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Vortices as fractons

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Fracton phases of matter feature local excitations with restricted mobility. Despite the substantial theoretical progress they lack conclusive experimental evidence. We discuss a simple and experimentally available realization of fracton physics. We note that superfluid vortices form a Hamiltonian system that conserves total dipole moment and trace of the quadrupole moment of vorticity; thereby establishing a relation to a traceless scalar charge theory in two spatial dimensions. Next we consider the limit where the number of vortices is large and show that emergent vortex hydrodynamics also conserves these moments. Finally, we show that on curved surfaces, the motion of vortices and that of fractons agree; thereby opening a route to experimental study of the interplay between fracton physics and curved space. Our conclusions also apply to charged particles in a strong magnetic field.

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Fracton phases of matter are characterized by the presence of immobile or partially mobile local excitations. The constraints on excitation mobility stem from the conservation laws of multipole moments of the charge density[1–3]. Phases that support fracton excitations were first discovered in exactly solvable quantum lattice models[4–6]. One systematic approach to characterization and classification of fracton phases is based on tensor[1,2,7–12] and multipole gauge theories (MGT)[3,13]. Recent years have witnessed a significant interest in development and classification of phases of quantum matter supporting fracton excitations[14–43], with possible applications ranging from quantum memory to quantum elasticity and quantum gravity. For recent reviews see[44,45]. Despite substantial theoretical progress and several proposals for experimental realization of the fracton physics[37,46–48] no conclusive experimental evidence of fracton physics exists.

One prominent yet down-to-earth example of excitations with restricted mobility is crystalline defects in quantum crystals and liquid crystals[49–57]. There, dislocations satisfy the glide constraint that forces them to move along their Burgers vector, while disclinations are immobile.

In this note, we point out that fracton physics is exhibited by superfluid vortices that have been experimentally observed for many decades. We show that vortices in two spatial dimensions share the mobility constraints with the traceless scalar charge theory (TSCT), which is a particular model of particles with restricted mobility. We review the Hamiltonian formulation of the vortex dynamics and show that it manifestly conserves dipole and (trace of) quadrupole moments of vorticity. In superfluids, the vorticity of individual vortices is quantized and locally conserved, which leads to identification of vorticity with the scalar charge. These conservation laws imply that isolated vortices are immobile, while vortex dipoles move perpendicular to their dipole moment. Both vortices and their dipoles can be readily created and studied experimentally in superfluid Helium[58], Bose–Einstein condensates[59,60], polariton superfluids[61,62], and non-linear media[63]. We then consider a hydrodynamic limit where the number of vortices becomes large; and collective, hydrodynamic description is applied to the vortices themselves. Remarkably, the resulting hydrodynamics admits a Hamiltonian formulation; with Poisson brackets realizing the classical w_∞ algebra[64]. We show that vortex hydrodynamics is also equivalent to scalar charge theory and provide a microscopic collective field theory expression for the rank-2 symmetric current. Finally, we discuss the behavior of vortices and fractons on curved manifolds, which can be realized as curved ^{4}He films.

Results and discussion

Vortices. We consider a two-dimensional incompressible ideal fluid. It is described by the Euler equations

$$(\partial_0 + u_i\partial_i)u_j = -\partial_j P \ , \tag{1}$$

where P is the pressure and u_i is the velocity field. The combination $\partial_0 + u_i\partial_i$ is known as the material derivative. The incompressibility condition implies that $\partial_i u_i = 0$. Taking the curl of (1) we obtain the Helmholtz equation

$$(\partial_0 + u_i\partial_i)\omega = 0 \ , \tag{2}$$

where $\omega = \epsilon^{ij}\partial_i u_j$ is the vorticity. Equation (2) admits solutions where the vorticity is concentrated in a finite number of point vortices. The complex velocity field $u_z = u_1 - iu_2$ takes form

$$u_z(z) = -i\sum_{\alpha=1}^{N}\frac{\gamma_\alpha}{z - z_\alpha(t)} \ , \quad \omega(z) = \sum_{\alpha=1}^{N}\gamma_\alpha\delta^2(z - z_\alpha(t)) \ , \tag{3}$$

where $z_\alpha(t) = x_1^\alpha(t) + ix_2^\alpha(t)$ (we will switch between complex and Cartesian coordinates at will) is time-dependent position of the αth vortex and $2\pi\gamma_\alpha$ is its circulation; while $\gamma = |\gamma_\alpha|$ is the vortex strength. We have assumed that vorticity is quantized in units of γ, which is the case in superfluids[58].

Remarkably, the vortex coordinates $x_i^\alpha(t)$ form a Hamiltonian system[65]

$$H = -2\pi\sum_{\alpha<\beta}\gamma_\alpha\gamma_\beta \ln|\mathrm{x}^\alpha - \mathrm{x}^\beta| \ , \tag{4}$$

$$\left\{x_1^\alpha, -2\pi\gamma_\beta x_2^\beta\right\} = \delta^{\alpha\beta} \ , \tag{5}$$

where $\alpha, \beta = 1, 2, \ldots, N$ label the vortex strength. We refer the reader to[66,67] for an in-depth review of the vortex systems.

Dynamical system (4)-(5) also describes charged particles moving in a strong magnetic field, in the limit of infinite cyclotron frequency, or equivalently, on the lowest Landau level. Consequently, all our results apply verbatim to the charged plasma in a strong magnetic field (see Supplementary Discussion for the details).

In dealing with (4) and (5) it is useful to use the complex coordinates z_α. In complex notations, the only non-trivial Poisson bracket takes the form[65]

$$\left\{z_\alpha, \bar{z}_\beta\right\} = i(\pi\gamma_\alpha)^{-1}\delta_{\alpha\beta} \ . \tag{6}$$

The equations of motion are[65]

$$\dot{\bar{z}}_\alpha = -i\sum_{\beta=1,\beta\neq\alpha}^{N}\frac{\gamma_\beta}{z_\alpha - z_\beta} \ . \tag{7}$$

It is worth emphasizing that H is not just the potential energy. Due to the non-trivial commutations relations between z_α and $\bar{z}_\alpha$, H can be viewed as the kinetic energy.

Conservation laws. Hamiltonian H is translation and rotation invariant. The corresponding integrals of motion are known as impulse, $_{Pi}$ and angular impulse, L[67,68]. They are given by

$$P_i = -2\pi\epsilon_{ij}\sum_\alpha\gamma_\alpha x_j^\alpha \ , \qquad L = 2\pi\sum_\alpha\gamma_\alpha x_i^\alpha x_j^\alpha\delta_{ij} \ . \tag{8}$$

We recognize in Eq. (8) that impulse is related to the dipole moment of vorticity D_i (also known as center of circulation), while angular impulse corresponds the trace of the quadrupole moment of vorticity, Q_{ij} (also known as moment of circulation), according to

$$P_i = -\epsilon_{ij}D_j \ , \qquad L = \delta_{ij}Q_{ij} \ . \tag{9}$$

Together, the quantities P_i, L, D_i, Q_{ij} form a multipole algebra

$$\begin{aligned} \{L, P_i\} &= 2\epsilon_{ij}P_j \ , & \left\{P_i, D_j\right\} &= -\delta_{ij}\Gamma \ , \\ \{L, D_i\} &= -2\epsilon_{ij}D_j \ , & \left\{P_i, \delta_{jk}Q_{jk}\right\} &= -2D_i \ , \end{aligned} \tag{10}$$

where we have introduced the total vortex strength

$$\Gamma = \sum_{\alpha=1}^{N}\gamma_\alpha \ . \tag{11}$$

Thus, the vortices are equivalent to a TSCT; where the total charge as well as dipole and trace of the quadrupole moments are conserved[1]. Isolated charges are immobile; while isolated dipoles move perpendicular to their dipole moment. We emphasize that the conservation of dipole and trace of quadrupole moment does not originate from an internal symmetry[3] as in all previously studied cases with $\mathbb{Z}$-valued charge. Instead, it originates from spatial symmetries and non-commutativity of the configuration space. We surmise that there is a deeper relation between non-commutative field theories and fracton physics.

Traceless Scalar Charge Theory (TSCT). We briefly pause to discuss some properties of the TSCT. More details can be found in[1,45]. TSCT describes particles that conserve a $U(1)$ charge as well as dipole and trace of the quadrupole moments. These conservation laws are succinctly summarized by the following equations

$$\dot{\rho} + \partial_i \partial_j J^{ij} = 0 \ , \qquad \mathrm{Tr}\,(J_{ij}) = 0 \ , \tag{12}$$

where ρ is the density of the $U(1)$ charge and J^{ij} is the symmetric, traceless rank-2 tensor. The indices are raised with the spatial metric g_{ij}, which is assumed to be flat and rotationally invariant $g_{ij} = \delta_{ij}$, unless specified otherwise. Denoting $j^i = \partial_j J^{ij}$ we observe that ρ satisfies ordinary continuity equation $\partial_0 \rho + \partial_i j^i = 0$; confirming the charge conservation. Furthermore, we can find that dipole moment and trace of quadrupole moment are conserved, by multiplying Eq. (12) with x_i and $x_i x^i$ respectively; and integrating over space.

$$\partial_0 D_k = \partial_0 \int x_k \rho = \int x_k \partial_i \partial_j J_{ij} = 0 \ , \tag{13}$$

$$\partial_0 \, \mathrm{Tr}\,(Q_{ij}) = \partial_0 \int x^2 \rho = \int \mathrm{Tr}\,(J_{ij}) = 0 \ . \tag{14}$$

These conservation laws imply that charge dipoles can only move perpendicular to their dipole moment[1].

One may wonder what kind of microscopic theory would support Eq. (12) as the conservation laws. In the present paper, we argue that vortices in incompressible superfluid obey these conservation equations. Furthermore, using the ideas from[69] it is clear that the following Lagrangian fits the bill

$$\begin{aligned}\mathcal{L} = \dot{\Phi}^\star \dot{\Phi} + g_1 |D_1(\Phi)|^2 + g_3 |D_3(\Phi)|^2 + g_1' \, \mathrm{Re}\left[(\Phi^\star)^2 \mathrm{D}_1(\Phi)\right] \\ + g_3' \, \mathrm{Re}\left[(\Phi^\star)^2 \mathrm{D}_3(\Phi)\right] + \mu |\Phi|^2 + \ldots \ \ ,\end{aligned} \tag{15}$$

where … stands for the higher-order terms and Φ is a complex scalar. The derivative operators $D^I(\Phi)$ are defined as

$$D^I(\Phi) = \sigma^I_{ij} \Big(\partial_i \Phi \partial_j \Phi - \Phi \partial_i \partial_j \Phi \Big) \ , \tag{16}$$

where σ^I_{ij} are the Pauli matrices. A restricted version of the Lagrangian (15) can be used to describe the defects in two-dimensional elasticity[49].

For generic values of g_I, g_I', the theory (15) is invariant under C_4, but not $SO(2)$. Though it is $SO(2)$ invariant for the special case of $g_1 = g_2$ and $g_1' = g_2'$. More importantly, the theory is invariant under the following transformation

$$\Phi' = e^{if(x)} \Phi \ , \qquad f(x) = \lambda + \lambda_k x_k + \zeta |x|^2 \ , \tag{17}$$

where the parameters $\lambda, \lambda_k, \zeta$ are arbitrary. Noether's theorem then implies that the corresponding conservation laws are precisely (12). The density is given by the usual expression $\rho = \Phi^\star \Phi$, while the general expression for the current is quite lengthy and not enlightening.

We discuss the chiral version of the above theory in the Supplementary Discussion.

Mobility constraints. Conservation laws (8) imply that motion of many vortices is constrained to preserve the dipole and quadrupole moment. Moreover, since the conserved quantities ($H, D_i D^i, \delta^{ij} Q_{ij}$) are in involution, the problem of N vortices is integrable for $N \leq 3$. Other typical cases are chaotic[67]. We discuss the "fractonic" motion of vortices next.

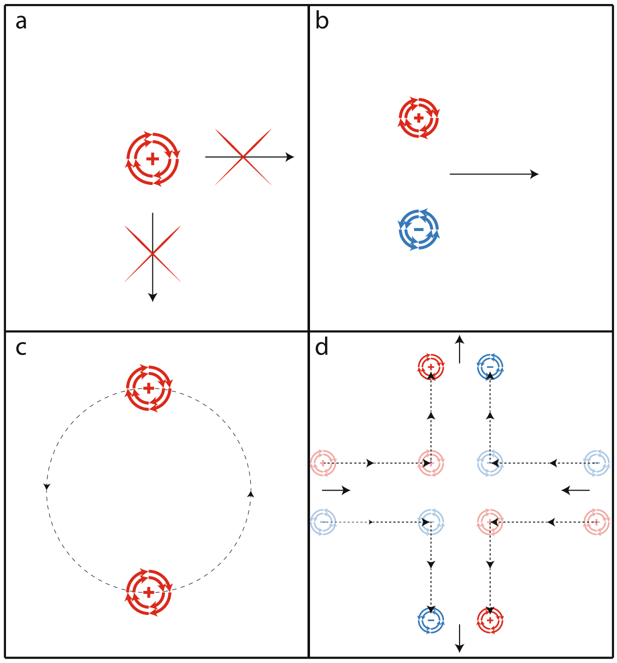

Fig. 1 Motion of point vortices. a An isolated vortex is immobile and corresponds to a fracton. **b** A neutral dipole moves perpendicular to its dipole moment—it is a "lineon". **c** A charge 2-dipole moves around the center of vorticity. In fractonic context this motion is also possible, albeit never discussed: a pair of identical charges can rotate by constantly emitting dipoles that cancel. **d** Scattering of two dipoles of opposite dipole moments. Upon scattering the dipole makes a $\pi/2$ turn.

A single or well-isolated vortex is immobile. Analogously to fractons, the mass of an isolated vortex is not well-defined. A broad class of definitions[70] leads to a diverging mass, which agrees with fracton ideas.

Dipole consisting of two vortices with opposite vorticities moves in a straight line perpendicular to its dipole moment. At low temperatures vorticity-neutral systems "condense" into a gas of neutral dipoles[71]. The dipole of two vortices with the same vorticities moves in a closed orbit around their "center of vorticity", while keeping the distance between the two vortices constant. Motion of dipole is illustrated in Fig. 1. Relative distances can only change if the number of vortices is $N \geq 3$[67].

The quadrupole of two vortex-dipoles exhibits a variety of complex dynamics. One common type of interactions (particularly at low temperature) is scattering between two dipoles as shown in Fig. 1. As a result of scattering a vortex dipole makes a $\pi/2$ turn, which agrees with phenomenology of TSCT.

Statistical mechanics. Although many-vortex dynamics are chaotic, for certain vortex configurations the relative positions of vortices are completely frozen. Such configurations are called vortex crystals or vortex equilibria[72]. The examples include N identical collinear vortices situated in the roots of N-th Hermite polynomial as well as Adler-Moser polynomials, identical vortices located at the vertices of a regular N-polygon, etc. There are many other examples (see[72] for a review). Vortex crystals can move as rigid objects, in which case they are referred to as relative equilibria, or can be stationary. Such configurations explore a very small fraction of the phase space. This is immediately obvious since for a vortex system phase space coincides with the configuration space. Vortex crystals emerge experimentally after relaxation of highly turbulent two-dimensional flows[73,74].

It is tempting to compare vortex crystals to the Hilbert space fragmentation seen in quantum dipole conserving systems[48,75–78]. There, the Hilbert space "shatters" into many disconnected subspaces; within each such subspace either integrability or thermalization is possible.

Mobility constraints combined with the phase space reduction lead to an exotic statistical mechanics of vortices[79,80]. In particular, above certain critical energies vortices experience "negative temperature"[79,81], which follows from the structure of the phase space. At negative temperature the vortices of the same vorticity tend to clamp together, which nicely corresponds to gravitational attraction of fractons discussed in[24]. Vortex crystals may be an obstruction to ergodicity: Clusters of vortices take a very long time to merge[80]. To the best of our knowledge, the ergodicity of vortex system is still an open problem[82].

Vortex hydrodynamics. Next we would like to consider the limit where the number of vortices is very large. Due to the chaotic behavior and strong interactions between the vortices, this limit admits a description in terms of an emergent hydrodynamics[64]. We will show that in hydrodynamic limit the dipole and trace of the quadrupole moments are conserved. These conservation laws will be made manifest by re-writing the continuity equation in the form (12), where the conserved $U(1)$ density is related to the vorticity $\rho=(2\pi\gamma)^{-1}\omega$.

We would like to emphasize one subtle difference between traditional TSCT and vortices: The former is non-chiral, while the latter is chiral. In TSCT a dipole moves perpendicular to its dipole moment; while for a vortex dipole, the dipole moment and the direction of motion form a right pair.

Vortex hydrodynamics for the chiral flow (i.e., when all vortices are of the same vorticity, $\gamma_\alpha=\gamma$) was derived by Wiegmann–Abanov in[64]. The continuum limit of the vortex Hamiltonian (4) is

$$H_{\rm WA}=\frac{1}{2}\int\left[v^2-\eta^2(\partial_i\ln\rho)(\partial_i\ln\rho)\right]d^2r\ , \tag{18}$$

where v_i is the vortex velocity and $\eta=\frac{\gamma^2}{4}$. Vortex fluid is incompressible: $\partial_i v_i=0$ and v_i is completely determined by the density through[64]

$$\epsilon_{ij}\partial_i v_j=2\pi\gamma\rho+\eta\Delta\ln\rho\ . \tag{19}$$

The Poisson brackets form the classical w_∞ algebra[83]

$$\{\rho(x),\rho(x')\}=\epsilon_{rs}\partial'_r\partial_s[\rho(x)\delta(x-x')]\ , \tag{20}$$

where $\partial'_i=\frac{\partial}{\partial x'_i}$. Brackets between velocity and density are deduced from (19)

$$\begin{aligned}\{v_k(x),\rho(x')\}=&-\partial'_k(\rho(x)\delta(x-x'))\\&-\eta\epsilon_{kj}\partial_j\left[\frac{1}{\rho}\epsilon_{rs}\partial'_r\partial_s(\rho(x)\delta(x-x'))\right].\end{aligned} \tag{21}$$

We are interested in computing the equation of motion for the density ρ

$$\dot\rho(x)=\{H_{\rm WA},\rho(x)\}\ . \tag{22}$$

Direct calculation gives the continuity equation

$$\dot\rho+\partial_k j_k=0\quad\Longleftrightarrow\quad\dot\rho+v_k\partial_k\rho=D_0\rho=0\ , \tag{23}$$

where $j_k=\rho v_k$. This is consistent with Helmholtz equation (2). The consistency is non-trivial since (2) includes the material derivative with u_i, while the material derivative contains v_i in (23). The equivalence of (2) and (23) is established using the relation between u_i and v_i[64]

$$v_i=u_i-\frac{\gamma}{4}\epsilon_{ij}\partial_j\ln\rho\ . \tag{24}$$

Using the identity

$$2\pi\gamma u_i\rho=\epsilon_{ik}\left[\partial_j(u_ju_k)-\frac{1}{2}\partial_k(u_ju_j)\right]\ , \tag{25}$$

with either (2) or (23) we find

$$j_i=\partial_jJ_{ij}\ ,\qquad J_{ij}=\frac{1}{2\pi\gamma}\left(\epsilon_{ik}u_ju_k-\frac{1}{2}\epsilon_{ij}u^2\right)-\frac{\gamma}{4}\epsilon_{ij}\rho\ . \tag{26}$$

The anti-symmetric part of J_{ij} drops out from (12). In the chiral case, an equivalent relation was derived in[84].

Emergent hydrodynamics for vortices of positive and negative vorticity was developed by Yu-Bradley[85]. The conservation of the impulse and angular impulse holds in their model as well. The number and charge (vortex-sign) densities are treated separately in this case. Note that the conservation laws discussed here apply to the charge density, not the number density. We derive the tensor current based on their hydrodynamics in the Supplementary Discussion. We will discuss an independent collective field theory derivation of the rank-2 conservation law (12) for arbitrary number of vortices next.

Collective field theory of vortices. We now turn to the collective form of (7). Vorticities are allowed to take both positive and negative values: $\gamma_\alpha=\pm\gamma$.

Density and current fields are defined as follows

$$\begin{aligned}\rho(z)&=\frac{1}{\gamma}\sum_\alpha\gamma_\alpha\delta(z-z_\alpha)\ ,\\ j_z(z)=\rho(z)v(z)&=\frac{1}{\gamma}\sum_\alpha\gamma_\alpha\dot{\bar z}_\alpha\delta(z-z_\alpha)\ .\end{aligned} \tag{27}$$

We will need the complex notation $j_z=j_1-ij_2$ and the δ-function identity

$$\partial_z\frac{1}{\bar z}=\partial_{\bar z}\frac{1}{z}=\pi\delta(z)\ . \tag{28}$$

The time derivative of the density is given by

$$\dot\rho=-\frac{1}{\gamma}\sum_{\alpha=1}^{N}[\gamma_\alpha\dot z_\alpha\partial_z\delta(z-z_\alpha)+\gamma_\alpha\dot{\bar z}_\alpha\partial_{\bar z}\delta(z-z_\alpha)]\ . \tag{29}$$

Using (7) this is transformed into

$$\dot\rho+\partial_z\partial_zJ_{\bar z\bar z}+\partial_{\bar z}\partial_{\bar z}J_{zz}=0\ , \tag{30}$$

where we have introduced a traceless symmetric tensor current

$$J_{zz}=\frac{1}{2\pi i\gamma}\left(\left(\sum_\alpha\frac{\gamma_\alpha}{z-z_\alpha}\right)^2+\partial_z\sum_\alpha\frac{\gamma_\alpha^2}{z-z_\alpha}\right)\ , \tag{31}$$

and $J_{\bar z\bar z}=\bar J_{zz}$. It is crucial that in (31) the second order poles cancel. In Cartesian components the symmetric tensor current is given by

$$J_{ij}=\frac{1}{2\pi\gamma}\left(\epsilon_{ik}u_ju_k-\frac{1}{2}\epsilon_{ij}u^2\right)-\frac{\gamma}{4}\epsilon_{ij}n\ , \tag{32}$$

$$n=(2\pi\gamma)^{-1}\sum_\alpha|\gamma_\alpha|\delta(z-z_\alpha)\ , \tag{33}$$

where we introduced the vortex number density $n(z)$. This is the central result of the present work: The continuity equation takes form (12). The above derivation is general and applies to hydro with vortices of both kinds present. In particular, it applies to the case when total vorticity is 0. We derive (32) in the Supplementary Discussion.

Curved space. Symmetric tensor gauge theories do not remain gauge invariant on a curved space[50]. Furthermore, the conservation law of dipole moment cannot remain unchanged on a curved space. Below, we show that, in curved space, the dynamics of vortices and the mobility constraints change. Vortices on a curved space have been studied in[86–88] and can be experimentally realized in thin ^{4}He films. Vortex hydrodynamics of chiral flows was generalized to curved spaces in[84]. Vortex problem on a surface of a sphere is also relevant for geophysical and atmospheric applications. The Helmholtz equation on a curved surface takes form[84]

$$\dot{\rho} + u_i \nabla_i \left(\rho + \frac{s}{4\pi} R \right) = 0 \ , \tag{34}$$

where ∇_i is a covariant derivative, R is the Ricci curvature and $s - \frac{1}{2}$ is the geometric spin of a vortex. Eq.(34) also takes form (12) with slightly modified J_{ij}[84]

$$J_{ij}(R) = J_{ij}|_{R=0} + \left[\frac{1}{2\pi\gamma} \nabla_i u_j + \frac{\gamma}{2} \epsilon_{ij} \left(\rho - \frac{s}{4\pi} R \right) \right] \ . \tag{35}$$

Note that the last term in (35) contributes to the equations of motion only when curvature is inhomogeneous. We can draw the following conclusion from (34)-(35). On a surface of constant curvature an isolated vortex remains immobile[87,89], which is consistent with[50]. A dipole moves along a geodesic that is perpendicular to the dipole moment; which is consistent with the corresponding fracton observations made in[90].

On a surface of variable curvature an isolated vortex does move: the dipole conservation law is broken and fractonic property is lost; in agreement with[50]. The potential force acting on an isolated vortex is obtained by differentiating the Robin function[91]. The dipole moves along a geodesic in the general case[92].

Conclusions. We have established an equivalence between vortex dynamics in two-dimensional superfluids and TSCT. We have shown that vortices provide a Hamiltonian realization of fracton dynamics for any finite number of vortices as well as in the hydrodynamic limit. Thus superfluid vortices provide a readily available platform for experimental realization of fracton quasiparticles. Vortices and vortex-dipoles are experimentally available with the present day technology. Another new platform may rely on chiral active fluids[93].

Similar conservation laws hold in three dimensions for vortex lines. We leave the exploration of higher dimensional case, discussion of more refined probes of fracton dynamics in superfluids and BECs, such as role of the trap and finite lifetime, generalization to chiral superfluids such as ^{3}He and many other open question to future work. Theory of vortices plays central role in statistical approach to turbulence[79]; where the questions of ergodicity and validity of statistical mechanics are central[82]. It would be very interesting to see if fracton-inspired ideas can lead to new insight into quantum and classical turbulence as well as the problem of quantization of vortex dynamics. Finally, dynamics of electrons residing in the lowest Landau level is formally identical to that of vortices; consequently we expect applications of fracton-inspired ideas to the physics of fractional quantum Hall effect.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Code availability

Code sharing is not applicable to this article as no code is developed during the current study.

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Author contributions

A.G. conceived the work. A.G. and D.D. performed the calculations and wrote the paper.

Competing interests

The authors declare no competing interests.

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