}[f] + C_{ns}[f, \rho_s], \tag{2}$$

where $C_{nn}[f]$ and $C_{ns}[f, \rho_s]$ are the collision integrals of normal fermion-normal fermion and normal fermion-superfluid pair, respectively. They are a functional of $f(\mathbf{r}, \mathbf{k}, t)$ and density of pairs $\rho_s(\mathbf{r})$ near below T_c. $\mathbf{k} = \mathbf{p}/\hbar$, and $\hbar = h/2\pi$ with h being the Plank constant. ε is the mean-field interaction energy. Many physical quantities are very sensitive to the equation of state which is given by $U(\rho) = \frac{\partial}{\partial \rho}[\rho\varepsilon(\rho)] = (\gamma + 1)c\rho^\gamma$ in a polytropic approximation $\varepsilon(\rho) = c\rho^\gamma$ with c being the constant with given η. Here γ is an index defined as the logarithmic derivative $\gamma = \partial \ln \varepsilon / \partial \ln \rho$[55], and ρ is the particle density for each component of Fermi gas. Interparticle interactions enter Eq. (2) in two different ways. One way is to modify the effective potential through the mean-field term ε which affects the streaming part of the Boltzmann kinetic equation. The mean-field term is essential linear in a_{sc} and this theory has no dissipative term. Another way is to consider the two-body interactions in the collision integral $-I[f, \rho_s]$ which is quadratic in the scattering length and describes the dissipative processes.

The two collision terms in Eq. (2) at Born approximation level are given by[67,68]

$$C_{nn}[f_1] = \frac{2g_n^2}{(2\pi)^5 \hbar} \int d\mathbf{k}_2 \int d\mathbf{k}_3 \int d\mathbf{k}_4 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$$
$$\times [(1 - f_1)(1 - f_2)f_3 f_4 - f_1 f_2 (1 - f_3)(1 - f_4)], \tag{3}$$

$$C_{ns}[f, \rho_s] = \frac{2g_{ns}^2 \rho_s}{(2\pi)^5 \hbar k_F^3} \int d\mathbf{k}_1 \int d\mathbf{k}_2 \int d\mathbf{k}_3 \int d\mathbf{k}_4$$
$$\times \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \delta(\epsilon_b + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$$
$$\times [\delta(\mathbf{k} - \mathbf{k}_1) - \delta(\mathbf{k} - \mathbf{k}_2) - \delta(\mathbf{k} - \mathbf{k}_3) - \delta(\mathbf{k} - \mathbf{k}_4)]$$
$$\times [(1 - f_1)f_2 f_3 f_4 - f_1(1 - f_2)(1 - f_3)(1 - f_4)], \tag{4}$$

with $f_i \equiv f(\mathbf{r}, \mathbf{k}_i, t)$ for $i = 1, 2, 3, 4$. Here $g_n = g = 4\pi\hbar a_{sc}/m$. g_{ns} is the fermion-pair coupling constant determined below. The binding energy ϵ_b of a fermion pair is always negative below T_c.

In order to study the damping rate, we need to calculate the viscous relaxation rate $1/\tau = 1/\tau_{nn} + 1/\tau_{ns}$. In the normal phase at $T \geq T_c$, $\tau = \tau_{nn}$ as usual, while in the superfluid phase at $T < T_c$, we have to estimate τ_{ns}. We first write $f = f^0 + \delta f$ where $f^0 = 1/[e^{(\hbar^2 k^2/2m + \hat{V}(\mathbf{r}) - \mu)/k_B T} + 1]$ is the equilibrium distribution function without the interactions between two particles and δf is its deviation with the interactions. The Boltzmann-Vlasov kinetic equation (2) is a differential and integral equation, and it is hard to find its solution exactly. In the relaxation time approximation, $I[\Phi] = (f - f^0)/\tau$ where Φ is defined by $\delta f = f^0 (1 - f^0)\Phi$. So

$$\frac{1}{\tau} = \frac{-\langle \Phi^* (C_{nn}[\Phi] + C_{ns}[\Phi, \rho_s]) \Phi \rangle}{\langle |\Phi|^2 f^0 (1 - f^0) \rangle} \tag{5}$$

for any trial function Φ by using a variational method. Here $\langle \cdots \rangle$ means an integral over all phase space (both coordinate and momentum space), i.e., $\int d\mathbf{r} \int \frac{d\mathbf{k}}{(2\pi)^3}(\cdots)$. We will model the scatting process and calculate Eq. (5) below.

The scale solutions of Boltzmann-Vlasov kinetic equation. We investigate the low-lying collective oscillations with both the mean-field and the dissipative contributions by means of the scaling factor method. We take ansatz[17,69]

$$R_i = r_i/b_i, \; V_i = (v_i - \dot{b}_i r_i/b_i)/\theta_i^{1/2}, f(\mathbf{r}, \mathbf{v}, t) = \Gamma(t)f^0[\mathbf{R}(t), \mathbf{V}(t)], \tag{6}$$

where $\Gamma(t) = 1/\prod_{j=1}^{3} b_j(t)\theta_j^{1/2}(t)$. The particle density takes the form of $\rho(\mathbf{r}, t) = \rho(\mathbf{R})/\prod_{j=1}^{3} b_j(t)$. The dependence on time is contained in the parameters b_i and θ_i. Following refs 17,69, we substitute this ansatz into Eq. (2), integrate in phase space, and calculate the average moment of $R_i V_i$. This leads to

$$\ddot{b}_i + \omega_i^2 b_i - \omega_i^2 \zeta_i \frac{\theta_i}{b_i} + \omega_i^2 \xi_i \left(\frac{\theta_i}{b_i} - \frac{1}{b_i \left(\prod_{j=1}^{3} b_j \right)^\gamma} \right) = \omega_i^2 \eta_i. \tag{7}$$

Here

$$\zeta_i \equiv \frac{\langle V_i^2 \rangle}{\langle \omega_i^2 R_i^2 \rangle}, \; \xi_i \equiv \frac{\gamma \langle \varepsilon \rangle}{\langle m\omega_i^2 R_i^2 \rangle}, \tag{8}$$

and

$$\eta_i \propto \int \tau^{-1}(f - f^0) R_i dR_i d\mathbf{V} \tag{9}$$

does not vanish at $T \leq T_c$ but is hard to calculate directly. The average means in the equilibrium density of $\rho(\mathbf{R})$ within the atom cloud. The average moment of V_i^2 leads to

$$\dot{\theta}_i + 2\frac{\dot{b}_i}{b_i}\theta_i = \frac{1}{\tau}\left(\frac{1}{3}\sum_{j=1}^{3} \theta_j - \theta_i \right). \tag{10}$$

Eq. (10) depends explicitly on the viscous relaxation time $1/\tau = 1/\tau_{nn} + 1/\tau_{ns}$.

By linearizing the Eqs. (7) and (10) around equilibrium ($b_i = \theta_i = 1$) which gives the generalized Virial theorem of $1 - \zeta_i - \xi_i = \eta_i$ with $i = (\perp, z)$ in the axial symmetry external potential, we find the following dispersion law

$$[\overline{\omega}^2 - \overline{\omega}_{hd}^2 + i\overline{\omega}\overline{\tau}(\overline{\omega}^2 - \overline{\omega}_{cl}^2)][(\overline{\omega}^2 - \overline{\omega}_{hd+}^2)(\overline{\omega}^2 - \overline{\omega}_{hd-}^2) + i\overline{\omega}\overline{\tau}(\overline{\omega}^2 - \overline{\omega}_{cl+}^2)(\overline{\omega}^2 - \overline{\omega}_{cl-}^2)] = 0. \tag{11}$$

Here subscripts *hd* and *cl* denote the hydrodynamic ($\overline{\tau} \to 0$) and collisionless ($\overline{\tau} \to \infty$) regimes, respectively. And

$$\overline{\omega}_{hd}^2 \equiv 1 + \zeta_\perp + \xi_\perp, \; \overline{\omega}_{cl}^2 \equiv 1 + 3\zeta_\perp + \xi_\perp, \; \overline{\omega}_{hd\pm}^2 \equiv \mathcal{A} \pm \sqrt{\mathcal{A}^2 - \mathcal{B}}, \; \overline{\omega}_{cl\pm}^2 \equiv \mathcal{C} \pm \sqrt{\mathcal{C}^2 - \mathcal{D}}, \tag{12}$$

$\mathcal{A} = \frac{2}{3}\zeta_\perp + \gamma\xi_\perp + \frac{1}{2}\lambda^2\chi_z + \frac{1}{2}\chi_\perp + \frac{1}{3}\lambda^2\zeta_z$, $\mathcal{B} = \frac{4}{3}\lambda^2\zeta_\perp(\chi_z - \gamma\xi_z) + \lambda^2\gamma\xi_\perp(\chi_z - 2\gamma\xi_z - \frac{2}{3}\zeta_z) + \lambda^2\chi_\perp\chi_z$ $+ \frac{2}{3}\lambda^2\zeta_z\chi_\perp$, $\mathcal{C} = \zeta_\perp + \frac{1}{2}\gamma\xi_\perp + \frac{1}{2}\lambda^2\chi_z + \frac{1}{2}\chi_\perp + \lambda^2\zeta_z$, $\mathcal{D} = \lambda^2(2\zeta_\perp + \gamma\xi_\perp + \chi_\perp)(2\zeta_z + \chi_z) - 2\lambda^2\gamma^2\xi_\perp\xi_z$, with $\chi_\perp = \zeta_\perp + (\gamma+1)\xi_\perp + 1$ and $\chi_z = \zeta_z + (\gamma+1)\xi_z + 1$. When we study the collective modes in the absence of the interacting term of ε but with the dissipative contribution ($\xi_i = 0$ and $\zeta_i = 1$ at $T \geq T_c$, the collision integral represents the two-body interactions), Eq. (12) becomes $\overline{\omega}_{hd} = \sqrt{2}$, $\overline{\omega}_{cl} = 2$, $\overline{\omega}_{hd\pm}^2 = \frac{1}{3}\left(5 + 4\lambda^2 \pm \sqrt{25 - 32\lambda^2 + 16\lambda^4} \right)$, $\overline{\omega}_{cl\pm}^2 = 2(1 + \lambda^2 \pm |1 - \lambda^2|)$. The mode frequency and damping are sensitive to the dimensionless viscous relaxation rate $1/\overline{\tau}$ as a function of (T, η, λ, N). For example, the equation from the first factor of the left-hand side of Eq. (11) gives the quadrupole mode frequencies in the transverse directions. This equation is the same with the one obtained early in refs 69–71. It is clear that the transition between the collisionless ($\overline{\omega}_{cl} = 2$) and hydrodynamic ($\overline{\omega}_{hd} = \sqrt{2}$) modes is determined by the behavior of the dimensionless viscous relaxation time $\overline{\tau}$. The damping exists due to the finite value of $i\overline{\tau}$ in those equations. The second factor of the left-hand side of Eq. (11) gives the frequencies of the two compression modes labeled by $+$ and $-$ as the radial and axial modes, respectively. Thus, our work is to determine ξ_i and ζ_i ($i = \perp, z$), and $\tau^{-1} = \tau_{nn}^{-1} + \tau_{ns}^{-1}$, as functions of (T, η, λ, N) near the phase transition. This extends the results in refs 69–71 and allows a direct comparison with experiments. To get these quantities, we adopt the *T*-matrix approximation at $T \geq T_c$ and the mean-field approximation at $T < T_c$.

T-matrix fluctuation theory above the critical temperatures T_c.   For a trapped system, we use the local density approximation (LDA) and write the chemical potential as $\mu(\mathbf{r}) = \mu_0 - \hat{V}(\mathbf{r})$ where μ_0 is chemical potential at the center of trap. μ is determined by $N = \int d\mathbf{r}\rho(\mathbf{r})$. The total potential energy is $V = \int d\mathbf{r}\hat{V}(\mathbf{r})\rho(\mathbf{r})$. To find a expression for $\rho(\mathbf{r})$ and discuss the strong-coupling effects in the BCS-BEC crossover above T_c, we include pairing fluctuations within the T-matrix approximation[29–35,37–44]. In ladder approximation, the single-particle thermal Green's function is $G_{\mathbf{k},\mathbf{r}}(i\omega_n) = 1/[i\omega_n - \xi_{\mathbf{k},\mathbf{r}} - \sum(\mathbf{k},\mathbf{r},i\omega_n)]$, where $\omega_n = (2n+1)\pi/\beta$ is the fermion Matsubara frequency for $n = 0, \pm 1, \pm 2, \cdots$, $\xi_{\mathbf{k},\mathbf{r}} = \varepsilon_{\mathbf{k}} - \mu(\mathbf{r})$, $\varepsilon_{\mathbf{k}} = \hbar^2\mathbf{k}^2/2m$, and $G^0_{\mathbf{k},\mathbf{r}}(i\omega_n) = 1/(i\omega_n - \xi_{\mathbf{k},\mathbf{r}})$ is the free fermion Green's function. The self-energy correction $\sum(\mathbf{k},\mathbf{r},i\omega_n) = k_B T \int \frac{d\mathbf{q}}{(2\pi)^3} \sum_{\nu_n} \Gamma(\mathbf{q},\mathbf{r},i\nu_n) G^0_{\mathbf{k}-\mathbf{q},\mathbf{r}}(i\nu_n - i\omega_n) e^{(i\nu_n - i\omega_n)\delta}$ describes effects of pairing fluctuations, where $\nu_n = 2n\pi/\beta$ is the boson Matsubara frequency and δ is the infinitesimal positive constant. The particle-particle scattering matrix is given by

$$\Gamma(\mathbf{q},\mathbf{r},i\nu_n) = \frac{4\pi a_{sc}}{m} \frac{1}{1 + \frac{4\pi\hbar^2 a_{sc}}{m}\prod(\mathbf{q},\mathbf{r},i\nu_n)}. \tag{13}$$

This describes the fluctuations in the Cooper channel. Here $\Pi(\mathbf{q},\mathbf{r},i\nu_n)$ is the pair propagator expressed as

$$\prod(\mathbf{q},\mathbf{r},i\nu_n) = \int \frac{d\mathbf{k}}{(2\pi)^3}\left[\frac{1 - f_- - f_+}{\xi_{\mathbf{k}+\mathbf{q}/2,\mathbf{r}} + \xi_{-\mathbf{k}+\mathbf{q}/2,\mathbf{r}} - i\nu_n} - \frac{1}{2\varepsilon_{\mathbf{k}}}\right], \tag{14}$$

with $f_{\mp}$ being the equilibrium Fermi distribution function f^0 for $\mathbf{p} = \hbar(\mp k + \mathbf{q}/2)$ in the absence of the inter-particle interactions. The superfluid phase transition temperature T_c is given as the temperature at which the Thouless criterion[72] is satisfied in the trap center ($r = 0$). The resulting T_c equation is given by $\Gamma^{-1}(\mathbf{q} = 0, \mathbf{r} = 0, i\nu_n = 0) = 0$.

We now calculate μ, ξ_i and ζ_i for $i = (\perp, z)$ at $T > T_c$. For the harmonic trap within LDA, $\mu = \mu_0(1 - \bar{r}^2)$ under the Thomas-Fermi approximation with the dimensionless spherical polar coordinate $\bar{r}$ in the radial direction. At $T > T_c$, the chemical potential μ is determined by the condition

$$\lambda N = 8\pi \int_0^1 \bar{r}^2 d\bar{r}\rho(\bar{\mathbf{r}}), \tag{15}$$

and the particle density $\rho(\bar{\mathbf{r}}) = \int \frac{d\mathbf{k}}{(2\pi k_F)^3\beta}\sum_{\omega_n} G_{\mathbf{k},\bar{\mathbf{r}}}(i\omega_n) e^{i\omega_n\delta}$. One of our calculations of μ is identical to the one described in refs 40,41, and another is the numerical Fourier transformation similar with refs 30,39 where more details are given, and the resulting multidimensional integrals are evaluated by using a Monte Carlo routine. The mean potential energy is always determined by

$$V = \frac{8\pi}{\lambda}\int_0^1 \hat{V}\bar{r}^2 d\bar{r}\rho(\bar{\mathbf{r}}). \tag{16}$$

At high enough temperature $T \gg T_c$, the mean kinetic energy is $K = 3Nk_BT/2$ and the mean interaction energy U is determined by the Virial theorem: $\xi_i = 1 - \zeta_i$. At $T \geq T_c$, U is determined by the definition from BCS limit to unitary limit

$$U = \frac{4\pi}{\lambda}g_n \int_0^1 \bar{r}^2 d\bar{r}\rho^2(\bar{\mathbf{r}}), \tag{17}$$

and K is determined by the Virial theorem: $\zeta_i = 1 - \xi_i$. Therefore, we can obtain ξ_i and ζ_i in whole BCS side of BCS-BEC crossover at $T \geq T_c$.

Mean-field theory below T_c.   We choose the mean-field theory to describe the system below T_c. In order to calculate the viscous relaxation rate $1/\tau_{ns}$ and ratios ξ_i and ζ_i for $i = (\perp, z)$, we express the fermion-pair and pair-pair coupling constants g_{ns} and g_s by combining the mean-field microscopic theory with the two-fluid approach that the system consists of N_n normal fluid fermions and N_s superfluid fermion pairs at $T < T_c$. On the one hand, we calculate the physical quantities within the mean-field theory, such as the particle number density, chemical potentials, energy gap, excitation energies, and the total energy. On the another hand, the two-fluid approach gives the effective energy $H_{eff} = K + V + U$, where K, V and U are the kinetic energy, potential energy and interacting energy, respectively. We make a connection between H_{eff} and A in the Leggett's mean-field theory to make sure they describe the same system at the level of E, μ, and $\rho(\mathbf{r})$. Thus we can obtain the expressions for g_{ns}, g_s, and $\rho_s(\mathbf{r})$ in this approach. Meanwhile, we can obtain the ratios ξ_i and ζ_i for $i = (\perp, z)$ in this way.

Firstly, the one-channel microscopic theory with mean-field approximation gives the action

$$A = \int d\mathbf{r} \{ \frac{\beta \Delta_{\mathbf{r}}^2}{g} - \int \frac{d\mathbf{k}}{(2\pi)^3} \sum_n \ln [\beta^2 (\omega_n^2 + E_{\mathbf{k},\mathbf{r}}^2)] \},$$

(18)

with $E_{\mathbf{k},\mathbf{r}} = \sqrt{\xi_{\mathbf{k},\mathbf{r}}^2 + \Delta_{\mathbf{r}}^2}$ being the BCS-Bogliubov excitation energy and $\beta = 1/k_B T$. The gap $\Delta_{\mathbf{r}}$ is determined by $\delta A / \delta \Delta_{\mathbf{r}} = 0$:

$$\frac{m}{4\pi a_{sc}} = \int \frac{d\mathbf{k}}{(2\pi)^3} \int d\mathbf{r} \left(\frac{1}{2\varepsilon_{\mathbf{k}}} - \frac{\text{th}\beta E_{\mathbf{k},\mathbf{r}}/2}{2E_{\mathbf{k},\mathbf{r}}} \right).$$

(19)

In the mean-field approximation, the total particle density distribution in the phase space is

$$f_{MF}(\mathbf{k}, \mathbf{r}) = 1 - \frac{\xi_{\mathbf{k},\mathbf{r}}}{E_{\mathbf{k},\mathbf{r}}} \text{th} \frac{1}{2} \beta E_{\mathbf{k},\mathbf{r}}.$$

(20)

Then $\rho(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^3} f_{MF}(\mathbf{k}, \mathbf{r})$ and $N = \int \rho(\mathbf{r}) d\mathbf{r} = \int \frac{d\mathbf{k}}{(2\pi)^3} \int d\mathbf{r} f_{MF}(\mathbf{k}, \mathbf{r})$ can be used to determine the chemical potential μ. Since the free energy is $F = A/\beta$ and the entropy is $S = -k_B \partial(TA)/\partial T$, the energy has the form

$$E = F + TS = -k_B T^2 \partial A / \partial T.$$

(21)

Then in order to calculate the viscous relaxation time τ_{ns} in the normal fermion scatting with super-fluid pairs, we want to get the fermion-pair coupling constant g_{ns} in the order phase. As described above for getting the expression for g_{ns}, we combine the one-channel microscopic theory with the two-liquid approach. We suppose that this system can also be described phenomenally as a mixture of normal fluid formed by N_n fermions and superfluid formed by N_s pairs composed of two fermions in the whole regimes with the total atomic number conservation: $N = N_n + 2N_s$. This is just the two-fluid model. For a trapped system, we have

$$\rho(\mathbf{r}) = \rho_n(\mathbf{r}) + 2\rho_s(\mathbf{r}).$$

(22)

The effective energy density is $\hat{\mathcal{H}}_{eff} = \hat{\mathcal{H}}_0 + \hat{\mathcal{U}}$, with $\hat{\mathcal{U}} = \frac{1}{4} g_n \rho_n^2 + \frac{1}{2} g_s \rho_s^2 + g_{ns} \rho_n \rho_s$ being the interaction density. Since we lack the knowledge of the expressions for g_s and g_{ns} in the one-channel microscopic model even at the mean-field level, we derive them in the phenomenological way. The mean superfluid pair-superfluid pair(normal fermion-normal fermion) interaction energy per particle is

$$u_{s(n)} = N_{s(n)}^{-1} \int d\mathbf{r} \rho_{s(n)} \partial \hat{\mathcal{U}} / \partial \rho_{s(n)} = N_{s(n)}^{-1} \int d\mathbf{r} [g_s \rho_s^2 (g_n \rho_n^2 / 2) + g_{ns} \rho_n \rho_s].$$

(23)

We consider the distribution function of normal fermion fluid in the phase space as a simple form

$$f_n(\mathbf{k}, \mathbf{r}) = \frac{1}{e^{\beta(\varepsilon_{\mathbf{k}} - \mu_F + \hat{V}(\mathbf{r}) + u_n)} + 1}.$$

(24)

We take the same approximation as in the mean-field theory of the one-channel microscopic model, and assume the N_s pairs have no fluctuations and are all frozen at the ground state with $\mathbf{q} = 0$. So only fermions in the normal fluid contribute to K. The kinetic energy is only from N_n fermions and the potential energy is from all N fermions:

$$K = 2 \int d\mathbf{r} \int \frac{d\mathbf{k}}{(2\pi)^3} \varepsilon_{\mathbf{k}} f_n(\mathbf{k}, \mathbf{r}), \quad V = \int \hat{V}(\mathbf{r}) \rho(\mathbf{r}) d\mathbf{r}.$$

(25)

The total energy is

$$E = K + V + U + N_s \epsilon_b.$$

(26)

Here the interacting energy is $U = \int d\mathbf{r} \hat{\mathcal{U}}$ and the binding energy is ϵ_b. From the chemical equilibrium condition, we have $\mu = \mu_F = \mu_B/2$. Since $\mu_B = u_s + \epsilon_b$, we have $\epsilon_b = 2\mu - u_s$. Because $\Delta_{\mathbf{r}} \to 0$ when $r \to \infty$, there is always $u_n = 0$. In this case, g_n/g_{ns} is determined by

$$\frac{g_n}{g_{ns}} = -\frac{\int d\mathbf{r} \rho_n \rho_s}{2 \int d\mathbf{r} \rho_n^2}.$$

(27)

g_s is determined by equaling Eq. (21) to Eq. (26). So far all parameters are well-defined and we can use them to calculate $\tau_{nn}, \tau_{ns}, (K, V, U)$ and further ξ_i and ζ_i for $i = (\perp, z)$ in the order phase at $T < T_c$. Since

two viscous relaxation time τ_{nn} and τ_{ns} are within the Boltzmann-Vlasov kinetic theory, our approach is combined with the two-fluid physical picture and the one-channel microscopic theory. This theory is within both the T-matrix fluctuation theory in the normal phase and within the mean-field theory in the order phase.

Remarkably, this mean-field theory in the one-channel microscopic model is the simplest theory to study the BCS-BEC crossover system at $T < T_c$. It catches the main characteristics of the system and includes the pairing and depairing effects. Strictly speaking, the mean-field theory only makes sense at the deep BCS side and BEC limit because it omits the pair fluctuations which play an important role near the unitary limit. However, we only use the Boltzmann-Vlasov kinetic theory to study the collective mode and damping rate in this work. The mean-field theory proves a qualitative correct picture at the BCS side where the phase transition locates and includes the pairing and depairing effects which is our main interest. So our theory makes sense for it combines the phenomenological Boltzmann-Vlasov kinetic theory with the mean-field theory at $T < T_c$ to study the collective mode and damping rate including the pairing and depairing effects.

Two viscous relaxation time of a trapped strongly interacting Fermi gas. We now calculate the viscous relaxation time. Take the trial function[68] $\Phi = \sum_{i,j}(k_i k_j - \delta_{ij} k^2/3)$, the denominator on the right side of Eq. (5) turns to

$$D \equiv \langle |\Phi|^2 f^0 (1 - f^0) \rangle = \frac{4m}{\beta\hbar^2} \int d\mathbf{r} \int \frac{d\mathbf{k}}{(2\pi)^3} k^2 f^0. \tag{28}$$

Here we have used the fact in the symmetry of the integral functions that $|\Phi|^2 \propto 12 k_x^2 k_y^2$. After the solid angle integral in the denominator, we have $\int d\Omega\, |\Phi|^2 \propto \frac{4}{5} k^4$.

Under the Galilean invariance of the collision process, it is convenient to use the center-of-mass wave vector $\mathbf{k}_0$ and the relative wave vector coordinates $\mathbf{k}_r$ and $\mathbf{k}_r'$ instead of the wave vectors of the incoming and outgoing particles $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$, $\mathbf{k}_4$:

$$\mathbf{k}_0 = \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4,\ \mathbf{k}_r = \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2),\ \mathbf{k}_r' = \frac{1}{2}(\mathbf{k}_3 - \mathbf{k}_4). \tag{29}$$

For atomic gases close to a Feshbach resonance under consideration in this work, a multichannel effective theory[20,73] for atom-atom scattering must be taken into account. Under the broad band approximation close to a resonance, the particle-particle scattering cross section is

$$\sigma(k_r, k_0, \mathbf{r}) = \frac{4\pi a_{sc}^2}{\left| 1 + \frac{4\pi\hbar^2 a_{sc}}{m} \prod (k_r, k_0, \mathbf{r}) \right|^2}, \tag{30}$$

where $\prod(k_r, k_0, \mathbf{r})$ is the pair propagator and it is expressed on shell version as

$$\prod (k_r, k_0, \mathbf{r}) = \int \frac{d\mathbf{q}}{(2\pi)^3} \left[\frac{1 - f_- - f_+}{\hbar^2 (k_r^2 - q^2)/m + i\delta} + \frac{1}{\hbar^2 q^2/m - i\delta} \right]. \tag{31}$$

Here $f_\mp$ is the equilibrium Fermi distribution function f^0 with momentum $\mathbf{p} = \hbar\left(\frac{1}{2}\mathbf{k}_0 \pm \mathbf{q}\right)$, and δ is an infinitesimal positive constant. In a vacuum, $\prod(k_r) = imk_r/4\pi\hbar^2$, and $\sigma(k_r) = \frac{4\pi a_{sc}^2}{1 + (k_r a_{sc})^2}$, as well known, is the unitarized vacuum scattering cross section.

At $T \geq T_c$, only τ_{nn} contributes to τ. This term is still important at $T < T_c$. From[67], the first numerator on the right side of Eq. (5) turns to

$$\Gamma_{nn} \equiv \frac{4\hbar^3}{(2\pi)^7 m^2} \int d\mathbf{r} \int d\mathbf{k}_1 \int d\mathbf{k}_2 \int d\mathbf{k}_3 \int d\mathbf{k}_4 \sigma(k_r, k_0, \mathbf{r}) \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$$

$$\times \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) f_1 f_2 (1 - f_3)(1 - f_4) \Phi_1(\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4). \tag{32}$$

Using the symmetry of the collision integral under interchange of incoming and outgoing particle momenta, we may write $\Phi_1(\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4) = \frac{1}{4} |\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4|^2$. In the means of the integrals of the azimuthal angles of the vectors $\mathbf{k}_r$ and $\mathbf{k}_r'$, $|\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4|^2$ becomes proper to $k_r^4 [(1 + 2y^2 - 3y^4) + (1 + 2y'^2 - 3y'^4)]$ and $y^4 + y'^4$ becomes $2y^2 y'^2$. Here $y = \cos\theta_r$ and $y' = \cos\theta_r'$ with θ_r and θ_r' being the polar angles of the vectors $\mathbf{k}_r$ and $\mathbf{k}_r'$. So that $|\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4|^2$ becomes proper to $1 + y^2 + y'^2 - 3y^2 y'^2$. The general dimensionless viscous fermion-fermion relaxation rate $1/\overline{\tau}_{nn} = \Gamma_{nn}/\omega_\perp D$ can be expressed as

$$\frac{1}{\overline{\tau}_{nn}} = \frac{\sqrt{2}}{5\pi}\left(\frac{k_B T a_{sc}}{\hbar\omega_\perp a_\perp}\right)^2 \frac{\int \overline{I}_c[z(\mathbf{r})]\,d\mathbf{r}}{\int d\mathbf{r}\overline{D}(\mathbf{r})},\tag{33}$$

where $\overline{D}(\mathbf{r}) = \frac{3\sqrt{\pi}}{8}f_{5/2}(z(\mathbf{r}))$, $z(\mathbf{r}) = e^{(\mu-\hat{V}(\mathbf{r}))/k_B T}$ is the fugacity in the scattering, and $f_\alpha(z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{x^{\alpha-1}dx}{z^{-1}e^x+1}$ is the Fermi function for $\alpha > 0$. $\overline{I}_c$ is the same as the one in[65,66]:

$$\overline{I}_c = \int_0^\infty dx_r \int_0^\infty dx_0 \int_0^1 dy \int_0^1 dy' x_0^2 x_r^7 \overline{\sigma}(x_r, x_0) F(x_r, x_0, y, y')(1 + y^2 + y'^2 - 3y^2 y'^2),\tag{34}$$

where $\overline{\sigma}(x_r, x_0) = \sigma(x_r, x_0)/(4\pi a_{sc}^2)$ and $x_{r,0} = \hbar k_{r,0}/(mk_B T)^{1/2}$ are all dimensionless. F is expressed as

$$F(x_r, x_0, y, y') = \frac{z^2 e^{-(x_0^2+x_r^2)}}{(1 + ze^{-x_1^2})(1 + ze^{-x_2^2})(1 + ze^{-x_3^2})(1 + ze^{-x_4^2})},\tag{35}$$

where the dimensionless variables $x_i = \hbar k_i/(2mk_B T)^{1/2}$ $(i = 1, 2, 3, 4)$ are the function of x_r, x_0, y and y':

$$x_1^2 = \frac{1}{2}(x_0^2 + 2x_0 x_r y + x_r^2),\; x_2^2 = \frac{1}{2}(x_0^2 - 2x_0 x_r y + x_r^2),$$
$$x_3^2 = \frac{1}{2}(x_0^2 + 2x_0 x_r y' + x_r^2),\; x_4^2 = \frac{1}{2}(x_0^2 - 2x_0 x_r y' + x_r^2).\tag{36}$$

There are at least two channels contribute to τ_{ns}. One is elastic with unbreakable pairs, the other is un-elastic with breakable pairs after scattering. The first channel only changes the momentum of fermions and fermion pairs, and it is important to τ_{ns} at $T \ll T_c$. It will complicates the calculations duo to the pair fluctuations. The second channel changes the momentum and energy of fermions as well as the fermion's particle numbers of the normal fluid or superfluid, and it contributes to τ_{ns} near T_c. In this work we only consider the second channel and write the second numerator on the right side of Eq. (5) to

$$\Gamma_{ns} = -\int d\mathbf{r}\int \frac{d\mathbf{k}}{(2\pi)^3}\Phi^* C_{ns}[\Phi, \rho_s]\Phi$$
$$= -\frac{8g_{ns}^2}{(2\pi)^8 \hbar k_F^3}\int d\mathbf{r}\rho_s(\mathbf{r})\int d\mathbf{k}_1 \int d\mathbf{k}_2 \int d\mathbf{k}_3 \int d\mathbf{k}_4 \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$$
$$\times\, \delta(\epsilon_b + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)(1 - f_1)f_2 f_3 f_4 \Phi_1(\Phi_2 + \Phi_3 + \Phi_4 - \Phi_1).\tag{37}$$

Here the incoming particles are a fermion pair with zero momentum and energy ϵ_b and a fermion with wave vector $\mathbf{k}_1$ and energy ε_1, while the outgoing particles are three fermions with wave vectors $\mathbf{k}_i$ and energies ε_i for $i = 2, 3, 4$. Since the kinetic energies do not conserve due to $\epsilon_b \neq 0$ in the scatting process, we can't easily choose a center-of-mass frame to simplify the calculations of delta function. The conversions of the momentum and energies in the scattering processes are expressed as

$$\mathbf{k}_0 \equiv \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4,\; \varepsilon_0 \equiv \epsilon_b + \varepsilon_1 = \varepsilon_2 + \varepsilon_3 + \varepsilon_4\tag{38}$$

In the multidimensional momentum integral of Eq. (37), according to the symmetry of the integral function, the factor $\Phi_1(\Phi_2 + \Phi_3 + \Phi_4 - \Phi_1)$ is proper to $\sum_{i=1}^4(k_{ix}^2 k_{iy}^2 + k_{iy}^2 k_{iz}^2 + k_{iz}^2 k_{ix}^2)$. We first perform the momentum integral $\int d\mathbf{k}_4 \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)h(\mathbf{k}_4) = h(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$ for a given function h in Eq. (37). We then choose the first wave vector elements $d\mathbf{k}_1 = k_{1z}^2 dk_{1z} \sin\theta_1 d\theta_1 d\phi_1$ in spherical polar coordinates $(k_{1z}, \theta_1, \phi_1)$ with the radial wave vector fixed along the z–axial k_{1z} in Cartesian coordinates; the second wave vector elements $d\mathbf{k}_2 = k_{2x}dk_{2x}d\phi_2 dk_{2z}$ in cylindrical coordinates with plane polar coordinates (k_{2x}, ϕ_2) and z–axial coordinate k_{1z}; and the combination $\delta(\epsilon_b + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)d\mathbf{k}_3 = \frac{2m}{\hbar^2 k_F}dk_{3x}\delta(k_{3y} - k_{3y}^\pm)dk_{3z}$ for the third wave vector elements in the Cartesian coordinates (k_{3x}, k_{3y}, k_{3z}), where $k_{3y}^\pm$ is determined by Eq. (38) of the conservation of the energies in the scattering processes. The denominator is the same as that in τ_{nn}. The general dimensionless viscous fermion-pair relaxation rate is $1/\overline{\tau}_{ns} = \Gamma_{ns}/\omega_\perp D$ and the numerator is written as

$$\Gamma_{ns} = -\frac{16mg_{ns}^2}{(2\pi)^7\hbar^2 k_F^4} \int_0^\infty \rho_s(\mathbf{r})\, r^2 dr \int_{K_{1r}}^\infty k_{1z}^2 dk_{1z} \int_0^{\pi/2} \sin\theta_1 d\theta_1 \int_0^{2\pi} d\phi_1$$

$$\times \int_0^{K_{2x}} k_{2x} dk_{2x} \int_0^{2\pi} d\phi_2 \int_{K_{2z}^-}^{K_{2z}^+} dk_{2z} \int_{K_{3x}^-}^{K_{3x}^+} dk_{3x} \int_{K_{3z}^-}^{K_{3z}^+} dk_{3z} F(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3), \tag{39}$$

where $F(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = (1 - f_1)f_2 f_3 f_4 \sum_{i=1}^4 (k_{ix}^2 k_{iy}^2 + k_{iy}^2 k_{iz}^2 + k_{iz}^2 k_{ix}^2)$ with $\mathbf{k}_4 = \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3$ in Eq. (38). Note that $\rho_s(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^3} \left[\frac{1}{2} f_{MF}(\mathbf{k}, \mathbf{r}) - f_n(\mathbf{k}, \mathbf{r})\right]$ combines the mean-field theory with the two-fluid mixture approach.

The integral limits of Eq. (39) are given below. Firstly, for large enough k_{1z} to both break the dimer and scatter them into three outgoing particles with wave vectors $(\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$, the minimum of the total energy ε_0 requires both $k_{2r} = k_{3r} = k_{4r} = k_{1r}/3$ and $\epsilon_0 \equiv -k_b^2 + k_{1r}^2 \geq 3(k_{1r}/3)^2$. Therefore the minimum k_{1r} is $K_{1r} = \sqrt{3}k_b/2$, else the scattering process is forbidden. In the same method, $\epsilon_1 \equiv \epsilon_0 - k_{2z}^2 \geq 2[(k_{1z} - k_{2z})/2]^2$ and the up and down integral limits of k_{2z} are $K_{2z}^\pm = \frac{1}{3}(k_{1z} \pm \sqrt{2k_{1z}^2 - 3k_b^2})$. From $\epsilon_1 - k_{3z}^2 \geq (k_{1z} - k_{2z} - k_{3z})^2$, we also have $K_{3z}^\pm = \frac{1}{2}(k_{1z} - k_{2z} \pm \sqrt{k_{1z}^2 - 3k_{2z}^2 - 2k_b^2 + 2k_{1z}k_{2z}})$. Secondly, since $\epsilon_2 \equiv \epsilon_1 - k_{3z}^2 - k_{4z}^2 \geq k_{2x}^2 + k_{3x}^2 + k_{4x}^2$ with the momentum conservation in the x–direction $(k_{4x} = -k_{2x} - k_{3x})$ and $\epsilon_3 \equiv \epsilon_2 - k_{2x}^2 \geq 2(-k_{2x}/2)^2$, we can obtain the maximum $K_{2x} = \sqrt{2\epsilon_2/3}$, else the scattering process is also forbidden. From $\epsilon_3 - k_{3x}^2 \geq (-k_{2x} - k_{3x})^2$, we have the up and down integral limits of k_{3x} as $K_{3x}^\pm = \frac{1}{2}(-k_{2x} \pm \sqrt{2\epsilon_3 - 3k_{2x}^2})$. And finally, from the energy conservation $\epsilon_4 \equiv \epsilon_3 - k_{3x}^2 - k_{4x}^2 = k_{3y}^2 + k_{4y}^2$ with the momentum conservation in the y–direction $(k_{3y} = -k_{4y})$, we get the fixed $k_{3y}^\pm = \pm\sqrt{\epsilon_4/2}$. Consequently, the integral function $F(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ and the integral limits are all given above and we can carry out the calculation of Γ_{ns} in Eq. (39).

Comparison with experiments.   We have first used the T-matrix fluctuation theory to calculate the critical temperature T_c as a function of the interaction strength. With the experimental parameters, we then have calculated the gap Δ, coupling constants g_s and g_{ns} and density of bosons $\rho_s(\mathbf{r})$ below T_c, and later the chemical potential μ, density of fermions $\rho(\mathbf{r})$, ratios $\zeta_\perp$, $\xi_\perp$, ζ_z, and ξ_z, index γ, and two viscous relaxation time τ_{nn} and τ_{ns}, and so on, around T_c between the BCS and unitary limits. Finally, we have solved Eq. (11) with different physical parameters (T, η, λ, N). The following is our numerical results and discussions on the properties of the system. Numerical calculations show in Fig. 1 for the temperature dependent (a) frequency Reω and (b) damping Imω for the quadrupole mode (upper plots) and compressions mode (lower plots) at the unitarity limit, respectively. The curves are given for $T \geq T_c = 0.3T_F$. The red solid lines are the numerical results with the full scattering matrix in Eq. (31). As a comparison, we also show the experimental points and theoretical plots from[10]. Our results are essentially consistent with the experimental values.

Figure 2 shows (a) frequency Reω and (b) damping Imω in the units of $\omega_\perp$ versus η at $T = 0.1T_F$ for the quadrupole mode. The points are experimental values from[1,14], while the red full lines represent our computing results. From Fig. 2, we can see that η around $\eta_0 \simeq -0.8$, the frequency exhibits a pronounced jump from the hydrodynamic $(\eta > \eta_0)$ to the collisionless $(\eta < \eta_0)$ frequency due to the maximum of $1/\tau_{ns}$ at $T \leq T_c$ [74]. This transition is accompanied by a pronounced maximum of the damping rate. This is a striking transition from hydrodynamic to collisionless behavior and it comes from the minimum of τ_{ns} at T_c. The present work provides strong evidence that quasistatic hydrodynamic theory does not apply to collective modes of a strongly interacting fermionic superfluid, when the oscillation frequencies approach the pairing gap. The sharp transition occurs at the maximum of viscous relaxation rate of normal fermion-superfluid pair collision at phase transition point. Above this point the pairing gap is breaking. Meanwhile, the strong damping is due to the fast varying of the frequency of collective modes along the BCS-unitarity crossover.

From the BCS limit to the left of the shaded region in Fig. 2, the system is at the normal phase and behaves as a collisionless Fermi gas at $T > T_c$; while from the unitary limit to the right of the shaded region, the system is at the order phase and behaves as superfluid fermion pairs at $T < T_c$. Inside the shaded region in Fig. 2 both above approaches for $\eta_i = 0$ and $\eta_i \neq 0$ are not applicable with $\eta_i = 1 - \zeta_i - \xi_i$ for $i = (\perp, z)$. In this region the system is neither normal phase nor superfluid phase, which is a critical region and the system will have complex behaviors.

Figure 3 shows (a) frequency Reω and (b) damping Imω versus η for the radial compressions mode (upper plots) and axial compression mode (lower plots). In the regime of a strongly interacting Fermi gas, an abrupt change in the collective excitation frequency occurs, we show that it is a signature of a transition from a superfluid to a collisionless phase. The measurements on the radial compression mode show three surprises[8]. The abrupt change of the excitation frequency and the large damping rate are not expected in a normal degenerate Fermi gas, where the collective excitation frequency is expected to vary smoothly from the hydrodynamic regime to the collisionless one. Furthermore, for the damping rate of
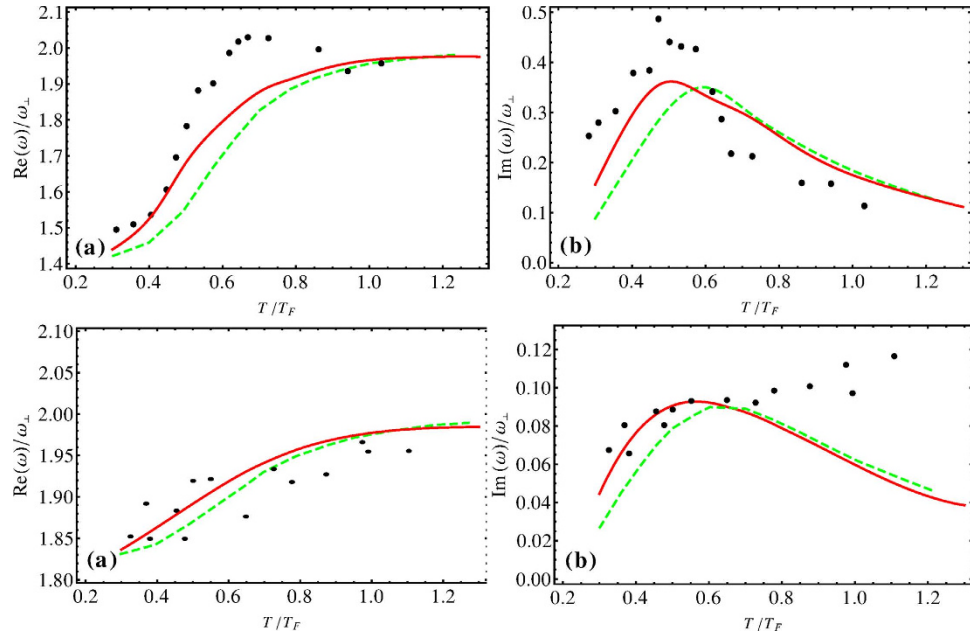
Figure 1. (**a**) Frequency Reω and (**b**) damping Imω in the units of $\omega_\perp$ versus the reduced temperature in the unit of Fermi temperature T_F for the quadrupole mode (upper plots) and compressions mode (lower plots) at the unitarity limit. The points are experimental values from[10], while the red solid lines represent our computing results with many-body effects. The green dashed lines are[10]'s results with both Pauli blocking and pairing effects. Ours red solid lines conform better to the experimental results with the parameters $\eta = -10^{-3}$, $N = 6 \times 10^5$, $\lambda = 32/1800$ (upper plots) and $\lambda = 26/1100$ (lower plots).
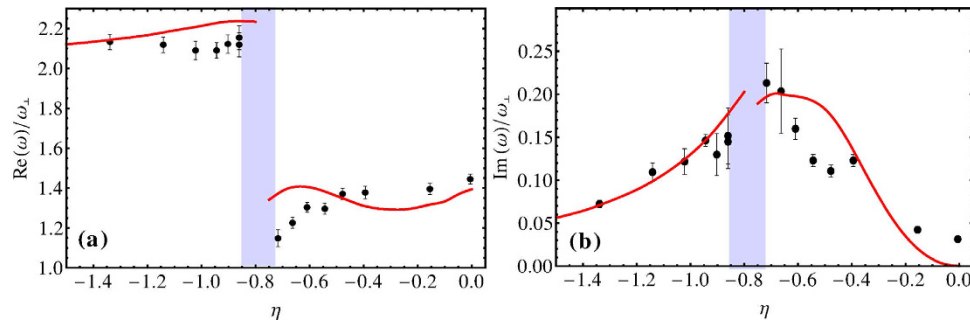
Figure 2. (**a**) Frequency Reω and (**b**) damping Imω in the units of $\omega_\perp$ versus the inverse reduced interaction strength η for the quadrupole mode near the phase transition. The points are experimental values from[1,14], the red solid lines represent our computing results with many-body effects. Ours red solid lines conform to the experimental results with the parameters $T = 0.1T_F$, $N = 4 \times 10^5$, $\lambda = 22/370$.

the radial compressions mode in the transition regime, a maximum value is Im$\omega/\omega_\perp \simeq 0.2$. The measured damping rate of Im$\omega/\omega_\perp \simeq 0.5$ is clearly inconsistent with our prediction for the normal hydrodynamic regime. Of course, for the experimental parameters of $T = 0.1T_F$, $N = 2 \times 10^5$, and $\lambda = 70/1500$ from[6], the transition occurs at $\eta_0 = 0.79$ with a smooth value of frequency Reω and a maximum damping rate of Im$\omega = 0.2$. Anyway, both the sudden change of the collective frequency and a strong damping may due to a transition from the superfluid phase to the normal phase and we need to study the superfluid phase in more details.

In summary, we have determined the transition and damping of collective modes in a trapped Fermi gas near the unitarity limit, based on the Boltzmann-Vlasov kinetic equation, combined with the many-body T-matrix fluctuation approximation in the normal phase and the many-body mean-field approximation in the order phase, and joined the two-fluid approach to connect the microscopic and phenomenal theories. We have calculated the dependence of temperature and scattering length on the frequency and damping of the collective modes, by using both theoretical and available experimental knowledge of the equation of state and two theoretical viscous relaxation time with the collision
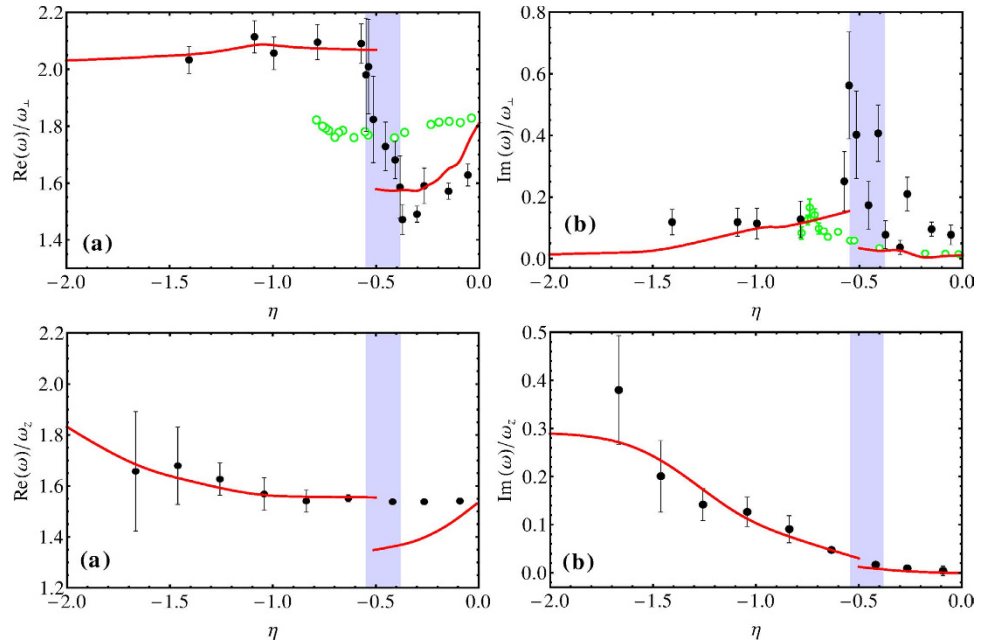
Figure 3. (**a**) Frequency Reω and (**b**) damping Imω in the units of $\omega_\perp$ (upper plots) and ω_z (lower plots) versus the inverse reduced interaction strength η for the radial compression mode (upper plots) and axial compression mode (lower plots) near the phase transition. The points are experimental values from[8], the red solid lines represent our computing results with many-body effects, with the parameters $T = 0.085 T_F$, $N = 4 \times 10^5$, the radial vibration frequency $\omega_\perp = 2\pi \times 750$ Hz, and the axial vibration frequency $\omega_z \simeq 2\pi \times [0.601(837 - 142\eta) + 11]^{1/2}$ Hz. Our red solid lines conform better to the experimental results[8]. The green circles in the upper plots are experimental values from[6], different both from the experimental values of[8] or our red solid lines, except near the unitarity limit.

probability of fermion-fermion scatting with and without fermion pairs, including a many-body scattering effect. Our results agree well with the experimental observations, not only qualitatively but also quantitatively, particularly for the sharp transition and strong damping of the collective modes in the BCS side near the phase transition when breaking the pairing gap. This theory provides a link between the microscopic and kinetic descriptions of many-body Feshbach scattering. Notes that we have not considered the superfluid dynamics below T_c, and omit the pair fluctuations which have the important effects near the Feshbach resonance. We may propose a new theory valid in the whole regime by adding these aspects further.

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Author Contributions

Y.L.M. conceived the idea and supervised the overall research. H.D. performed the numerical computations. All authors analyzed numerical results and contributed to the preparation of the manuscript.

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