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Reply to 'Entanglement growth in diffusive systems with large spin'

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REPLYING TO T. Rakovszky et al. *Communications Physics* https://doi.org/10.1038/s42005-021-00594-4 (2021)

The question discussed[1,2] is about the growth of the second Rényi entropy S_2 at long times in systems with a diffusive degree of freedom (DOF). One in particular wants to distinguish between the diffusive $S_2 \sim \sqrt{t}$, found for any diffusive system with the local Hilbert space dimension $q = 2$ and spatial dimension[3] $d = 1$, and ballistic $S_2 \sim t$ growth that is generic[2] for $q > 2$. Contested[1] is the case of single diffusive DOF like charge, density, or spin in systems with $q > 2$.

The correct thermodynamic limit (TDL) is to first let the system size to infinity, $L \to \infty$, only then $t \to \infty$. In order to be able to unanonimously distinguish different asymptotic powers in $S_2 \sim t^{\alpha}$ one must have an infinite range of possible values that S_2 can take (at least in principle; in practice one often has to deal with behavior at finite times, however, in order to be able to claim a given power the time-window should be reasonably large, e.g., span more than say an order of magnitude), which necessarily implies that the size of the subsystem A must be infinite. Therefore, (a) we study situations where the size of the subsystem A diverges in the TDL. We will also use guiding principles of Science[4]: (b) description of the physical world, (c) unbiased observations (present all evidence, not just the one in favor of a chosen narrative), (d) experimentation (verifiability).

Rakovszky et al.[1] incorrectly say that Žnidarič[2] claims the diffusive growth of S_2 appears only in $d = 1$ and $q = 2$). Žnidarič[2] presents arguments and shows a compelling numerical evidence that for $q > 2$ one will in general observe ballistic growth of S_2, however, it does not exclude diffusive growth for $q > 2$ and $d = 1$. It presents a number of examples ($q = 3$ and $q = 4$) that in fact do display diffusive growth. While presenting an additional diffusive system[1] is useful in delineating cases with $q > 2$ where diffusive growth nevertheless does occur, it does not invalidate the main message of Žnidarič[2].

The main point of Rakovszky et al.[1] seems to be (mentioned, e.g., in the first paragraph) that there are "more" diffusive cases in $q > 2$. We all agree that there are diffusive cases, but to be able to say which are more one has to count, and Rakovszky et al.[1] do not specify how they count Hamiltonians. I will instead focus on well-defined questions.

That being said, I do use a word "generic", so let me explain what it means. It means generic in the sense of point (b)—describing nature. Elementary particles all have small internal dimension and so lattice models with large q will be typically obtained by a direct product of such elementary DOF. Large q will be a consequence of having multiple (interacting) DOF at a single lattice site (e.g., spin, charge, multiple fermion species, bosons,...), each of which can or can not be diffusive. For instance, a canonical example is the Hubbard model with $q = 4$ due to spin-up and spin-down fermions. If in such models only a single 2-level DOF is conserved and diffusive, and there are no additional constrains, one will observe $S_2 \sim t$—and that is the main message of Žnidarič[2] (which is not in conflict with refs. [3,5]). Simply put, a single diffusive DOF is in itself a weak constraint in a large Hilbert space. Rakovszky et al.[1] on the other hand focuses only on large-spin models. They of course do exist and are important, however, viewing large q as solely due to a large spin is less generic. If spin models do show diffusive growth that would be interesting. So far no hard evidence has been presented in support of that. Numerical results[2], albeit on small systems, are compatible with ballistic growth of S_2 also in spin $S = 1$ systems.

Žnidarič[2] presents Floquet Hubbard-like models that do conform with the above. They are dismissed[1] as having a "particular symmetry". Let me give another spin-1/2 ladder example ($q = 4$; one can view the two legs as representing spin-up and spin-down fermions) that has hopefully less "symmetry". The Floquet dynamics uses two gates, one is a density correlated hopping on the lower leg $U_{\mathrm{zzXX}} = \exp(-\mathrm{i}\frac{\pi}{4}\sigma_1^z\sigma_2^z(\tau_1^x\tau_2^x + \tau_1^y\tau_2^y))$ (or an analogous $\tilde{U}_{\mathrm{zzXX}}$ with the correlated hopping on the upper leg[2]), the other a non-conserving gate U_{G} on the lower leg[2]. At each step (there are $2(L-1)$ per unit of time; L is the number of rungs), we apply a unitary on a randomly selected plaquete, and compare two models: (i) apply either U_{zzXX} or $\tilde{U}_{\mathrm{zzXX}}$, (ii) apply either U_{zzXX} or $\tilde{U}_{\mathrm{zzXX}} U_{\mathrm{G}}$. The model (i) conserves the total spin on the upper and the lower leg (two diffusive DOF), while (ii) conserves spin only on the upper leg (one diffusive DOF). In Fig. 1 we see that, in line with Žnidarič[2], the model (ii) displays ballistic $S_2 \sim t$ rather than diffusive growth. In fact, even the model (i) can display ballistic growth if the bipartite cut is parallel to the direction of diffusive spreading!

Rakovszky et al.[1] shows numerical data for a particular random circuit, suggesting $S_2 \sim \sqrt{t}$ for $q = 3$. The only evidence is the numerically calculated $\mathrm{d}S_2/\mathrm{d}t$, which is compatible with $\sim 1/\sqrt{t}$ in a window $\mathrm{d}S_2/\mathrm{d}t \in [0.3, 0.4]$ (or $t \in [25, 50]$). Considering a fitting of a power-law in a tiny window I would not quite call that a "direct refutation"[1], however, let us nevertheless assume $S_2 \sim \sqrt{t}$ is correct.

Such observation could possibly be explained by special properties of random circuits. In particular, it has been known for some time that random circuits with 2-qubit gates and the full 1-

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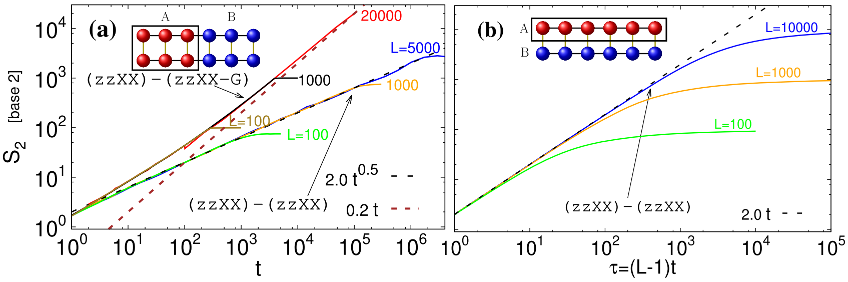

Fig. 1 Ballistic growth of S_2 in systems with single diffusive degree of freedom (DOF). a Floquet ladder model with a single diffusive DOF (zzXX-(zzXX-G)) vs. diffusive growth for two diffusive DOF (zzXX-zzXX). **b** Even if spin on both legs is diffusive one can get the ballistic growth for the shown bipartition to A and B. Blue and red balls sketch spins in ladders and subsystems A and B. The initial state is $(|0\rangle + |1\rangle)^{\otimes 2L}$ and different colored curves denote different system sizes L.

site invariance (Haar random single-qubit gates) can be mapped to Markovian chains on a reduced space[6,7]—due to the invariance the operator space for the average dynamics has dimension 2 instead of the full q^2—on top of that, this average dynamics can be for $q = 2$ and either random or specific fixed 2-qubit gates described by integrable spin chains[6]. The average dynamics of such random circuits is, therefore, doubly special—it lives on a reduced space on which it is described by an integrable model. Similar dimensionality reduction could be at play also in random circuits with conserved DOFs[3]. If true, this could explain diffusive growth of S_2; even-though one seemingly has $q = 3$, the average dynamic is effectively that of a two-level model.

Rakovszky et al.[1] argue that $S_2 \sim \sqrt{t}$ is expected in diffusive systems due to Huang[5], which shows that a so-called "frozen regions" (FR)[1] present a bottleneck to evolution also in $q > 2$, eventually resulting in diffusive growth of S_2. In line with point (c), one needs to mention that the proof works only under rather specific conditions. The main ingredient (Condition 1, called "diffusion" in Huang[5]) is, schematically, a property that if one starts with a state that contains a FR $|0\ldots 0\rangle$, like $|\psi(0)\rangle = |\phi_1\rangle|00\ldots 0\rangle|\phi_2\rangle$, the FR must remain frozen for any $\phi_{1,2}$ up-to times of order $t < m^2/x_0$, where m is the length of the FR and x_0 a state-independent constant. It is a short-time property upto which dynamics stays factorized (nontrivial dynamics happens on longer times; to scale t one has to increase m). While the FR condition might hold in some diffusive systems, it is, importantly, different and not equal to diffusion. Examples of system with diffusive DOF that violate it are abundant, e.g., all examples in Žnidarič[2] and here. Standard diffusion has been demonstrated in many systems, the FR property on the other hand has not been shown for any with $q > 2$. One does not expect that a single 2-level diffusive DOF will in general cause the FR effect, i.e., blocking the whole large Hilbert space.

Žnidarič[2] never claimed that his results are inconsistent with Rakovszky et al.[3], as Rakovszky et al.[1] may suggest. They[1] also "complain" about the chosen units of times, repeating what has been mentioned in Žnidarič[2] (caption in Table I and on p. 2). As explained[2], the chosen units of time have no influence on any of the conclusions and are such as to make finite-size analysis—a must for any serious claims about the asymptotic behavior (point (a))—easier. Namely, with chosen units the curves for S_2 and different L overlap, facilitating a read-out of the asymptotic power-law exponent. If any other units would be chosen one would have to each time rescale plots of S_2 for different L's in order to have an overlapping curves, making analysis cumbersome. While time units of course do influence the crossover time from ballistic to diffusive growth in $d > 1$, it has no influence on the value of S_2 at which this happens. The result stressed[2] is that for $d > 1$ and in the TDL one will always observe the linear growth of S_2 at any finite value of S_2. The diffusive growth of S_2 is in the TDL pushed to infinitely large values of S_2 (even in finite systems diffusive growth is pushed to large $S_2 \approx 10^4$, e.g. Fig. 2c in ref. [2]).

Everything that the authors of Rakovszky et al.[1] say in their last paragraph about purity e^{S_2} is correct, however, they have crucially omitted that Žnidarič[2] presents the explanation as "A non-rigorous intuitive...", i.e., meant for non-specialists not familiar with the concept. While the statement is not true for special states, it is correct for most states from the Hilbert space (e.g., random states according to the Haar measure all have $S_r = \ln(N_A) - c(r)$, where $c(r)$ is independent of the Hilbert space size N_A).

Data availability

Data is available upon reasonable request from the author.

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Competing interests

The author declares no competing interests.

Additional information

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