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Network Robustness: Detecting Topological Quantum Phases

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SUBJECT AREAS:

PHASE TRANSITIONS
AND CRITICAL
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COMPLEX NETWORKS

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Can the topology of a network that consists of many particles interacting with each other change in complexity when a phase transition occurs? The answer to this question is particularly interesting to understand the nature of the phase transitions if the distinct phases do not break any symmetry, such as topological phase transitions. Here we present a novel theoretical framework established by complex network analysis for demonstrating that across a transition point of the topological superconductors, the network space experiences a homogeneous-heterogeneous transition invisible in real space. This transition is nothing but related to the robustness of a network to random failures. We suggest that the idea of the network robustness can be applied to characterizing various phase transitions whether or not the symmetry is broken.

Phases of matter can be distinguished by using Landau's approach, which characterizes phases in terms of underlying symmetries that are spontaneously broken. The information we need to understand phase transitions is usually encoded in appropriate correlation functions, e.g. the correlation length would diverge close to a quantum critical point. Particularly, the low-lying excitations and the long-distance behavior of the correlations near the critical phase are believed to be well described by quantum field theory. A major problem is, however, that in some cases it is unclear how to extract important information from the correlation functions if these phases do not break any symmetries, such as topological phase transitions^{1–4}.

Complex network theory has become one of the most powerful frameworks for understanding network structures of many real-world systems^{5–9}. According to graph theory, the elements of a system often are called nodes and the relationships between them, which a weight is associated with, are called links. Decades ago, this unnoticed idea constructing a weighted network from condensed matters had been proposed in quantum Hall systems¹⁰. Recently, there is another proposal in which complex network has been applied to examining quantum phase transitions^{11,12}. In the language of network analysis, therefore, each network of N nodes can be described by the $N \times N$ adjacency matrix $\hat{A}$. In what follows, we consider lattice sites as the nodes of the weighted network of which each weighted link between nodes i and j is expressed by the element of the adjacency matrix $\hat{A}_{ij}$. In Fig. 1(a), we start from a square-lattice example of size $N = 9$ in which the nodes form a simple regular network. Following the procedure [Fig. 1(b) and (c)], we then reconnect them by using some correlation functions as weights of network links to generate a complete network with the link-weight distribution (see Fig. 1(d)). The links now carry the weights containing information about important relationship between particles in many-body systems.

In this report, focusing on the topological superconductors in one (1D) and two (2D) dimensions^{13,14}, we explore the possibilities of detecting the topological phase transitions by using the novel network analysis. Our analysis reveals that (i) a homogeneous-heterogeneous transition occurs in network space from a topologically trivial phase to a topologically non-trivial phase, which is accompanied by a hidden symmetry breaking (namely, a reduction of the network robustness), and (ii) the complex many-body network analysis can be applied to other phase transitions without a prior knowledge of the system's symmetry.

Results

1D p-wave superconductor. The first model we consider was introduced by Kitaev¹³. The Hamiltonian for L spinless fermions in a chain with periodic boundary conditions is

$$H_{1D} = - \sum_i \left(\hat{c}_i^\dagger \hat{c}_{i+1} + \hat{c}_i^\dagger \hat{c}_{i+1}^\dagger + H.c. \right) + \mu \hat{c}_i^\dagger \hat{c}_i, \quad (1)$$

where μ is chemical potential. The simplest superconducting (SC) model system shows the two-fold ground-state degeneracy stemming from an unpaired Majorana fermion at the end of the chain with open boundary

Figure 1 | An example of the construction of complex many-body networks: (a) The system is a 3×3 square lattice with periodic boundary conditions; (b) The system is replotted based on the same topology as (a); (c) All links are removed from (b); (d) All nodes are reconnected by using some correlations between particles. The thickness of links denotes the magnitude of the correlation.

conditions. This model has two phases sharing the same physical symmetries: a topologically trivial (strong pairing) phase for $\mu > \mu_c (=2)$ and a topologically non-trivial (weak pairing) phase for $\mu < \mu_c$. The transition between them is the topological phase transition identified by the presence or absence of unpaired Majorana fermions localized at each end.

The Hamiltonian in momentum space is quadratic of fermionic operators $\hat{c}_k$, given by

$$\sum_{\mathbf{k}} \begin{pmatrix} \hat{c}_{\mathbf{k}}^\dagger & \hat{c}_{-\mathbf{k}} \end{pmatrix} \begin{pmatrix} \epsilon_{\mathbf{k}} & -i \sin \mathbf{k} \\ i \sin \mathbf{k} & -\epsilon_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} \hat{c}_{\mathbf{k}} \\ \hat{c}_{-\mathbf{k}}^\dagger \end{pmatrix}, \quad (2)$$

where $\epsilon_{\mathbf{k}} = -\frac{\mu}{2} - \cos \mathbf{k}$. By using the standard Bogoliubov transformation, $\gamma_{\mathbf{k}} = \cos(\theta_{\mathbf{k}}/2) \hat{c}_{\mathbf{k}} - i \sin(\theta_{\mathbf{k}}/2) \hat{c}_{-\mathbf{k}}^\dagger$ where $\theta_{\mathbf{k}} = \sin \mathbf{k} / \epsilon_{\mathbf{k}}$, the Hamiltonian can be diagonalized. The excitation spectrum of the form, $E_{\mathbf{k}} = \sqrt{(2\epsilon_{\mathbf{k}})^2 + \sin^2 \mathbf{k}}$, remains fully gapped except at the critical point μ_c . The SC ground state is the state annihilated by all $\gamma_{\mathbf{k}}$:

$$|\Psi_{GS}\rangle = e^{\frac{1}{2} \sum_{ij} G_{ij} \hat{c}_i^\dagger \hat{c}_j^\dagger} |0\rangle, \quad (3)$$

where G_{ij} represents the pairing amplitude given by ref. 15

$$G_{ij} = \frac{1}{L} \sum_{\mathbf{k}} \tan(\theta_{\mathbf{k}}/2) e^{i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)}. \quad (4)$$

A possible choice of the adjacency matrix of the 1D superconductor is the normalized pairing amplitude in which the non-local property between spinless fermions is concealed. The adjacency matrix can serve as an intuitive definition for the network of spinless fermions with p -wave Cooper pairing:

$$\hat{A}_{ij} = \frac{|G_{ij}|}{\max G_{ij}}. \quad (5)$$

The node i or j stands for a given lattice site. The weights of links contain information about the pairing strength between spinless fermions.

In Fig. 2(a), one can see the change of network topologies for different μ in a short chain of $L = 10$. Below μ_c , the topologically non-trivial phase displays irregular patterns of the complete network, where each node is connected to all other nodes with different link weights. As increasing μ above μ_c , the topologically trivial phase demonstrates a ring structure comprised of the nodes with the largest link weight that is called “highways” of the network. The obvious change of topologies of the network across the critical point is intimately related to the critical behavior observed in real space.

We now recall the pairing amplitude in real space shown in Eq.(4). Consider translational invariance, Figure 2(b) shows how the normalized pairing amplitude $A_r \equiv |r_i - r_j| (= \hat{A}_{ij})$ changes as the topological phase transition occurs. For $\mu < \mu_c$, the weak pairing phase indicates that the size of the Cooper pair is infinite, leading to $A_r \sim \text{const}$. At the critical point, the critical phase has power-law correlations at large distances. Above the critical point, i.e. $\mu > \mu_c$, the strong pairing phase instead shows that the pairing amplitude is exponentially decaying with distances: $A_r \sim e^{-r/\xi}$. The Cooper pairs form molecules from two fermions bound in real space over a length scale ξ . The exponentially decaying pairing amplitude in real space results in a ring structure in network space.

To further analyze the network structure, we examine how the probability distribution $p(w)$ of the weights w of network links evolves over contiguous topological phases. In Fig. 2(c), the weights distribute like a delta function for $\mu < \mu_c$. Namely, the weights homogeneously distribute in network space. As further increasing μ , the distributions begin to lose weight but still remain nearly homogeneous. It is noteworthy that the distribution at the critical point possesses a decaying function with a heavy tail. Hence the weight

Figure 2 | (a) The evolution of network topologies of the 1D p -wave superconductor from chemical potential $\mu = 0.2$ to 4.2. The chain length $L = 10$. The thickness and color of links represent their weights. Color scale: Blue (Red) indicates the largest (smallest) weights. (b) Pairing amplitude A_r as a function of distance r for different μ . (c) The probability distribution $p(w)$ of the weights w of network links for different μ . The bin size is chosen for clear demonstrations. The chain length $L = 1000$.

distribution of network links becomes more heterogeneous. In the strong pairing regime ($\mu > \mu_c$), the distribution moves to almost zero weight and recovers a sharp peak at $w \sim 0$. The tail of the distribution in the strong pairing regime thus looks much more heterogeneous than the weak pairing regime.

This observation reminds us of a well-known fact in real-world networks that a network with the heterogeneous weight distribution of links is robust to random failures^{16,17}. The robustness of the network originating from its heterogeneity seems to indicate a hidden symmetry in network space. More precisely, the hidden symmetry describes a phenomenon that the network function and structure remain unchanged or invariant under random removal of its links. Thus, there exists a homogeneous-heterogeneous network transition hidden in the topological phase transition. It may allow us to define a topological order parameter in network space for identifying the phase transition without any local order parameter.

2D $p + ip$ superconductor. Consider now a 2D time-reversal symmetry breaking superconductor, the $p + ip$ superconductor for N spinless fermions:

$$H_{2D} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \hat{c}_{\mathbf{k}}^\dagger \hat{c}_{\mathbf{k}} + \left(\Delta_{\mathbf{k}} \hat{c}_{\mathbf{k}}^\dagger \hat{c}_{-\mathbf{k}}^\dagger + H.c. \right), \quad (6)$$

where the single-particle dispersion $\epsilon_{\mathbf{k}} = -2(\cos k_x + \cos k_y) - \mu$ and the gap function $\Delta_{\mathbf{k}} = \sin k_x + i \sin k_y$. For the spinless fermions, the gap function has odd parity symmetry, $\Delta_{-\mathbf{k}} = -\Delta_{\mathbf{k}}$. One can see that the excitation spectrum has gapless nodes at time-reversal invariant momenta: $(0, 0)$, $(0, \pi)$, $(\pi, 0)$, (π, π) .

Following the same analysis as the 1D example, the SC ground-state wave function has the same form as Eq.(3). However, the pairing amplitude G_{ij} is now given by

$$G_{ij} = \frac{1}{N} \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} e^{i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)}, \quad (7)$$

where $v_{\mathbf{k}}$ and $u_{\mathbf{k}}$ are BCS coherence factors (More details can be found in Ref. 14). Note that the system preserves the particle-hole

symmetry so that only $\mu \geq 0$ will be considered later. This model also has two topological distinct phases: a topologically trivial phase for $\mu > \mu_c (=4)$ and a topologically non-trivial phase for $\mu < \mu_c$. Other than the 1D case, however, there is the other transition point at $\mu'_c = 0$ due to the bulk gap closure at $(\pi, 0)$ and $(0, \pi)$.

We now investigate how the network analysis performs in the face of the 2D topological phase transition. The definition for the adjacency matrix $\hat{A}_{ij}$ in the 2D superconductor is still the same as Eq.(5). The complex topologies of the weighted network for the topological trivial and non-trivial phases are shown in Fig. 3(a) in a square lattice of $N = 16$. The topologically non-trivial phase ($\mu < \mu_c$) gives rise to a weighted complete network with the link-weight distribution. It would be just a trivial complete network if the network were unweighted. For $\mu > \mu_c$, each node only has four highways to its neighbors, hence the network topology is equivalent to a torus which corresponds to a square lattice with periodic boundary conditions.

As in the 1D example the difference between the strong pairing phase and the weak pairing phase can be distinguished by examining the pairing amplitude. In Figure 3(b), with $\mu < \mu_c$ a weakly paired condensate forms from Cooper pairs loosely bound in real space, which gives $A_r \sim r^{-1}$ at large distance. As for $\mu > \mu_c$, the pairing amplitude falls exponentially for large r , $A_r \sim e^{-r/\xi}$, because the pairs of the strong pairing phase are tightly bound in real space. Thus, the exponentially decaying pairing amplitude gives rise to the torus structure in network space.

Let us turn to discussing the probability distribution of the weights of network links. For $\mu < \mu_c$, Fig. 3(c) shows that the distribution still shows a bell-like function due to powerlaw decaying pairing amplitude. Most links centering around a moderate weight behave homogeneous in network space. Similar to the 1D case, the heterogeneity of the distribution appears as further approaching μ_c . For $\mu > \mu_c$, a majority of links lose their weights, hence, the distribution becomes much more heterogeneous and exhibits a long tail. The 2D superconductor shows much broader weight distribution of network links than the 1D case. Even so, a homogeneous-heterogeneous transition is still observed in the network space.

Figure 3 | (a) The evolution of network topologies of the 2D $p + ip$ superconductor from chemical potential $\mu = 2.5$ to 6. The lattice size $N = 16$. (b) Pairing amplitude A_r as a function of distance r for different μ . (c) The probability distribution $p(w)$ of the weights w of network links for different μ . The lattice size $N = 1600$.

Network measures. So far we have not introduced the topological order parameter for the topological superconductors in 1D and 2D yet. We turn our attention to two network measures that could be used in these topological quantum systems. One is the so-called small-world phenomena¹⁸. Many of real-world networks have the property of relatively short average path length defined by a shortest route running along the links of a network. A small-world network including not only strong clustering but also short path length has also been introduced to describe real-world networks¹⁹.

Instead of the weighted clustering coefficient C^{20} and the average path length D^{21} commonly used in network analysis (see more details in Methods), a measure of the small-world property called “small-worldness” has been recently proposed^{22,23}. It is defined as

$$S \equiv \frac{C}{D}, \quad (8)$$

which is based on the maximal tradeoff between high clustering and short path length. A network with larger S has a higher small-world level. The small-worldness seems to be appropriate to describing the universal critical properties because it can extract information about both locality (weighted clustering coefficient) and non-locality (average path length) from network space.

In Fig. 4(a), we illustrate the critical behavior of the small-worldness in the 1D p -wave superconductor. One can see that the small-worldness drops to zero when the 1D superconductor comes from the topologically non-trivial phase to the topologically trivial phase. This coincidence convinces us that the network topology enables the small-worldness, akin to an order parameter in the theory of conventional phase transitions, to expose the change of nontrivial topology inherent in the weak pairing regime²³. For the 2D $p + ip$ superconductor, however, the small-worldness displays the notorious finite size effect for the topologically non-trivial phase as a result of its much broader weight distribution. In order to overcome this hassle, we define the normalized small-worldness as $S^* = S/\max S(\mu)$. In Fig. 4(b), near the critical point μ_c the normalized small-worldness vanishes in the strong pairing regime as well. Surprisingly, the other

critical point μ'_c seems to be also characterized by a decline in the normalized small-worldness.

There is an alternative way to understand the disappearance of the small-worldness in the strong pairing regime. As approaching the strong pairing phase, most links start to lose weights and the weight distribution shows more heterogeneous, thus leading to the reduction of the small-worldness. The phenomenon that most links are like slow traffic lanes results in vanishing small-worldness. In other words, the regular network of the strong pairing phase has a very small weighted clustering coefficient and much longer average path length as the system size goes to infinity. The same reasoning from the weight distribution of network links can be also applied to other many-body systems with/without local order parameters. Hence, this result strongly suggests that the small-worldness can be considered as a topological order parameter in the network space.

We have to mention a point now in passing. In the network representation, there exists a hidden symmetry corresponding to a heterogeneous network with a long-tail weight distribution. The homogeneous-heterogeneous transition of the network topology observed is intimately related to the hidden symmetry breaking. The hidden symmetry is nothing but the robustness of a complex network. The heterogeneity of network links implies that a weighted network is more robust against random failures¹⁷, accompanied by higher hidden symmetry in network space. Conversely, the homogeneity of network links means that a weighted network becomes fragile to random failures, and thus breaks the hidden symmetry. The other network measure we have to take is just to quantify the hidden symmetry.

The network structure and function strongly rely on its structural robustness, i.e. the ability of a network to maintain the connectivity when a fraction of nodes or links are randomly removed. A variety of network measures have been proposed to detect the structural robustness^{24–27}. Recently the concept of natural connectivity derived from the graph spectrum has been introduced to measure the structural robustness^{28–30} (also see the details in Methods). We can further extend the concept of natural connectivity to the weighted network.

Figure 4 | (a) Small-worldness S and (c) network robustness R of the 1D p -wave superconductor vs chemical potential μ for different chain length L . (b) Normalized small-worldness S^* and (d) network robustness R of the 2D $p + ip$ superconductor for different lattice size N as a function of μ . The blue lines indicate the critical points.

The natural connectivity in a weighted network represent the “strength” of loops of all lengths instead of the number of loops. Thus we call the natural connectivity as the network robustness R , that can be given by

$$R = \ln \left(\frac{1}{n} \sum_{i=1}^n e^{\bar{\lambda}_i} \right), \quad (9)$$

where $\bar{\lambda}_i (\equiv \lambda_i / \max \lambda_i)$ stands for the normalized eigenvalue of the adjacency matrix $\hat{A}$ to avoid the enhancement of the network robustness as increasing the number of nodes $n (=L$ in 1D and N in 2D).

In Fig. 4(c) and (d), we analyze the homogeneous-heterogeneous transition of network topology by plotting the network robustness R for the topological superconductors. As we expect, in both 1D and 2D cases the strong pairing phase always exhibits more robust network structure than the weak pairing phase, owing to its more heterogeneous weight distribution. For the homogeneous-heterogeneous transition in network space, the hidden symmetry breaking at the critical point indicates that the network loses its robustness ($R = 0$, namely, the hidden symmetry is broken), further leading to an appearing order parameter: the small-worldness ($S \neq 0$). This is a clear picture that Landau’s symmetry breaking theory works well even in network space. More interestingly, the symmetry-breaking idea is successfully applied to identifying the topological phase transitions in the topological superconductors. It is worthy to be mentioned that the concept behind the hidden symmetry breaking can also provides significant information to comprehend traditional phase transitions with Landau’s symmetry breaking in condensed matter systems.

Discussion

By using complex network analysis we have addressed how to read useful information from the pairing amplitude to characterize the topological phases in 1D and 2D topological superconductors. We have illustrated that a network measure, small-worldness, plays a significant role as a topological order parameter in network space, relied on Landau symmetry-breaking picture. The evolution of the weight distribution of network links across the critical point is responsible for the change of the small-worldness, which is

analogous to the change of the speed limit on a road network from the highway to the slow traffic lane. The phenomenon that the structure of the weighted network varies from heterogeneity to homogeneity implies a hidden symmetry broken—or, to put it another way, the disappearance of the network robustness to random failures. The hidden symmetry breaking has been successfully described by another network measure, network robustness. The robustness of a complex network is able to uncover a wealth of topological information underneath the pairing amplitude, and further comprehend the mechanism of the phase transitions without local order parameters. We thus suggest that complex network analysis can be a valuable tool to investigate quantum or classical phase transitions in condensed matters.

Relied on the success of complex network analysis in this report, other topological phase transitions beyond the change of BCS pairing structure should be carefully examined. For example, we can define the reduced density matrix of interacting topological systems as a weighted adjacency matrix and further construct the corresponding network space. We can also analyze the spectrum of the graph and the strength of loops of length n in the network representation, which is closely related to the ground-state entanglement spectrum³⁴ and the Rényi entropy of order n , respectively. A very interesting direction that we leave for the future is detecting other topologically ordered phases with the same topological entanglement entropy in microscopic Hamiltonians, such as the toric code and the double semion model^{35,36}.

Methods

Weighted clustering coefficient. In graph theory, the clustering coefficient of an unweighted network captures the degree to which the neighbors of a given node link to each other²¹. More precisely, the word “clustering” means that a triangular cluster will be formed if the two neighbors of a given node connect to each other. The clustering coefficient thus refers to the local property of a network. Instead, the degree of clustering of a weighted network can be described by the weighted clustering coefficient²⁰,

$$C = \frac{1}{N} \sum_i \frac{[\hat{A}^{(1/3)}]_{ii}^3}{k_i(k_i-1)}, \quad (10)$$

where $\hat{A}^{(1/n)}$ is a matrix obtained from $\hat{A}$ by taking the n -th root of its individual elements and N the number of nodes. k_i is the number of neighbors of a given node i .

Notice that the unweighted and weighted clustering coefficients are equivalent as the weights become binary. Therefore, one can regard Eq.(10) as the probability with two neighbors of a randomly selected node linking to each other in a weighted network.

Average path length. We now turn to another fundamental concept in graph theory which is the shortest path between two nodes. Physical distance in a network is usually irrelevant, and can be replaced by the path length. A path is a route that runs along the links of a network. The path length is thus defined by the inverse of the link weights the path contains. This definition seizes the intuitive idea that strongly coupled nodes are close to each other. It is noteworthy, then, that the path length can represent the non-local property of a network. Hence the path length is the average of the shortest path lengths in a weighted network³¹, defined as

$$D = \frac{1}{C^N} \sum_{i < j} \hat{d}_{ij}, \quad (11)$$

where $\hat{d}_{ij}$ is the sum of $(\hat{A}_{\mu\nu})^{-1}$ along the shortest weighted path including nodes μ and ν between nodes i and j . Here we compute the matrix $\hat{d}_{ij}$ by using the well-known graph search algorithm called Dijkstra's algorithm³².

Natural connectivity. Natural connectivity has been recently proposed to represent the robustness of complex networks²⁸. The concept of natural connectivity is based on the Estrada index of a graph used to characterize subgraph centrality³³. In a simple graph without multilinks or self-loop, when one route fails, the network robustness guarantees that the two nodes can still communicate through other alternative routes. The more alternative routes between two nodes a network contains, the more robust the connectedness between them becomes. On the other hand, it can be expected that the number of loops of all lengths in a network is closely related to the redundancy of alternative routes. Therefore, the natural connectivity is able to quantify network robustness in terms of the number of loops of all lengths.

Consider shorter loops have more influence on the redundancy of alternative paths than longer loops, we define a weighted sum of loops of all lengths,

$$W = \sum_{l=1}^{\infty} \frac{N_l}{l!} \quad (12)$$

where $N_l = \text{Tr}(\hat{A}^l) = \sum_{i=1}^N \lambda_i^l$ is the number of loops of length l . The set $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ denote the eigenvalues of the adjacency matrix $\hat{A}$, which is called the graph spectrum²¹. With a little algebra, we obtain

$$W = \sum_{i=1}^N e^{\lambda_i}. \quad (13)$$

Hence the weighted sum of loops of all lengths can be easily derived from the graph spectrum. In order to avoid the size effect of the weighted sum, the natural connectivity λ is further defined as follows:

$$\lambda = \ln\left(\frac{W}{N}\right). \quad (14)$$

The natural connectivity can be also interpreted as the Helmholtz free energy of a network³⁰. Consequently, it is reasonable to measure the natural connectivity as the robustness of network topology as they have proposed²⁸.

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