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Entanglement growth in diffusive systems with large spin

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Recent works[1,2] argued that Rényi entropies S_α with indices $\alpha > 1$ exhibit a sub-ballistic, $\propto \sqrt{t}$, growth in systems with diffusive transport. A subsequent work by Žnidarič[3] claims that such sub-ballistic growth occurs only in certain cases (in particular, $d = 1$ dimensional systems with $q = 2$ states per site) and is generically replaced by ballistic growth in the absence of further fine-tuning. Below, we try to make precise the conditions needed for diffusive (rather than ballistic) Rényi growth and argue that they apply to a much wider class of systems than what is suggested by Žnidarič[3]. In particular, the examples presented by Žnidarič that avoid diffusive growth do so because of the presence of additional non-conserved degrees of freedom, not merely their larger local Hilbert space.

Žnidarič[3] considers U(1)-symmetric Floquet systems and claims that to have $S_{\alpha>1} \sim \sqrt{t}$ requires that all on-site diagonal operators (in some preferred basis) correspond to conserved quantities, with transport behavior that is diffusive (or slower). This is automatically satisfied in a system with $q = 2$ states per site (e.g., a spin-1/2 system) with a single U(1) symmetry (e.g., $\sum_j S_j^z$ being conserved), but it is not generally the case for $q > 2$, leading to the claim that in such systems, $S_{\alpha>1} \sim t$, unless additional conserved densities are present (e.g., if $\sum_j (S_j^z)^2$ is also separately conserved). We now present evidence that this claim is incorrect and that generically the conservation of S^z alone is sufficient to induce $\sqrt{t}$ growth for arbitrary finite q. We also highlight the assumptions Žnidarič makes that we expect are responsible for this disagreement.

A direct refutation of the above claim is obtained by evaluating S_2 in a system with $q = 3$. This is readily achieved in a random circuit model, extending earlier results where the same was done for a $q = 2$ chain[1]. To be concrete, we consider a chain where the on-site Hilbert space resembles a Hubbard model in the infinite interaction limit (i.e., with double occupancies projected out): the three on-site states correspond to an empty site ($|0\rangle$), or a site occupied by a spin-up/spin-down particle ($|\uparrow\rangle, |\downarrow\rangle$ respectively). We evolve the system with a brick-wall circuit of 2-site random unitaries which conserve the total number of particles but not the spin.

In this case, Žnidarič[3] would predict $S_{\alpha>1} \sim t$ because $q = 3$ and there is only a single conserved density. On the contrary, calculating the annealed average of S_2 numerically[1] we find (Fig. 1) that $S_2^{(a)} \propto \sqrt{t}$ for initial states that are superpositions of both empty and occupied sites (if all sites are occupied then the circuit precisely reduces to a $q = 2$ random circuit without symmetries, which has $\propto t$ growth[4,5]; however, such initial states are finely-tuned).

In fact, this result is expected: the proof[2] of $\sqrt{t}$ growth originally derived for $q = 2$ extends straightforwardly to this model, and to many other $q > 2$ systems[6]. Rather than the particular value of q, the relevant condition for the proof is the existence of 'empty' regions where no dynamics can occur due to the symmetry; let us call this the "frozen region condition" (FRC). For example, in our $q = 3$ model, in an empty region, the state cannot evolve until some particles propagate into the region from the outside.

A more precise definition of the FRC follows. Consider a system of some finite size and make use of the symmetries to block-diagonalize the time-evolution operator. We say that the FRC is satisfied if there exist blocks containing only a single state for any system size. Entanglement growth should be at most $\propto \sqrt{t \log t}$ for any such system, as argued previously[1,2,6]. Note that FRC is not a property of the dynamics per se, but only of how the U(1) symmetry is represented on the local Hilbert space.

The FRC is satisfied for a large class of systems, including many experimentally relevant ones. For example, it holds for any model where the local degrees of freedom are spin-S variables and $\sum_j S_j^z$ is conserved: states that are fully polarized in the z directions are frozen. As seen above, it also applies to systems of fermionic particles (or hard-core bosons), as long as their total number is conserved. On the other hand, this discussion highlights why certain systems do have $S_{\alpha>1} \sim t$ despite their U(1) symmetry: it can be the case that the symmetry only acts on some subset of the degrees of freedom, while others are unconstrained by it, such that no frozen regions can exist. This happens for example in a two-leg ladder if only the magnetization on one of the legs is conserved[1,3]. While such exceptions exist, they are in fact much less common than what would be implied by the conditions stated by Žnidarič[3].

Let us comment on the source of the above-mentioned disagreement. Žnidarič[3] provides a non-rigorous theoretical argument, aiming to connect the growth of S_2 to the decay of correlations of diagonal operators. The assumption made in this argument is that, while densities of explicitly conserved quantities (e.g., S_j^z) have power-law decaying correlations, most diagonal operators are insensitive to the symmetries and would decay exponentially. For example, according to Žnidarič[3], $(S_j^z)^2$ should

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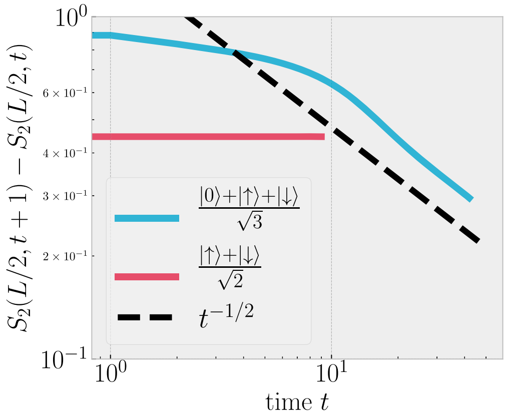

Fig. 1 Time derivative of the half-chain annealed average S_2 in a random circuit with $q = 3$ states per site and only a single U(1) symmetry. The red curve corresponds to an initial state which is a superposition of spin-up and spin-down particles only, with no empty sites; in this case the derivative is constant (ballistic). For a generic initial state that is the superposition of all three possible states (blue curve) the derivative decays as $1/\sqrt{t}$ (diffusive).

have exponential correlations, unless $\sum_j (S_j^z)^2$ is explicitly conserved; this leads to the statement about $S_2 \sim t$.

We believe that in fact the conservation of $\sum_j S_j^z$ is sufficient to cause power law decaying correlations ('hydrodynamic tails') in $(S_j^z)^2$. For example, $(S_j^z)^2$ can evolve into operators of the form $S_k^z S_l^z$ $(l \neq k)$, i.e., the product of two conserved densities, which decay as t^{-d} in a diffusive system. This is similar to the standard arguments that show the existence of long-time tails in the current operator itself[7]. For this reason, we expect that the class of operators with power-law decaying correlations is much larger than expected by Žnidarič[3], which helps explain why $S_{\alpha>1} \sim \sqrt{t}$ is in fact much more prevalent.

We now list two further and distinct criticisms of Žnidarič's work[3]. Firstly, apart from the role of the size of the local Hilbert space q, Žnidarič[3] also raises the question about the validity of the $\sqrt{t}$ result in dimensions $d > 1$. Žnidarič's numerical results are in fact consistent with the claim, made in our earlier work[1], that the sub-ballistic growth is present in any dimension. However, the way these results are presented could make this appear to be a finite-size effect. This is due to a choice of units: Žnidarič[3] measures time in units that depend on the overall system size (effectively rescaling $t \to tL^{1-d}$). In the more standard time units set by the microscopic couplings, the diffusive growth sets in at a system-size independent timescale.

Secondly, we make one final remark about the interpretation of these results that we believe might be confusing for readers of Žnidarič[3]. There, it is claimed that one can think of e^{S_2} as a measure of the number of degrees of freedom needed to describe the corresponding state. If this were so, the result $S_2 \sim \sqrt{t}$ would be rather powerful, indicating that systems obeying the FRC are much easier to simulate on a classical computer than other types of dynamics. However, this is not so. It was one of the important insights of earlier works[1,2] that the long-time dynamics of $S_{\alpha>1}$ is dominated by the largest eigenvalue of the reduces density matrix. As such, knowing about these higher entropies tells us very little about the full complexity of the state; the lower Rényi entropies $(S_{\alpha\leq 1})$ might give a better characterization of the information in a state[8], and indeed these appear to grow linearly in time[1,9].

Data availability

Data are available from the authors upon reasonable request.

Code availability

Code supporting this analysis is available from the authors upon reasonable request.

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Author contributions

T.R. performed numerical simulations. All three authors contributed to the theoretical development of this critique.

Competing interests

The authors declare no competing interests.

Additional information

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