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## $q$ -Rung orthopair fuzzy 2-tuple linguistic WASPAS algorithm for patients' prioritization based on prioritized Maclaurin symmetric mean aggregation operators

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Due to the fuzziness of the medical field,  $q$ -rung orthopair fuzzy 2-tuple linguistic ( $q$ -RF2L) set is the privileged way to aid medical professionals in conveying their assessments in the patient prioritization problem. The theme of the present study is to put forward a novel approach centered around the merging of prioritized averaging (PA) and the Maclaurin symmetric mean (MSM) operator within  $q$ -RF2L context. According to the prioritization of the professionals and the correlation among the defined criteria, we apply both PA and MSM to assess priority degrees and relationships, respectively. Keeping the pluses of the PA and MSM operators in mind, we introduce two aggregation operators (AOs), namely  $q$ -RF2L prioritized Maclaurin symmetric mean and  $q$ -RF2L prioritized dual Maclaurin symmetric mean operators. Meanwhile, some essential features and remarks of the proposed AOs are discussed at length. Based on the formulated AOs, we extend the weighted aggregated sum product assessment methodology to cope with  $q$ -RF2L decision-making problems. Ultimately, to illustrate the practicality and effectiveness of the stated methodology, a real-world example of patients' prioritization problem is addressed, and an in-depth analysis with prevailing methods is performed.

**Keywords**  $q$ -Rung orthopair fuzzy 2-tuple linguistic term set, Maclaurin symmetric mean, Prioritized averaging, WASPAS, MCGDM

The global population is increasing, but healthcare resources are limited, posing challenges for healthcare institutions in meeting public health needs. Prioritizing timely patient treatment is crucial, yet these institutions grapple with efficiently allocating scarce medical resources among patients<sup>1,2</sup>. Ghanaian public healthcare facilities face a similar dilemma, as the country's population growth has led to a rise in emergency cases, while the availability of sick beds remains insufficient. The shortage of sick beds in public healthcare institutions has sparked debates among various healthcare stakeholders in Ghana. Research by Sun et al.<sup>3</sup> shows that limited medical resources often result in extended waiting times and treatment delays, increasing the likelihood of patients developing complications and, in some cases, leading to fatalities<sup>4</sup>. Numerous scholars<sup>2,3,5</sup> emphasize the crucial importance of prioritizing patients promptly for effective casualty treatment. This involves addressing several complex research questions in the context of multi-criteria group decision-making (MCGDM): How can we prioritize patients in situations where decision-making involves uncertainty and imprecision? What strategies can be employed to address patient prioritization when multiple experts with priority relationships are involved? How do we tackle patient prioritization when there are interdependencies among multiple input factors and priority relationships among evaluation criteria?

Numerous prior studies<sup>2-4,6,7</sup> have approached the issue of patient prioritization from diverse angles. For instance, Ashour and Kremer<sup>6</sup> introduced a dynamic patient grouping and prioritization method to enhance emergency department efficiency. Zhang et al.<sup>7</sup> employed fuzzy multi-criteria decision-making (MCDM) techniques to investigate patient prioritization, utilizing a hesitant fuzzy linguistic-VIekriterijumsko KOMPromisno Rangiranje (VIKOR) method. Similarly, Zhang et al.<sup>2</sup> proposed an intuitionistic multiplicative rangement et

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synthèse de données relationnelles (ORESTE) method for hospitalization patient prioritization. Davodabadi et al.<sup>4</sup> developed a fuzzy-technique for order of preference by similarity to ideal solution (TOPSIS) method with discrete event simulation for ICU patient prioritization. Nonetheless, these studies did not delve into the crucial aspect of evaluating the interdependence of criteria in real-world decision-making (DM). In contrast, Sun et al.<sup>3</sup> addressed patient prioritization using the hesitant fuzzy linguistic projection-based multi-attributive border approximation area comparison (MABAC) method and enhanced the traditional MABAC method by incorporating the Bonferroni mean to account for the relationship between two input factors. Nevertheless, these models do not tackle the interrelationships among multiple input factors. In sum, existing research has developed models for patient prioritization, but none have comprehensively addressed all the research challenges outlined above. Hence, there is a pressing need to develop an appropriate technique to effectively handle patient prioritization for hospital admissions in healthcare institutions while considering interdependencies.

DM processes frequently encounter inherent uncertainty and vagueness<sup>8–11</sup>. Based on their level of expertise, health experts may hesitate in their assessments and opt to employ fuzzy data sets to express patient evaluations for hospital admission. To assist experts in making informed decisions, Attansov<sup>12</sup> introduced the concept of intuitionistic fuzzy sets (IFS). IFS enables experts to convey their expressions by incorporating both membership grade (MG) and non-membership grade (NMG), providing a more nuanced representation of their assessments. Szmiedt and Kacprzyk<sup>13</sup> pioneered the exploration of medical applications using IFSs. Subsequently, Yager<sup>14</sup> developed the Pythagorean fuzzy set (PyFS) concept. PyFS models exhibit greater robustness than IFS models in handling practical applications<sup>15–18</sup>. However, both models encounter limitations. For instance, when DM scenarios involve opinions like (0.8, 0.9), both PyFSs and IFSs demonstrate an inability to apply their methodologies effectively. To address these limitations, Yager introduced<sup>19</sup> a more advanced and generalized form of fuzzy set, termed the  $q$ -rung orthopair fuzzy set ( $q$ -ROFS). This variant stands out because it incorporates the parameter ' $q$ ', which significantly expands the information space. Consequently, various methods have been formulated and refined within the  $q$ -ROFS framework by researchers<sup>20–23</sup>. Akram and Shahzadi<sup>20</sup> advanced the notion of  $q$ -ROFS by developing a series of aggregation operators (AOs) grounded in Yager's norm. They also thoroughly explored and provided numerical evidence to support the relevant characteristics of these operators. Seikh and Mandal<sup>21</sup> established an MCDM method utilizing their Frank-operations-based AOs in a  $q$ -rung orthopair fuzzy ( $q$ -ROF) environment with unknown weights. Recently, Kumar and Gupta<sup>24</sup> addressed the challenge of project selection with their innovative DM algorithm, distinguished by its ability to incorporate confidence levels for the alternatives. Numerous other scholars have also introduced novel AOs and their implementations in the context of MCDM<sup>25–28</sup>.

The 2-tuple linguistic (2TL) learning approach, known for its efficacy in minimizing information distortion and loss, was first conceptualized by scholars in<sup>29</sup>. This paradigm, integral for dealing with human language in DM, has led to the development of numerous methodologies and 2TL AOs. In this realm, Deng et al.<sup>30</sup> introduced generalized 2TL Pythagorean fuzzy Heronian mean AOs, integrating generalized Heronian mean with 2TL Pythagorean fuzzy numbers. Wei and Gao<sup>31</sup> further contributed by proposing Pythagorean fuzzy 2TL power AOs, merging power geometric and average operations with Pythagorean fuzzy 2TL data, aimed at resolving MCDM issues. Wei et al.<sup>32</sup> expanded this framework by introducing the  $q$ -rung orthopair fuzzy 2TL ( $q$ -RF2L) sets, allowing the expression of MG and NMG of elements in relation to 2TL variables, along with operational laws. They also developed various  $q$ -RF2L Heronian mean AOs. Abbas et al.<sup>33</sup> pioneered a clustering algorithm that processes data in the  $q$ -RF2L format, adeptly handling situations with unspecified weight information. Complementing the advancements of  $q$ -RF2L set, Ju et al.<sup>34</sup> formulated a strategy for tackling MCGDM problems using  $q$ -RF2L input. This strategy includes  $q$ -RF2L weighted averaging and geometric operators and the novel  $q$ -RF2L Muirhead mean and dual Muirhead mean operators. However, all these existing AOs within the  $q$ -RF2L framework fall short in addressing scenarios where criteria and experts simultaneously exhibit interdependence and varying priorities. This gap is particularly relevant in real-world situations, such as prioritizing patient admissions in hospitals.

Maclaurin<sup>35</sup> initially introduced the Maclaurin symmetric mean (MSM), later expanded by Detemple and Robertson<sup>36</sup>. MSM effectively models the interconnections among multiple inputs provided by experts. This concept led to Qin and Liu<sup>37</sup> proposing the dual Maclaurin symmetric mean (DMSM), which, alongside MSM, has garnered significant attention in the field of information aggregation. For instance, Mu et al.<sup>38</sup> introduced an interval-valued Pythagorean fuzzy power MSM operator for MCGDM, and Ali<sup>39</sup> developed partitioned MSM operators using hesitant fuzzy numbers. However, many MSM and DMSM operators overlook the prioritization of experts and criteria. Addressing this, Yager<sup>40</sup> proposed the prioritized averaging (PA) operator. Ali<sup>41</sup> then utilized PA's prioritizing capabilities to create probabilistic hesitant bipolar fuzzy Hamacher prioritized AOs for DM. Similarly, Akram et al.<sup>42</sup> explored MCGDM with spherical fuzzy prioritized weighted AOs, and Rong et al.<sup>43</sup> applied complex  $q$ -RF2L MSM and DMSM operators to emergency program selection. Despite these advancements, the PA operator has not yet been adapted to the  $q$ -RF2L context. Given the strengths of PA, MSM, and DMSM, there's a clear opportunity to integrate these operators, developing new  $q$ -RF2L AOs for MCGDM methods that can effectively tackle prioritized DM challenges.

From the discussions and analyses previously outlined, the motivations for this study can be summarized as follows:

- The  $q$ -RF2L theory represents a hybrid model for information representation, skillfully integrating the notable aspects of  $q$ -ROFS and the 2-tuple linguistic model into a cohesive framework. This model allows for a precise representation of assessment values within predefined LTS in uncertain environments. Despite the distinct attributes of the  $q$ -RF2LS, to date, only a few scholarly works<sup>32–34</sup> explore this context.

- Information fusion is crucial in consolidating the preference information of decision experts. Moreover, many practical issues require consideration of the correlation between identified criteria while also necessitating the modeling of priority relationships. However, the current  $q$ -RF2L AOs lack the capability to effectively manage such scenarios, rendering them inadequate for numerous practical applications.
- The WASPAS method is renowned for its ability to assess and rank alternatives with a heightened level of reliability, as evidenced in various studies<sup>44–46</sup>. Given these strengths, there's a compelling need to expand the WASPAS approach into the  $q$ -RF2L setting. This extension would not only leverage the method's inherent strengths but also enhance its applicability in more complex and uncertain environments to capitalize further on its assessment and ranking capabilities in this advanced context.
- Taking into account the benefits of  $q$ -RF2LS, it is likely that health experts would favor this framework for patient assessment. The WASPAS method has demonstrated its efficacy in DM for such issues. However, there is a noticeable research gap in applying this method within a  $q$ -RF2L environment specifically for addressing the problem of patient prioritization.

Motivated by the considerations mentioned above, this paper aims to devise an innovative decision-making approach for evaluating patient prioritization issues. To fulfill this goal, we initially develop appropriate AOs to aggregate  $q$ -RF2L information. Subsequently, our focus shifts to constructing a decision algorithm that can effectively prioritize patients in emergency departments. Finally, it is essential to validate the effectiveness and advantages of the proposed decision algorithm from various perspectives. Consequently, the specific aims of this article are outlined as follows:

- To introduce two innovative AOs, namely  $q$ -RF2LPMSM and  $q$ -RF2LPDMSM, designed to account for the interdependence among criteria while simultaneously modeling their priority relationships.
- To thoroughly elucidate the characteristics of the proposed operators, focusing on its monotonicity and boundedness, and to detail special cases supported by rigorous proof.
- To devise an MCGDM method that integrates the WASPAS technique with the proposed operators. This method aims to address the traditional limitation of WASPAS by considering the interdependence of evaluation criteria in its decision process while also factoring in the priority relationships among these criteria.
- To showcase the practicality of the proposed framework by applying it to solve the problem of prioritizing patients for hospital admission.

### Preliminaries

This section mainly addresses several basic ideas and operators regarding LTS,  $q$ -RFLS, MSM, and DMSM operators.

Suppose  $S = \{\mathbb{J}_\phi | \phi = 0, 1, \dots, \ell\}$  represents an LTS with an odd cardinality. Any label,  $\mathbb{J}_\phi$  signifies the possible value for a linguistic variable, and it has to adhere with the stipulations<sup>29,47</sup> listed below:

1. *Ordered set*:  $\mathbb{J}_{\phi_1} \leq \mathbb{J}_{\phi_2} \Leftrightarrow \phi_1 \leq \phi_2$ ;
2. *Negation operator*:  $Neg(\mathbb{J}_{\phi_1}) = \mathbb{J}_{\phi_2}$ , such that  $\phi_1 + \phi_2 = \ell$ ;
3. *Max operator*:  $\max(\mathbb{J}_{\phi_1}, \mathbb{J}_{\phi_2}) = \mathbb{J}_{\phi_1}$  if  $\mathbb{J}_{\phi_1} \geq \mathbb{J}_{\phi_2}$ ;
4. *Min operator*:  $\min(\mathbb{J}_{\phi_1}, \mathbb{J}_{\phi_2}) = \mathbb{J}_{\phi_1}$  if  $\mathbb{J}_{\phi_1} \leq \mathbb{J}_{\phi_2}$ .

For instance,  $S$  can be defined as

$$S = \left\{ \begin{array}{l} \mathbb{J}_0 = \text{extremely poor}, \mathbb{J}_1 = \text{very poor}, \mathbb{J}_2 = \text{poor}, \mathbb{J}_3 = \text{medium}, \mathbb{J}_4 = \text{good}, \mathbb{J}_5 = \text{very good}, \\ \mathbb{J}_6 = \text{extremely good} \end{array} \right\}.$$

In light of the concept of symbolic translation, Herrera and Martinez<sup>29,47</sup> set up the 2-tuple fuzzy linguistic representation approach. It is employed to convey linguistic assessment information as a 2-tuple  $(\mathbb{J}_\phi, \Delta)$  where  $\mathbb{J}_\phi$  is a linguistic label from the pre-defined linguistic term set  $S$ ,  $\Delta$  is the measure of symbolic translation, and  $\Delta \in [-0.5, 0.5)$ .

**Definition 1**<sup>29,47</sup> Let  $\vartheta$  be the calculated result of an aggregation of the indices of a set of labels assessed in an LTS  $S$ , i.e., the outcome of a symbolic aggregation operation,  $\vartheta \in [0, \ell]$ , with  $\ell$  being the cardinality of  $S$ . If  $r = \text{round}(\vartheta)$  and  $\Delta = \vartheta - r$  are two numbers such that  $r \in [0, \ell]$  and  $\Delta \in [-0.5, 0.5)$ , then  $\Delta$  is known as a symbolic translation.

**Definition 2**<sup>29,47</sup> Let  $S = \{\mathbb{J}_\phi | \phi = 0, 1, \dots, \ell\}$  be an LTS and  $\vartheta \in [0, \ell]$  be a numerical value representing the linguistic symbolic aggregation outcome. Then, the function  $\lambda$  that retrieves the 2-tuple linguistic information equivalent to  $\vartheta$  is then described as

$$\lambda : [0, \ell] \longrightarrow S \times [-0.5, 0.5), \tag{1}$$

$$\lambda(\vartheta) = \begin{cases} \mathbb{J}_r, & r = \text{round}(\vartheta) \\ \Delta = \vartheta - r, & \Delta \in [-0.5, 0.5). \end{cases} \tag{2}$$

where round  $(\cdot)$  is the conventional round function,  $\lfloor \cdot \rfloor_r$  is the index label closest to  $\vartheta$ , and  $\Delta$  is the symbolic translation value.

**Definition 3<sup>29,47</sup>** Let  $S = \{\lfloor \cdot \rfloor_\Delta | \varphi = 0, 1, \dots, \ell\}$  be an LTS and  $(\lfloor \cdot \rfloor_r, \Delta)$  be a 2-tuple. There is always a function  $\Upsilon$  can be described, such that, from a 2-tuple  $(\lfloor \cdot \rfloor_r, \Delta)$  it yield its equivalent numerical value  $\vartheta \in [1, \ell]$ , which is

$$\Upsilon : S \times [-0.5, 0.5] \longrightarrow [0, \ell], \tag{3}$$

$$\Upsilon (\lfloor \cdot \rfloor_r, \Delta) = r + \Delta = \vartheta. \tag{4}$$

**Definition 4<sup>32</sup>** A  $q$ -RFLS  $\mathcal{F}$  on a universal set  $Z$  is described as

$$\mathcal{F} = \left\{ \left( \left( \lfloor \cdot \rfloor_{r(z_i)}, \Delta (z_i) \right), \langle m(z_i), n(z_i) \rangle \right) \right\}, \tag{5}$$

where  $\lfloor \cdot \rfloor_{r(z_i)} \in S, \Delta (z_i) \in [-0.5, 0.5], m(z_i), n(z_i) \in [0, 1]$ , with the restriction  $0 \leq m^q(z_i) + n^q(z_i) \leq 1 (q \geq 1) \forall z_i \in Z$ . The numbers  $m(z_i), n(z_i)$  symbolize the MG and NMG of the element  $z_i$  to linguistic variable  $(\lfloor \cdot \rfloor_{r(z_i)}, \Delta (z_i))$ , respectively. Furthermore,  $\pi_{\mathcal{F}}(z_i) = 1 - (m^q(z_i) + n^q(z_i))^{1/q}$  is named refusal grade, and the  $q$ -rung orthopair fuzzy 2-tuple linguistic number ( $q$ -RFLN) is marked by  $\vartheta = ((\lfloor \cdot \rfloor_r, \Delta), \langle m, n \rangle)$ .

**Definition 5<sup>32</sup>** Let  $\vartheta_1 = ((\lfloor \cdot \rfloor_{r_1}, \Delta_1), \langle m_1, n_1 \rangle)$  and  $\vartheta_2 = ((\lfloor \cdot \rfloor_{r_2}, \Delta_2), \langle m_2, n_2 \rangle)$  be two  $q$ -RFLNs. Then, their basic operational rules are listed as:

1.  $\vartheta_1 \oplus \vartheta_2 = \left( \lambda (\Upsilon (\lfloor \cdot \rfloor_{r_1}, \Delta_1) + \Upsilon (\lfloor \cdot \rfloor_{r_2}, \Delta_2)), \left\langle (m_1^q + m_2^q - m_1^q m_2^q)^{1/q}, n_1 n_2 \right\rangle \right)$ ;
2.  $\vartheta_1 \otimes \vartheta_2 = \left( \lambda (\Upsilon (\lfloor \cdot \rfloor_{r_1}, \Delta_1) \cdot \Upsilon (\lfloor \cdot \rfloor_{r_2}, \Delta_2)), \left\langle m_1 m_2, (n_1^q + n_2^q - n_1^q n_2^q)^{1/q} \right\rangle \right)$ ;
3.  $\alpha \vartheta_1 = \left( \lambda (\alpha \Upsilon (\lfloor \cdot \rfloor_{r_1}, \Delta_1)), \left\langle \left( 1 - (1 - m_1^q)^\alpha \right)^{1/q}, n_1^\alpha \right\rangle \right) \alpha > 0$ ;
4.  $\vartheta_1^\alpha = \left( \lambda ((\Upsilon (\lfloor \cdot \rfloor_{r_1}, \Delta_1))^\alpha), \left\langle m_1^\alpha, \left( 1 - (1 - n_1^q)^\alpha \right)^{1/q} \right\rangle \right) \alpha > 0$ ;
5.  $\vartheta_1^c = ((\lfloor \cdot \rfloor_{r_1}, \Delta_1), \langle n_1, m_1 \rangle)$ .

**Definition 6<sup>32</sup>** Let  $\vartheta_1 = ((\lfloor \cdot \rfloor_{r_1}, \Delta_1), \langle m_1, n_1 \rangle)$  be a  $q$ -RFLN. Then its score function  $\widehat{S}(\vartheta_1)$  and accuracy function  $\widehat{A}(\vartheta_1)$  are described as:

$$\widehat{S}(\vartheta_1) = \frac{\Upsilon (\lfloor \cdot \rfloor_{r_1}, \Delta_1) \cdot (1 + m_1^q - n_1^q)}{2\ell}, \tag{6}$$

$$\widehat{A}(\vartheta_1) = \Upsilon (\lfloor \cdot \rfloor_{r_1}, \Delta_1) \cdot (m_1^q + n_1^q). \tag{7}$$

**Definition 7<sup>32</sup>** Let  $\vartheta_1 = ((\lfloor \cdot \rfloor_{r_1}, \Delta_1), \langle m_1, n_1 \rangle)$  and  $\vartheta_2 = ((\lfloor \cdot \rfloor_{r_2}, \Delta_2), \langle m_2, n_2 \rangle)$  be two  $q$ -RFLNs. Then, they can be compared according to the following laws:

1. If  $\widehat{S}(\vartheta_1) > \widehat{S}(\vartheta_2)$ , then  $\vartheta_1 \succ \vartheta_2$ ;
2. If  $\widehat{S}(\vartheta_1) = \widehat{S}(\vartheta_2)$ , then:
  - (i) if  $\widehat{A}(\vartheta_1) > \widehat{A}(\vartheta_2)$ , then  $\vartheta_1 \prec \vartheta_2$ ;
  - (ii) if  $\widehat{A}(\vartheta_1) = \widehat{A}(\vartheta_2)$ , then  $\vartheta_1 = \vartheta_2$ .

**Definition 8<sup>48</sup>** Let  $\vartheta_1 = ((\lfloor \cdot \rfloor_{r_1}, \Delta_1), \langle m_1, n_1 \rangle)$  and  $\vartheta_2 = ((\lfloor \cdot \rfloor_{r_2}, \Delta_2), \langle m_2, n_2 \rangle)$  be two  $q$ -RFLNs. Then the Hamming distance between  $\vartheta_1$  and  $\vartheta_2$  is defined as

$$d(\vartheta_1, \vartheta_2) = \frac{1}{2\ell} \left[ \left| (1 + m_1^q - n_1^q) \cdot \Upsilon (\lfloor \cdot \rfloor_{r_1}, \Delta_1) - (1 + m_2^q - n_2^q) \cdot \Upsilon (\lfloor \cdot \rfloor_{r_2}, \Delta_2) \right| \right]. \tag{8}$$

**Definition 9<sup>40</sup>** Let  $\mathcal{E} = \{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$  be a collection of criteria with a prioritization relationship  $\epsilon_1 \succ \epsilon_2 \succ \dots \succ \epsilon_n$ , where  $\succ$  means ‘‘higher priority’’. The value  $\epsilon_j(a)$  represents the criterion value of the alternative  $a$  with respect to criteria  $\epsilon_j$  and satisfies the condition  $\epsilon_j(a) \in [0, 1]$ . If

$$PA(\mathcal{E}_i(a)) = \sum_{g=1}^n w_g \mathcal{E}_g(a), \tag{9}$$

then PA is called the prioritized averaging operator, where  $w_g = \frac{T_g}{\sum_{g=1}^n T_g}, T_g = \prod_{t=1}^{g-1} \epsilon_t(a) (g = 2, \dots, n)$  and  $T_1 = 1$ .

**Definition 10<sup>35</sup>** Let  $\mathcal{N}_i (i = 1, 2, \dots, \mathfrak{h})$  be a set of non-negative numbers, then the MSM operator is characterized as follows:

$$MSM^{(b)}(\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_{\mathfrak{h}}) = \left( \frac{\sum_{1 \leq i_1 \leq \dots \leq i_p \leq \mathfrak{h}} \left( \prod_{g=1}^p \mathcal{N}_{i_g} \right)^{\frac{1}{p}}}{C_{\mathfrak{h}}^b} \right)^{\frac{1}{p}} \tag{10}$$

where  $b$  is a parameter and  $b = 1, 2, \dots, \mathfrak{h}, i_1, i_2, \dots, i_p$  are  $b$  integer values taken from the collection  $\{1, 2, \dots, \mathfrak{h}\}$  of  $\mathfrak{h}$  integer values,  $C_{\mathfrak{h}}^b$  represent the binomial coefficient.

The MSM operator exhibits the following characteristics:

1.  $MSM^{(b)}(0, 0, \dots, 0) = 0$ ;
2.  $MSM^{(b)}(\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_{\mathfrak{h}}) = \mathcal{N}$ , if  $\mathcal{N}_i = \mathcal{N} (i = 1, 2, \dots, \mathfrak{h})$ ;
3.  $MSM^{(b)}(\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_{\mathfrak{h}}) \leq MSM^{(b)}(\check{\mathcal{N}}_1, \check{\mathcal{N}}_2, \dots, \check{\mathcal{N}}_{\mathfrak{h}})$ , if  $\mathcal{N}_i \leq \check{\mathcal{N}}_i$  for all  $i$ .
4.  $\min_i \{\mathcal{N}_i\} \leq MSM^{(b)}(\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_{\mathfrak{h}}) \leq \max_i \{\mathcal{N}_i\}$ .

According to the MSM operator’s definition, Qin and Liu<sup>37</sup> proposed its dual form, as articulated below.

**Definition 11<sup>37</sup>** Let  $\mathcal{N}_i (i = 1, 2, \dots, \mathfrak{h})$  be a set of non-negative numbers, then the DMSM operator is characterized as follows:

$$DMSM^{(b)}(\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_{\mathfrak{h}}) = \frac{1}{b} \left( \prod_{1 \leq i_1 < \dots < i_p \leq \mathfrak{h}} \left( \sum_{g=1}^p \mathcal{N}_{i_g} \right)^{\frac{1}{p}} \right)^{\frac{1}{b}} \tag{11}$$

where  $b$  is a parameter and  $b = 1, 2, \dots, \mathfrak{h}, i_1, i_2, \dots, i_p$  are  $b$  integer values taken from the collection  $\{1, 2, \dots, \mathfrak{h}\}$  of  $\mathfrak{h}$  integer values,  $C_{\mathfrak{h}}^b$  represent the binomial coefficient.

Similarly to the MSM operator, the DMSM operator also exhibits the aforementioned characteristics.

### q-RF2L prioritized aggregation operators

This section introduces the concept of the  $q$ -RF2LPMSM operator and the  $q$ -RF2LPDMSM operator. We delve into a comprehensive discussion of other appealing properties inherent to these operators, such as boundary and monotonicity, along with a detailed exploration of their specific instances.

### q-RF2L prioritized Maclaurin symmetric mean aggregation operators

Considering the context provided by Definitions 9 and 10, the definition of the  $q$ -RF2LPMSM operator is presented as follows:

**Definition 12** Let  $\partial_i = ((\llbracket r_i, \varkappa_i \rrbracket), \langle m_i, n_i \rangle) (i = 1, 2, \dots, \mathfrak{h})$  be a family of  $q$ -RF2LNs, then  $q$ -rung orthopair fuzzy 2-tuple linguistic prioritized Maclaurin symmetric mean ( $q$ -RF2LPMSM) is given by

$$q-RF2LPMSM^{(b)}(\partial_1, \partial_2, \dots, \partial_{\mathfrak{h}}) = \left( \frac{\oplus_{1 \leq i_1 < \dots < i_p \leq \mathfrak{h}} \left( \otimes_{g=1}^p \partial_{i_g}^{\mathfrak{q}_{i_g}} \right)^{\frac{1}{p}}}{C_{\mathfrak{h}}^b} \right)^{\frac{1}{b}} \tag{12}$$

where  $\mathfrak{q}_{i_g} = \frac{\mathcal{T}_{i_g}}{\sum_{g=1}^p \mathcal{T}_{i_g}}$ ,  $\mathcal{T}_{i_g} = \prod_{t=1}^{i-1} \mathcal{S}(\partial_t)$  ( $i = 2, \dots, \mathfrak{h}$ ),  $\mathcal{T}_1 = 1$ ,  $\mathcal{S}(\partial_t)$  is the score function value of  $\partial_t$  and  $b = 1, 2, \dots, \mathfrak{h}, i_1, i_2, \dots, i_p$  are  $b$  integer values taken from the collection  $\{1, 2, \dots, \mathfrak{h}\}$  of  $\mathfrak{h}$  integer values,  $C_{\mathfrak{h}}^b$  represent the binomial coefficient.

**Theorem 1** Let  $\partial_i = ((\llbracket r_i, \varkappa_i \rrbracket), \langle m_i, n_i \rangle) (i = 1, 2, \dots, \mathfrak{h})$  be a family of  $q$ -RF2LNs, where  $b = 1, 2, \dots, \mathfrak{h}$ , then the aggregated value by using the  $q$ -RF2LPMSM operator is also a  $q$ -RF2LN, and

$$q - RF2LPMSM^{(b)}(\partial_1, \partial_2, \dots, \partial_{\hbar}) = \left( \frac{\oplus_{1 \leq i_1 < \dots < i_b \leq \hbar} (\otimes_{g=1}^b \partial_{i_g}^{\mathbb{N}_{i_g}})}{C_{\hbar}^b} \right)^{\frac{1}{b}} = \left( \begin{array}{c} \wedge \left( \left( \frac{1}{C_{\hbar}^b} \sum_{\Omega} \prod_{g=1}^b (\vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g}))^{\mathbb{N}_{i_g}} \right)^{\frac{1}{b}} \right), \\ \left( \left( 1 - \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^b (m_{i_g}^{\mathbb{N}_{i_g}})^q \right)^{\frac{1}{C_{\hbar}^b}} \right)^{\frac{1}{q}} \right)^{\frac{1}{b}}, \\ \left( 1 - \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^b (1 - n_{i_g}^q)^{\mathbb{N}_{i_g}} \right)^{\frac{1}{C_{\hbar}^b}} \right)^{\frac{1}{q}} \right)^{\frac{1}{b}} \right) \end{array} \right)^{\frac{1}{q}} \tag{13}$$

where  $\Omega = 1 \leq i_1 < \dots < i_b \leq \hbar$  and  $\mathbb{N}_{i_g} = \frac{\mathcal{T}_{i_g}}{\sum_{g=1}^b \mathcal{T}_{i_g}}$ ,  $\mathcal{T}_{i_g} = \prod_{t=1}^{i-1} \mathcal{S}(\partial_t)(i = 2, \dots, \hbar)$ ,  $\mathcal{T}_1 = 1$ ,  $\mathcal{S}(\partial_t)$  is the score function value of  $\partial_t$  and  $b = 1, 2, \dots, \hbar, i_1, i_2, \dots, i_b$  are  $b$  integer values taken from the collection  $\{1, 2, \dots, \hbar\}$  of  $\hbar$  integer values,  $C_{\hbar}^b$  represent the binomial coefficient.

**Proof** By using operational laws, we formulate the aggregated results of  $q$ -RF2LPMSM as follows:

$$\begin{aligned} \partial_{i_g}^{\mathbb{N}_{i_g}} &= \left( \wedge \left( (\vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g}))^{\mathbb{N}_{i_g}} \right), \left\langle m_{i_g}^{\mathbb{N}_{i_g}}, \left( 1 - (1 - n_{i_g}^q)^{\mathbb{N}_{i_g}} \right)^{\frac{1}{q}} \right\rangle \right) \\ \otimes_{g=1}^b \partial_{i_g}^{\mathbb{N}_{i_g}} &= \left( \wedge \left( \prod_{g=1}^b (\vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g}))^{\mathbb{N}_{i_g}} \right), \left\langle \prod_{g=1}^b m_{i_g}^{\mathbb{N}_{i_g}}, \left( 1 - \prod_{g=1}^b (1 - n_{i_g}^q)^{\mathbb{N}_{i_g}} \right)^{\frac{1}{q}} \right\rangle \right) \\ \oplus_{\Omega} \otimes_{g=1}^b \partial_{i_g}^{\mathbb{N}_{i_g}} &= \left( \wedge \left( \sum_{\Omega} \left( \prod_{g=1}^b (\vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g}))^{\mathbb{N}_{i_g}} \right) \right), \left\langle \left( 1 - \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^b m_{i_g}^{\mathbb{N}_{i_g}} \right)^q \right) \right)^{\frac{1}{q}}, \right. \\ &\quad \left. \prod_{\Omega} \left( 1 - \prod_{g=1}^b (1 - n_{i_g}^q)^{\mathbb{N}_{i_g}} \right)^{\frac{1}{q}} \right\rangle \right) \\ \frac{1}{C_{\hbar}^b} (\oplus_{\Omega} (\otimes_{g=1}^b \partial_{i_g}^{\mathbb{N}_{i_g}})) &= \left( \wedge \left( \frac{1}{C_{\hbar}^b} \sum_{\Omega} \left( \prod_{g=1}^b (\vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g}))^{\mathbb{N}_{i_g}} \right) \right), \right. \\ &\quad \left. \left\langle \left( 1 - \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^b m_{i_g}^{\mathbb{N}_{i_g}} \right)^q \right)^{\frac{1}{C_{\hbar}^b}} \right)^{\frac{1}{q}}, \right. \right. \\ &\quad \left. \left. \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^b (1 - n_{i_g}^q)^{\mathbb{N}_{i_g}} \right)^{\frac{1}{C_{\hbar}^b}} \right)^{\frac{1}{q}} \right\rangle \right) \\ \left( \frac{1}{C_{\hbar}^b} \oplus_{\Omega} (\otimes_{g=1}^b \partial_{i_g}^{\mathbb{N}_{i_g}}) \right)^{\frac{1}{b}} &= \left( \wedge \left( \left( \frac{1}{C_{\hbar}^b} \sum_{\Omega} \prod_{g=1}^b (\vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g}))^{\mathbb{N}_{i_g}} \right)^{\frac{1}{b}} \right), \right. \\ &\quad \left( \left( 1 - \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^b m_{i_g}^{\mathbb{N}_{i_g}} \right)^q \right)^{\frac{1}{C_{\hbar}^b}} \right)^{\frac{1}{q}} \right)^{\frac{1}{b}}, \\ &\quad \left( 1 - \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^b (1 - n_{i_g}^q)^{\mathbb{N}_{i_g}} \right) \right)^{\frac{1}{C_{\hbar}^b}} \right)^{\frac{1}{q}} \right)^{\frac{1}{b}} \right) \end{aligned}$$

□

**Theorem 2** Let  $\partial_i = (\mathbb{I}_{r_i}, \mathcal{X}_i), \langle m_i, n_i \rangle (i = 1, 2, \dots, \tau)$  and  $\check{\partial}_i = (\check{\mathbb{I}}_{r_i}, \check{\mathcal{X}}_i), \langle \check{m}_i, \check{n}_i \rangle (i = 1, 2, \dots, \tau)$  be two collections of  $q$ -RF2LNs, if  $\partial_i \leq \check{\partial}_i, \forall i$ . Then

$$q - RF2LPMSM(\partial_1, \partial_2, \dots, \partial_\tau) \leq q - RF2LPMSM(\check{\partial}_1, \check{\partial}_2, \dots, \check{\partial}_\tau). \tag{14}$$

**Proof** Because  $\partial_i \leq \check{\partial}_i$ , then we have  $(\mathbb{I}_{r_i}, \mathcal{X}_i) \leq (\check{\mathbb{I}}_{r_i}, \check{\mathcal{X}}_i), m_i \leq \check{m}_i$  and  $n_i \geq \check{n}_i \forall (i = 1, 2, \dots, \tau)$ . Then

$$\left( \frac{1}{C_\tau^b} \sum_{g=1}^b \prod_{\Omega} (\vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g}))^{q_{i_g}} \right)^{\frac{1}{b}} \leq \left( \frac{1}{C_\tau^b} \sum_{g=1}^b \prod_{\Omega} (\vee (\check{\mathbb{I}}_{r_{i_g}}, \check{\mathcal{X}}_{i_g}))^{q_{i_g}} \right)^{\frac{1}{b}}.$$

Further, since  $m_i \leq \check{m}_i$ , we have

$$\begin{aligned} \left( \prod_{i=1}^b m_i^{q_{i_g}} \right)^q &\leq \left( \prod_{i=1}^b \check{m}_i^{q_{i_g}} \right)^q \Rightarrow \prod_{\Omega} \left( 1 - \left( \prod_{i=1}^b \check{m}_i^{q_{i_g}} \right)^q \right) \geq \prod_{\Omega} \left( 1 - \left( \prod_{i=1}^b m_i^{q_{i_g}} \right)^q \right) \\ &\Rightarrow 1 - \left( \prod_{\Omega} \left( 1 - \left( \prod_{i=1}^b m_i^{q_{i_g}} \right)^q \right) \right)^{\frac{1}{C_\tau^b}} \leq 1 - \left( \prod_{\Omega} \left( 1 - \left( \prod_{i=1}^b \check{m}_i^{q_{i_g}} \right)^q \right) \right)^{\frac{1}{C_\tau^b}} \\ &\Rightarrow \left( 1 - \left( \prod_{\Omega} \left( 1 - \left( \prod_{i=1}^b m_i^{q_{i_g}} \right)^q \right) \right)^{\frac{1}{C_\tau^b}} \right)^{\frac{1}{b}} \leq \left( 1 - \left( \prod_{\Omega} \left( 1 - \left( \prod_{i=1}^b \check{m}_i^{q_{i_g}} \right)^q \right) \right)^{\frac{1}{C_\tau^b}} \right)^{\frac{1}{b}}. \end{aligned}$$

Similar to the testify of non-membership degrees, we attain

$$\left( 1 - \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{i=1}^b (1 - n_i^{q_{i_g}}) \right) \right)^{\frac{1}{C_\tau^b}} \right)^{\frac{1}{b}} \right)^q \geq \left( 1 - \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{i=1}^b (1 - (\check{n}_i)^{q_{i_g}}) \right) \right)^{\frac{1}{C_\tau^b}} \right)^{\frac{1}{b}} \right)^q.$$

From the above inequalities, we attain

$$q - RF2LPMSM(\partial_1, \partial_2, \dots, \partial_\tau) \leq q - RF2LPMSM(\check{\partial}_1, \check{\partial}_2, \dots, \check{\partial}_\tau).$$

That proves the monotonicity of  $q$ -RF2LPMSM operator. □

**Theorem 3** Let  $\partial_i (i = 1, 2, \dots, \tau)$  be a family of  $q$ -RF2LNs. If  $\partial_i^+ = \left( \max_{1 \leq i \leq \tau} (\mathbb{I}_{r_i}, \Delta_i), \left\langle \max_{1 \leq i \leq \tau} m_i, \min_{1 \leq i \leq \tau} n_i \right\rangle \right)$  and

$\partial_i^- = \left( \min_{1 \leq i \leq \tau} (\mathbb{I}_{r_i}, \Delta_i), \left\langle \min_{1 \leq i \leq \tau} m_i, \max_{1 \leq i \leq \tau} n_i \right\rangle \right)$ , then

$$q - RF2LPMSM(\partial_1^-, \partial_2^-, \dots, \partial_\tau^-) \leq q - RF2LPMSM(\partial_1, \partial_2, \dots, \partial_\tau) \leq q - RF2LPMSM(\partial_1^+, \partial_2^+, \dots, \partial_\tau^+). \tag{15}$$

**Proof** According to Theorem 1, we have

$$q - RF2LPMSM^{(b)}(\partial_1^-, \partial_2^-, \dots, \partial_{\hbar}^-) = \left( \begin{array}{c} \wedge \left( \left( \frac{1}{C_{\hbar}^b} \sum_{\Omega} \prod_{g=1}^b (\min \vee (\mathbb{I}_{r_{ig}}, \mathcal{N}_{ig}))^{q_{ig}} \right)^{\frac{1}{b}} \right), \\ \left\langle \left( \left( 1 - \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^b (\min n_{ig})^{q_{ig}} \right)^{\frac{1}{q}} \right)^{\frac{1}{b}} \right)^{\frac{1}{q}} \right), \right. \\ \left. \left( 1 - \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^b (1 - \max n_{ig})^{q_{ig}} \right) \right)^{\frac{1}{b}} \right)^{\frac{1}{q}} \right) \right\rangle \end{array} \right)$$

$$q - RF2LPMSM^{(b)}(\partial_1^+, \partial_2^+, \dots, \partial_{\hbar}^+) = \left( \begin{array}{c} \wedge \left( \left( \frac{1}{C_{\hbar}^b} \sum_{\Omega} \prod_{g=1}^b (\max \vee (\mathbb{I}_{r_{ig}}, \mathcal{N}_{ig}))^{q_{ig}} \right)^{\frac{1}{b}} \right), \\ \left\langle \left( \left( 1 - \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^b (\max n_{ig})^{q_{ig}} \right)^{\frac{1}{q}} \right)^{\frac{1}{b}} \right)^{\frac{1}{q}} \right), \right. \\ \left. \left( 1 - \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^b (1 - \min n_{ig})^{q_{ig}} \right) \right)^{\frac{1}{b}} \right)^{\frac{1}{q}} \right) \right\rangle \end{array} \right)$$

From Theorem 2, we have

$$q - RF2LPMSM(\partial_1^-, \partial_2^-, \dots, \partial_{\hbar}^-) \leq q - RF2LPMSM(\partial_1, \partial_2, \dots, \partial_{\hbar}) \leq q - RF2LPMSM(\partial_1^+, \partial_2^+, \dots, \partial_{\hbar}^+). \quad \square$$

In the next, several novel operators shall be attained through assigning diverse values of  $b$ .

1. If  $b = 1$ , the  $q$ -RF2LPMSM operator is yielded to  $q$ -RF2LP averaging ( $q$ -RF2LPA) operator, displayed as below:

$$\begin{aligned} q - RF2LPMSM^{(1)}(\partial_1, \partial_2, \dots, \partial_{\hbar}) &= \frac{1}{\oplus_{1 \leq i_1 \leq \hbar} \partial_{i_1}^{q_{i_1}}} \\ &= \frac{1}{\hbar} (\oplus_{i=1}^{\hbar} \partial_i^{q_i}) (\text{let } i_1 = i) = \left( \begin{array}{c} \wedge \left( \frac{1}{\hbar} \sum_{i=1}^{\hbar} (\vee (\mathbb{I}_{r_i}, \mathcal{N}_i))^{q_i} \right), \\ \left\langle \left( \left( 1 - \prod_{i=1}^{\hbar} \left( 1 - (n_i^{q_i}) \right)^{\frac{1}{q}} \right)^{\frac{1}{\hbar}} \right), \prod_{i=1}^{\hbar} (n_i^{q_i})^{\frac{1}{\hbar}} \right\rangle \end{array} \right) \end{aligned} \quad (16)$$

2. If  $b = 2$ , the  $q$ -RF2LPMSM operator is yielded to  $q$ -RF2LP Bonferroni mean ( $q$ -RF2LPBM) operator, displayed as below:

$$q - RF2LPMSM^{(2)}(\partial_1, \partial_2, \dots, \partial_{\hbar}) = \left( \frac{\oplus_{1 \leq i_1 < i_2 \leq \hbar} (\otimes_{g=1}^2 \partial_{i_g}^{q_{i_g}})}{C_{\hbar}^2} \right)^{\frac{1}{2}} = \left( \frac{1}{\hbar(\hbar-1)} \oplus_{\substack{i_1, i_2=1, \\ i_1 \neq i_2}}^b (\partial_{i_1}^{q_{i_1}} \otimes \partial_{i_2}^{q_{i_2}}) \right)^{\frac{1}{2}}$$



$$= \left( \begin{array}{c} \wedge \left( \frac{1}{\hbar(\hbar-1)} \left( \sum_{\substack{i_1, i_2=1, \\ i_1 \neq i_2}}^{\hbar} (\vee (\mathbb{I}_{r_{i_1}}, \mathcal{X}_{i_1}))^{\mathbb{I}_{i_1}} (\vee (\mathbb{I}_{r_{i_2}}, \mathcal{X}_{i_2}))^{\mathbb{I}_{i_2}} \right)^{\frac{1}{2}} \right), \\ \left( \left( 1 - \left( \prod_{\substack{i_1, i_2=1, \\ i_1 \neq i_2}}^{\hbar} (1 - (m_{i_1}^q)^{\mathbb{I}_{i_1}} (m_{i_2}^q)^{\mathbb{I}_{i_2}}) \right)^{\frac{2}{\hbar(\hbar-1)}} \right)^{\frac{1}{q}} \right)^{\frac{1}{2}}, \\ \left\langle \left( 1 - \left( 1 - \left( \prod_{\substack{i_1, i_2=1, \\ i_1 \neq i_2}}^{\hbar} (1 - (1 - n_{i_1}^q)^{\mathbb{I}_{i_1}} (1 - n_{i_2}^q)^{\mathbb{I}_{i_2}}) \right)^{\frac{1}{\hbar(\hbar-1)}} \right)^{\frac{1}{2}} \right)^{\frac{1}{q}} \right\rangle \end{array} \right). \tag{17}$$

3. If  $\mathfrak{p} = 3$ , the  $q$ -RF2LPMSM operator is yielded to  $q$ -RF2LP generalized Bonferroni mean ( $q$ -RF2LPGBM) operator, displayed as below:

$$q - RF2LPMSM^{(3)}(\partial_1, \partial_2, \dots, \partial_{\hbar}) = \left( \frac{\bigoplus_{\substack{1 \leq i_1 < i_2 \\ < i_3 \leq \hbar}}^3 (\otimes_{g=1}^3 \partial_{i_g}^{\mathbb{I}_{i_g}})}{C_{\hbar}^3} \right)^{\frac{1}{3}} = \tag{18}$$

$$\left( \frac{1}{\hbar(\hbar-1)(\hbar-2)} \bigoplus_{\substack{i_1, i_2, i_3=1, \\ i_1 \neq i_2 \neq i_3}}^{\hbar} (\partial_{i_1}^{\mathbb{I}_{i_1}} \otimes \partial_{i_2}^{\mathbb{I}_{i_2}} \otimes \partial_{i_3}^{\mathbb{I}_{i_3}}) \right)^{\frac{1}{3}} = q - RF2LPBM^{(1,1,1)}(\partial_1, \partial_2, \dots, \partial_{\hbar}).$$

4. If  $\mathfrak{p} = \hbar$ , the  $q$ -RF2LPMSM operator is yielded to  $q$ -RF2LP geometric mean ( $q$ -RF2LPGM) operator, displayed as below:

$$q - RF2LPMSM^{(\hbar)}(\partial_1, \partial_2, \dots, \partial_{\hbar}) = \left( \otimes_{i=1}^{\hbar} \partial_i^{\mathbb{I}_i} \right)^{\frac{1}{\hbar}} = \left( \begin{array}{c} \wedge \left( \left( \prod_{i=1}^{\hbar} (\vee (\mathbb{I}_{r_i}, \mathcal{X}_i))^{\mathbb{I}_i} \right)^{\frac{1}{\hbar}} \right), \\ \left\langle \left( \prod_{i=1}^{\hbar} m_i^{\mathbb{I}_i} \right)^{\frac{1}{\hbar}}, \left( 1 - \left( \prod_{i=1}^{\hbar} (1 - n_i^q)^{\mathbb{I}_i} \right)^{\frac{1}{\hbar}} \right)^{\frac{1}{q}} \right\rangle \end{array} \right). \tag{19}$$

**q-RF2L prioritized dual Maclaurin symmetric mean aggregation operators**

**Definition 13** Let  $\partial_i = ((\mathbb{I}_{r_i}, \mathcal{X}_i), (m_i, n_i)) (i = 1, 2, \dots, \hbar)$  be a family of  $q$ -RF2LNs, then  $q$ -rung orthopair fuzzy 2-tuple linguistic prioritized dual Maclaurin symmetric mean ( $q$ -RF2LPDMSM) is given by

$$q - RF2LPDMSM^{(b)}(\partial_1, \partial_2, \dots, \partial_{\hbar}) = \frac{1}{b} \left( \otimes_{1 \leq i_1 < \dots < i_b \leq \hbar}^b (\bigoplus_{g=1}^b \partial_{i_g}^{\mathbb{I}_{i_g}})^{\frac{1}{C_{\hbar}^b}} \right), \tag{20}$$

where  $\mathbb{I}_{i_g} = \frac{\mathcal{T}_{i_g}}{\sum_{g=1}^{\hbar} \mathcal{T}_{i_g}}$ ,  $\mathcal{T}_{i_g} = \prod_{t=1}^{i-1} \widehat{S}(\partial_t) (i = 2, \dots, \hbar)$ ,  $\mathcal{T}_1 = 1$ ,  $\widehat{S}(\partial_t)$  is the score function value of  $\partial_t$  and

$b = 1, 2, \dots, \hbar, i_1, i_2, \dots, i_b$  are  $b$  integer values taken from the collection  $\{1, 2, \dots, \hbar\}$  of  $\hbar$  integer values,  $C_{\hbar}^b$  represent the binomial coefficient.

**Theorem 4** Let  $\partial_i = ((\mathbb{I}_{r_i}, \mathcal{X}_i), (m_i, n_i)) (i = 1, 2, \dots, \hbar)$  be a family of  $q$ -RF2LNs, and  $p = 1, 2, \dots, \hbar$ , then the aggregated value by using the  $q$ -RF2LPDMSM operator is also a  $q$ -RF2LN, and

$$q\text{-RF2LPDMSM}^{(p)}(\partial_1, \partial_2, \dots, \partial_\hbar) = \frac{1}{p} \left( (\otimes_{1 \leq i_1 < \dots < i_p \leq \hbar} (\oplus_{g=1}^p \mathbb{I}_{i_g} \partial_{i_g}))^{\frac{1}{C_{\hbar}^p}} \right) = \left( \begin{array}{l} \wedge \left( \frac{1}{p} \left( \prod_{g=1}^p \left( \sum_{i_g} \mathbb{I}_{i_g} \vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g}) \right)^{\frac{1}{C_{\hbar}^p}} \right) \right), \\ \left\langle \left( 1 - \left( 1 - \left( \prod_{g=1}^p \left( 1 - \prod_{i_g} (1 - m_{i_g}^q)^{\mathbb{I}_{i_g}} \right) \right)^{\frac{1}{C_{\hbar}^p}} \right)^{\frac{1}{p}} \right)^{\frac{1}{q}}, \right. \\ \left. \left( \left( 1 - \left( \prod_{g=1}^p \left( 1 - \left( \prod_{i_g} n_{i_g}^q \right)^q \right)^{\frac{1}{C_{\hbar}^p}} \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \right) \right\rangle \end{array} \right), \tag{21}$$

where  $\Omega = 1 \leq i_1 < \dots < i_p \leq \hbar$  and  $\mathbb{I}_{i_g} = \frac{\mathcal{T}_{i_g}}{\sum_{g=1}^p \mathcal{T}_{i_g}}$ ,  $\mathcal{T}_{i_g} = \prod_{t=1}^{i-1} \mathcal{S}(\partial_t) (i = 2, \dots, \hbar)$ ,  $\mathcal{T}_1 = 1$ ,  $\mathcal{S}(\partial_t)$  is the score function value of  $\partial_t$  and  $p = 1, 2, \dots, \hbar, i_1, i_2, \dots, i_p$  are  $p$  integer values taken from the collection  $\{1, 2, \dots, \hbar\}$  of  $\hbar$  integer values,  $C_{\hbar}^p$  represent the binomial coefficient.

**Proof** By using operational laws, we formulate the aggregated results of  $q$ -RF2LPDMSM as follows:

$$\begin{aligned} \mathbb{I}_{i_g} \partial_{i_g} &= \left( \wedge (\mathbb{I}_{i_g} (\vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g}))), \left\langle \left( 1 - (1 - m_{i_g}^q)^{\mathbb{I}_{i_g}} \right)^{\frac{1}{q}}, n_{i_g}^{\mathbb{I}_{i_g}} \right\rangle \right) \\ \oplus_{g=1}^p \mathbb{I}_{i_g} \partial_{i_g} &= \left( \wedge \left( \sum_{g=1}^p \mathbb{I}_{i_g} (\vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g})) \right), \left\langle \left( 1 - \prod_{g=1}^p (1 - m_{i_g}^q)^{\mathbb{I}_{i_g}} \right)^{\frac{1}{q}}, \prod_{g=1}^p n_{i_g}^{\mathbb{I}_{i_g}} \right\rangle \right) \\ \otimes_{\Omega} \oplus_{g=1}^p \mathbb{I}_{i_g} \partial_{i_g} &= \left( \wedge \left( \prod_{\Omega} \left( \sum_{g=1}^p \mathbb{I}_{i_g} (\vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g})) \right) \right), \left\langle \frac{\prod_{\Omega} \left( 1 - \prod_{g=1}^p (1 - m_{i_g}^q)^{\mathbb{I}_{i_g}} \right)^{\frac{1}{q}}}{\left( 1 - \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^p n_{i_g}^q \right)^q \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}} \right\rangle \right) \\ \left( \otimes_{\Omega} \left( \oplus_{g=1}^p \mathbb{I}_{i_g} \partial_{i_g} \right) \right)^{\frac{1}{C_{\hbar}^p}} &= \left( \wedge \left( \left( \prod_{\Omega} \left( \sum_{g=1}^p \mathbb{I}_{i_g} (\vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g})) \right) \right)^{\frac{1}{C_{\hbar}^p}} \right), \right. \\ &\quad \left. \left\langle \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^p (1 - m_{i_g}^q)^{\mathbb{I}_{i_g}} \right)^{\frac{1}{q}} \right)^{\frac{1}{C_{\hbar}^p}}, \right. \right. \\ &\quad \left. \left. \left( 1 - \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^p n_{i_g}^q \right)^q \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} \right\rangle \right) \\ \frac{1}{p} \left( \otimes_{\Omega} \left( \oplus_{g=1}^p \mathbb{I}_{i_g} \partial_{i_g} \right) \right)^{\frac{1}{C_{\hbar}^p}} &= \left( \wedge \left( \frac{1}{p} \left( \prod_{\Omega} \sum_{g=1}^p \mathbb{I}_{i_g} (\vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g})) \right)^{\frac{1}{C_{\hbar}^p}} \right), \right. \\ &\quad \left. \left\langle \left( 1 - \left( 1 - \left( \prod_{g=1}^p \left( 1 - \prod_{i_g} (1 - m_{i_g}^q)^{\mathbb{I}_{i_g}} \right) \right)^{\frac{1}{C_{\hbar}^p}} \right)^{\frac{1}{p}} \right)^{\frac{1}{q}}, \right. \right. \\ &\quad \left. \left. \left( \left( 1 - \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^p n_{i_g}^q \right)^q \right)^{\frac{1}{C_{\hbar}^p}} \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \right\rangle \right) \end{aligned}$$

□

**Theorem 5** Let  $\partial_i = (\mathbb{I}_{r_i}, \mathcal{X}_i), \langle m_i, n_i \rangle (i = 1, 2, \dots, \tau)$  and  $\check{\partial}_i = (\check{\mathbb{I}}_{r_i}, \check{\mathcal{X}}_i), \langle \check{m}_i, \check{n}_i \rangle (i = 1, 2, \dots, \tau)$  be two collections of  $q$ -RF2LNs, if  $\partial_i \leq \check{\partial}_i, \forall i$ . Then

$$q - RF2LPDMSM(\partial_1, \partial_2, \dots, \partial_\tau) \leq q - RF2LPDMSM(\check{\partial}_1, \check{\partial}_2, \dots, \check{\partial}_\tau). \tag{22}$$

**Proof** Because  $\partial_i \leq \check{\partial}_i$ , then we have  $(\mathbb{I}_{r_i}, \mathcal{X}_i) \leq (\check{\mathbb{I}}_{r_i}, \check{\mathcal{X}}_i), m_i \leq \check{m}_i$  and  $n_i \geq \check{n}_i \forall (i = 1, 2, \dots, \tau)$ . Then

$$\wedge \left( \frac{1}{p} \left( \prod_{\Omega} \left( \sum_{g=1}^p \mathbb{I}_{i_g} \vee (\mathbb{I}_{r_{i_g}}, \mathcal{X}_{i_g}) \right)^{\frac{1}{c_{\tau}^p}} \right) \right) \leq \wedge \left( \frac{1}{p} \left( \prod_{\Omega} \left( \sum_{g=1}^p \mathbb{I}_{i_g} \vee (\check{\mathbb{I}}_{r_{i_g}}, \check{\mathcal{X}}_{i_g}) \right)^{\frac{1}{c_{\tau}^p}} \right) \right).$$

Further, since  $m_i \leq \check{m}_i$ , we have

$$\begin{aligned} \prod_{g=1}^p (1 - m_{i_g}^q)^{\mathbb{I}_{i_g}} &\geq \prod_{g=1}^p (1 - \check{m}_{i_g}^q)^{\mathbb{I}_{i_g}} \\ &\Rightarrow \prod_{\Omega} \left( 1 - \prod_{g=1}^p (1 - m_{i_g}^q)^{\mathbb{I}_{i_g}} \right) \leq \prod_{\Omega} \left( 1 - \prod_{g=1}^p (1 - \check{m}_{i_g}^q)^{\mathbb{I}_{i_g}} \right) \\ &\Rightarrow \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^p (1 - m_{i_g}^q)^{\mathbb{I}_{i_g}} \right) \right)^{\frac{1}{c_{\tau}^p}} \right)^{\frac{1}{p}} \geq \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^p (1 - \check{m}_{i_g}^q)^{\mathbb{I}_{i_g}} \right) \right)^{\frac{1}{c_{\tau}^p}} \right)^{\frac{1}{p}} \\ &\Rightarrow \left( 1 - \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^p (1 - m_{i_g}^q)^{\mathbb{I}_{i_g}} \right) \right)^{\frac{1}{c_{\tau}^p}} \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \leq \left( 1 - \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^p (1 - \check{m}_{i_g}^q)^{\mathbb{I}_{i_g}} \right) \right)^{\frac{1}{c_{\tau}^p}} \right)^{\frac{1}{p}} \right)^{\frac{1}{q}}. \end{aligned}$$

Similarly, for non-membership grades, we have

$$\left( \left( 1 - \left( \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^p n_{i_g}^q \right)^{\frac{1}{c_{\tau}^p}} \right) \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \right) \geq \left( \left( 1 - \left( \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^p \check{n}_{i_g}^q \right)^{\frac{1}{c_{\tau}^p}} \right) \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \right).$$

From the above inequalities, we get

$$q - RF2LPDMSM(\partial_1, \partial_2, \dots, \partial_\tau) \leq q - RF2LPDMSM(\check{\partial}_1, \check{\partial}_2, \dots, \check{\partial}_\tau).$$

That proves the monotonicity of  $q$ -RF2LPDMSM operator. □

**Theorem 6** Let  $\partial_i (i = 1, 2, \dots, \tau)$  be a family of  $q$ -RF2LNs. If  $\partial_i^+ = \left( \max_{1 \leq i \leq \tau} (\mathbb{I}_{r_i}, \Delta_i), \left\langle \max_{1 \leq i \leq \tau} m_i, \min_{1 \leq i \leq \tau} n_i \right\rangle \right)$  and

$\partial_i^- = \left( \min_{1 \leq i \leq \tau} (\mathbb{I}_{r_i}, \Delta_i), \left\langle \min_{1 \leq i \leq \tau} m_i, \max_{1 \leq i \leq \tau} n_i \right\rangle \right)$ , then

$$q - RF2LPDMSM(\partial_1^-, \partial_2^-, \dots, \partial_\tau^-) \leq q - RF2LPDMSM(\partial_1, \partial_2, \dots, \partial_\tau) \leq q - RF2LPDMSM(\partial_1^+, \partial_2^+, \dots, \partial_\tau^+). \tag{23}$$

**Proof** According to Theorem 4, we have

$$q - RF2LPDMSM^{(b)}(\partial_1^-, \partial_2^-, \dots, \partial_h^-) = \left( \begin{array}{c} \wedge \left( \frac{1}{b} \left( \prod_{\Omega} \left( \sum_{g=1}^b \mathbb{1}_{i_g} \min v(\mathbb{1}_{r_{i_g}}, \mathcal{X}_{i_g}) \right) \right)^{\frac{1}{c_h^b}} \right), \\ \left\langle \left( 1 - \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^b (1 - \min m_{i_g}^g) \right) \right)^{\frac{1}{c_h^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{q}}, \right\rangle \\ \left( \left( 1 - \left( \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^b \max n_{i_g}^g \right) \right)^q \right) \right)^{\frac{1}{c_h^b}} \right)^{\frac{1}{b}} \end{array} \right)$$

$$q - RF2LPDMSM^{(b)}(\partial_1^+, \partial_2^+, \dots, \partial_h^+) = \left( \begin{array}{c} \wedge \left( \frac{1}{b} \left( \prod_{\Omega} \left( \sum_{g=1}^b \mathbb{1}_{i_g} \max v(\mathbb{1}_{r_{i_g}}, \mathcal{X}_{i_g}) \right) \right)^{\frac{1}{c_h^b}} \right), \\ \left\langle \left( 1 - \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^b (1 - \max m_{i_g}^g) \right) \right)^{\frac{1}{c_h^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{q}}, \right\rangle \\ \left( \left( 1 - \left( \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^b \min n_{i_g}^g \right) \right)^q \right) \right)^{\frac{1}{c_h^b}} \right)^{\frac{1}{b}} \end{array} \right)$$

From Theorem 5, we have

$$q - RF2LPDMSM(\partial_1^-, \partial_2^-, \dots, \partial_h^-) \leq q - RF2LPDMSM(\partial_1, \partial_2, \dots, \partial_h) \leq q - RF2LPDMSM(\partial_1^+, \partial_2^+, \dots, \partial_h^+).$$

□

A few novel operators shall be attained through assigning diverse parameter values of  $b$ .

1. If  $b = 1$ , the  $q$ -RF2LPDMSM operator is yielded to  $q$ -RF2L prioritized geometric mean ( $q - RF2LPGM$ ) operator, displayed as below:

$$q - RF2LPDMSM^{(1)}(\partial_1, \partial_2, \dots, \partial_h) = \left( \otimes_{1 \leq i_1 \leq h} \mathbb{1}_{i_1} \partial_{i_1} \right)^{\frac{1}{h}} = \left( \otimes_{i=1}^h \mathbb{1}_i \partial_i \right)^{\frac{1}{h}} (\text{let } i_1 = i) = \left( \begin{array}{c} \wedge \left( \left( \prod_{i=1}^h \mathbb{1}_i (v(\mathbb{1}_{r_i}, \mathcal{X}_i)) \right)^{\frac{1}{h}} \right), \\ \left\langle \left( \prod_{i=1}^h m_i^g \right)^{\frac{1}{h}}, \left( 1 - \left( \prod_{i=1}^h (1 - n_i^g) \right)^{\frac{1}{h}} \right)^{\frac{1}{q}} \right\rangle \end{array} \right) \tag{24}$$

2. If  $b = 2$ , the  $q$ -RF2LPDMSM operator is yielded to  $q$ -RF2L prioritized geometric Bonferroni mean ( $q$ -RF2LPGBM) operator, displayed as below:

$$\begin{aligned}
 q - RF2LPDMSM^{(2)}(\partial_1, \partial_2, \dots, \partial_{\bar{h}}) &= \frac{1}{\bar{p}} \left( \left( \bigotimes_{\substack{1 \leq i_1 < \dots < i_p \leq \bar{h}}} \left( \bigoplus_{g=1}^{\bar{p}} \mathbb{I}_{i_g} \partial_{i_g} \right) \right)^{\frac{1}{\bar{p}}} \right) = \\
 \frac{1}{2} \left( \left( \bigotimes_{\substack{i_1, i_2=1, \\ i_1 \neq i_2}}^{\bar{p}} (\mathbb{I}_{i_1} \partial_{i_1} \oplus \mathbb{I}_{i_2} \partial_{i_2}) \right)^{\frac{1}{\bar{h}(\bar{h}-1)}} \right) &= \left( \left( \left( \frac{1}{2} \left( \sum_{\substack{i_1, i_2=1, \\ i_1 \neq i_2}}^{\bar{p}} \mathbb{I}_{i_1} (\vee (\mathbb{I}_{r_{i_1}}, \mathcal{X}_{i_1})) + \mathbb{I}_{i_2} (\vee (\mathbb{I}_{r_{i_2}}, \mathcal{X}_{i_2})) \right) \right)^{\frac{1}{\bar{h}(\bar{h}-1)}} \right), \right. \\
 \left. \left( 1 - \left( 1 - \left( \prod_{\substack{i_1, i_2=1, \\ i_1 \neq i_2}}^{\bar{p}} \left( 1 - (1 - m_{i_1}^q) (1 - m_{i_2}^q) \right) \right)^{\frac{1}{\bar{h}(\bar{h}-1)}} \right)^{\frac{1}{2}} \right)^{\frac{1}{q}} \right), \right. \\
 \left. \left( \left( 1 - \left( \prod_{\substack{i_1, i_2=1, \\ i_1 \neq i_2}}^{\bar{p}} \left( 1 - (n_{i_1}^q) (n_{i_2}^q) \right) \right)^{\frac{1}{\bar{h}(\bar{h}-1)}} \right)^{\frac{1}{2}} \right)^{\frac{1}{q}} \right) \right) &= q - RF2LPDMSM^{(1,1)}(\partial_1, \partial_2, \dots, \partial_{\bar{h}}).
 \end{aligned}
 \tag{25}$$

3. If  $\bar{p} = 3$ , the  $q$ -RF2LPDMSM operator is yielded to  $q$ -RF2L prioritized generalized geometric Bonferroni mean ( $q$ -RF2LPGGBM) operator, displayed as below:

$$\begin{aligned}
 q - RF2LPDMSM^{(3)}(\partial_1, \partial_2, \dots, \partial_{\bar{h}}) &= \frac{1}{\bar{p}} \left( \bigotimes_{\substack{1 \leq i_1 < i_2 < i_3 \leq \bar{h}}} \left( \bigoplus_{g=1}^{\bar{p}} \mathbb{I}_{i_g} \partial_{i_g} \right) \right)^{\frac{1}{\bar{p}}} = \\
 \frac{1}{3} \left( \bigotimes_{\substack{i_1, i_2, i_3=1, \\ i_1 \neq i_2 \neq i_3}}^{\bar{p}} (\mathbb{I}_{i_1} \partial_{i_1} \oplus \mathbb{I}_{i_2} \partial_{i_2} \oplus \mathbb{I}_{i_3} \partial_{i_3}) \right)^{\frac{1}{\bar{h}(\bar{h}-1)(\bar{h}-2)}} &= q - RF2LBM^{(1,1,1)}(\partial_1, \partial_2, \dots, \partial_{\bar{h}}).
 \end{aligned}
 \tag{26}$$

4. If  $\bar{p} = \bar{h}$ , the  $q$ -RF2LPDMSM operator is yielded to  $q$ -RF2L prioritized averaging mean ( $q$ -RF2LAM) operator, displayed as below:

$$\begin{aligned}
 q - RF2LPDMSM^{(\bar{h})}(\partial_1, \partial_2, \dots, \partial_{\bar{h}}) &= \frac{1}{\bar{h}} (\bigoplus_{i=1}^{\bar{h}} \partial_i) \\
 &= \left( \left( \frac{1}{\bar{h}} \sum_{i=1}^{\bar{h}} \mathbb{I}_i (\vee (\mathbb{I}_{r_i}, \mathcal{X}_i)) \right), \right. \\
 \left. \left( \left( 1 - \prod_{i=1}^{\bar{h}} (1 - m_i^q) \right)^{\frac{1}{\bar{h}}} \right)^{\frac{1}{q}}, \left( \prod_{i=1}^{\bar{h}} n_i^q \right)^{\frac{1}{\bar{h}}} \right) &= q - RF2LAM(\partial_1, \partial_2, \dots, \partial_{\bar{h}}).
 \end{aligned}
 \tag{27}$$

### A novel MCGDM approach with $q$ -RF2L information

This section elaborates on a novel approach called the  $q$ -RF2LP MCGDM method. We extend the existing WASPAS technique to accommodate situations that involve  $q$ -rung orthopair fuzzy 2-tuple linguistic contexts. Furthermore, we improve the WASPAS method by integrating the  $q$ -RF2LPMSM and  $q$ -RF2LPDMSM operators. These enhancements are made to efficiently tackle  $q$ -RF2LP MCGDM challenges.

#### Description of the MCGDM problem

Let's consider a scenario where we have a finite set of alternatives, denoted as  $\beta = \{\beta_1, \beta_2, \dots, \beta_m\}$ , and a set of criteria, denoted as  $\epsilon = \{\epsilon_1, \epsilon_2, \dots, \epsilon_{\bar{h}}\}$ . These criteria come with priority information indicating their relative importance, where  $\epsilon_1 > \epsilon_2 > \dots > \epsilon_{\bar{h}}$ , signifying that criterion  $\epsilon_g$  holds a higher priority than criterion  $\epsilon_z$  if  $g < z$ . Additionally, we have a group of decision-makers (DMs), denoted as  $\mathbb{C} = \{\mathbb{C}_1, \mathbb{C}_2, \dots, \mathbb{C}_p\}$ , and they are ranked in a linear order of importance:  $\mathbb{C}_1 > \mathbb{C}_2 > \dots > \mathbb{C}_p$ . In this ranking, DM  $\mathbb{C}_z$  holds a higher

priority than DM  $\mathbb{C}_t$  if  $z < t$ . Each DM is tasked with expressing their viewpoints using  $q$ -RF2LNs:  $\partial_{ig}^{(t)} = \left( \left( \mathbb{I}_{r_{ig}}^{(t)}, \Delta_{ig}^{(t)} \right), \left\langle m_{ig}^{(t)}, n_{ig}^{(t)} \right\rangle \right)$ , which represents the  $q$ -RF2L evaluations of the alternatives concerning the criteria, provided by the expert  $\mathbb{C}_t^{(t)}$ . Thus, the  $q$ -RF2LP matrix can be constructed as follows:

$$M_{m \times \mathfrak{h}}^{(t)} = \begin{matrix} & \begin{matrix} \mathbb{E}_1 & \cdots & \mathbb{E}_g & \cdots & \mathbb{E}_{\mathfrak{h}} \end{matrix} \\ \begin{matrix} \beta_1 \\ \vdots \\ \beta_i \\ \vdots \\ \beta_m \end{matrix} & \begin{pmatrix} \partial_{11}^{(t)} & \cdots & \partial_{1g}^{(t)} & \cdots & \partial_{1\mathfrak{h}}^{(t)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \partial_{i1}^{(t)} & \cdots & \partial_{ig}^{(t)} & \cdots & \partial_{i\mathfrak{h}}^{(t)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \partial_{m1}^{(t)} & \cdots & \partial_{mg}^{(t)} & \cdots & \partial_{m\mathfrak{h}}^{(t)} \end{pmatrix} \end{matrix}, \text{ where } \partial_{ig}^{(t)} = \left( \left( \mathbb{I}_{r_{ig}}^{(t)}, \mathbb{K}_{ig}^{(t)} \right), \left\langle m_{ig}^{(t)}, n_{ig}^{(t)} \right\rangle \right) \text{ symbolizes } q\text{-RFOLN}$$

provided by  $t$ th DM to which alternative  $\beta_i$  meets that the criteria  $\mathbb{E}_g$  having the constraint that  $0 \leq m_{ig}^q + n_{ig}^q \leq 1$ .

**WASPAS approach with  $q$ -RF2LNs**

Considering that the individual decision matrix  $M^{(t)}$  includes both benefit and cost criteria, it is essential to normalize these criteria. This normalization process involves converting the cost-type criteria into benefit-type criteria, which can be accomplished using the following equation:

$$\widetilde{\partial}_{ig}^{(t)} = \begin{cases} \partial_{ig}^{(t)}, & \mathbb{E}_g \text{ is benefit criteria,} \\ \partial_{ig}^{(t)c}, & \mathbb{E}_g \text{ is cost criteria,} \end{cases} \tag{28}$$

where  $\partial_{ig}^{(t)c}$  represents the complement of  $\partial_{ig}^{(t)}$  ( $i = 1, 2, \dots, m; g = 1, 2, \dots, \mathfrak{h}$ ).

Using Definition 9 and the normalized individual decision matrix  $M^t = \left( \widetilde{\partial}_{ig}^{(t)} \right)_{m \times \mathfrak{h}}$  as a basis, we proceed to compute the priority degrees among the experts in the following manner:

$$\mathcal{T}_{ig}^{(t)} = \begin{cases} 1 & ; t = 1 \\ \prod_{z=1}^{t-1} \widehat{S}(\partial_{ig}^{(z)}) & t = 2, \dots, p. \end{cases} \tag{29}$$

As per the setup of the MCGDM problem, it is necessary to aggregate the individual normalized decision matrices into a collective decision matrix. This collective decision matrix can be generated using either the  $q$ -RF2LPMSM or  $q$ -RF2LPDMSM operator. To calculate the aggregated value of alternative “ $\beta_i$ ” concerning the criteria “ $\mathbb{E}_g$ ”, we use the Eq. (13) or Eq. (21) in the following manner:

$$q\text{-RF2LPMSM}^{(b)} \left( \partial_{ig}^{(1)}, \partial_{ig}^{(2)}, \dots, \partial_{ig}^{(p)} \right) = \left( \frac{\bigoplus_{1 \leq i_1 < \dots < i_p \leq p} \left( \bigotimes_{t=1}^p \partial_{ig}^{(t)} \right)_{i_1, \dots, i_p}}{C_p^b} \right)^{\frac{1}{b}} = \left( \left\langle \begin{matrix} \wedge \left( \left( \frac{1}{C_p^b} \sum_{t=1}^p \left( \vee \left( \mathbb{I}_{r_{ig}}^{(t)}, \mathbb{K}_{ig}^{(t)} \right) \right)_{i_1, \dots, i_p} \right)^{\frac{1}{b}} \right) \\ \left( \left( 1 - \prod_{t=1}^p \left( 1 - \left( \prod_{t=1}^p m_{ig}^{(t)} \right)^q \right) \right)^{\frac{1}{q}} \right)^{\frac{1}{b}} \\ \left( 1 - \left( 1 - \left( \prod_{t=1}^p \left( 1 - \left( \prod_{t=1}^p \left( 1 - n_{ig}^{(t)q} \right) \right) \right)^{\frac{1}{q}} \right) \right)^{\frac{1}{b}} \end{matrix} \right) \right)^{\frac{1}{q}} \right)^{\frac{1}{b}}, \tag{30}$$

where  $\mathbb{I}_{ig}^{(t)} = \frac{\mathcal{T}_{ig}^{(t)}}{\sum_{i=1}^p \mathcal{T}_{ig}^{(t)}}$  or

$$q - RF2LPDMSM^{(b)}(\delta_{ig}^{(1)}, \delta_{ig}^{(2)}, \dots, \delta_{ig}^{(p)}) = \left( \begin{array}{c} \wedge \left( \frac{1}{b} \prod_{\Omega} \left( \sum_{i=1}^p \mathbb{I}_{ig}^{(t)} (\vee (s_{rig}^{(t)}, \mathcal{X}_{ig}^{(t)})) \right) \right)^{\frac{1}{c_p^b}}, \\ \left( 1 - \left( 1 - \prod_{\Omega} \left( \left( 1 - \prod_{i=1}^p (1 - m_{ig}^{(t)q}) \mathbb{I}_{ig}^{(t)q} \right) \right)^{\frac{1}{c_p^b}} \right)^{\frac{1}{q}}, \\ \left( 1 - \left( \prod_{\Omega} \left( 1 - \left( \prod_{i=1}^p n_{ig}^{(t)q} \right) \right)^{\frac{1}{c_p^b}} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} \end{array} \right)^{\frac{1}{b}} \quad (31)$$

Based on the outcomes from  $q$ -RF2LPMSM and  $q$ -RF2LPDMSM operators, we derive an integrated group decision matrix denoted as  $M = (\delta_{ig})_{m \times h}$ . The construction of the decision matrix  $M$  is accomplished as:

$$M_{m \times h} = \begin{matrix} & \begin{matrix} \epsilon_1 & \dots & \epsilon_g & \dots & \epsilon_h \end{matrix} \\ \begin{matrix} \beta_1 \\ \vdots \\ \beta_i \\ \vdots \\ \beta_m \end{matrix} & \begin{pmatrix} \delta_{11} & \dots & \delta_{1g} & \dots & \delta_{1h} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \delta_{i1} & \dots & \delta_{ig} & \dots & \delta_{ih} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \delta_{m1} & \dots & \delta_{mg} & \dots & \delta_{mh} \end{pmatrix} \end{matrix}$$

The element  $\delta_{ig} = ((\mathbb{I}_{rig}, \mathcal{X}_{ig}), (m_{ig}, n_{ig}))$  in the decision matrix  $M$  represents the collective assessment of alternative  $\beta_i$  concerning criterion  $\epsilon_g$ . Regarding Definition 6 and the collective decision matrix  $M$ , the process of determining the priority levels among the criteria is performed in the following manner:

$$\mathcal{T}_{ig} = \begin{cases} 1 & ; g = 1 \\ \prod_{z=1}^{g-1} \hat{S}(\delta_{iz}) & g = 2, \dots, h. \end{cases} \quad (32)$$

Using the foundational concept of the WASPAS method and the collective decision matrix  $M$ , we compute the measures of the  $q$ -RF2L prioritized sum model ( $q$ -RF2L-PSM)  $\bowtie_i^{(1)}$  for each alternative  $\beta_i$ . This calculation involves applying the  $q$ -RF2LPMSM operator in the following manner:

$$\bowtie_i^{(1)} = \left( \frac{\oplus_{1 \leq i_1 < \dots < i_b \leq h} (\otimes_{g=1}^b \delta_{ig}^{\mathbb{I}_{ig}})}{C_{h_i}^b} \right)^{\frac{1}{b}} = \left( \begin{array}{c} \wedge \left( \left( \frac{1}{C_{h_i}^b} \sum_{\Omega} \prod_{g=1}^b (\vee (\mathbb{I}_{rig}, \mathcal{X}_{ig}))^{\mathbb{I}_{ig}} \right) \right)^{\frac{1}{b}}, \\ \left( 1 - \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^b m_{ig}^{\mathbb{I}_{ig}} \right)^q \right)^{\frac{1}{c_{h_i}^b}} \right)^{\frac{1}{q}}, \\ \left( 1 - \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^b (1 - n_{ig}^q) \right) \right)^{\frac{1}{c_{h_i}^b}} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} \end{array} \right)^{\frac{1}{b}} \quad (33)$$

Similarly, we ascertain the measures of the  $q$ -RF2L prioritized product model ( $q$ -RF2L-PPM)  $\bowtie_i^{(2)}$  for each alternative  $\beta_i$  using the  $q$ -RF2LPDMSM operator in the following manner:

$$\bowtie_i^{(2)} = \frac{1}{\mathfrak{p}} \left( \otimes_{1 \leq i_1 < \dots < i_{\mathfrak{p}} \leq \mathfrak{p}} \left( \oplus_{g=1}^{\mathfrak{p}} \mathfrak{q}_{ig} \otimes \partial_{ig} \right) \right)^{\frac{1}{c_{\mathfrak{h}}^{\mathfrak{p}}}} = \left( \left\langle \begin{array}{l} \left( \wedge \left( \frac{1}{\mathfrak{p}} \prod_{\Omega} \left( \left( \sum_{g=1}^{\mathfrak{p}} \mathfrak{q}_{ig} \cdot \left( \vee \left( \mathbb{I}_{r_{ig}}, \mathfrak{K}_{ig} \right) \right) \right)^{\frac{1}{c_{\mathfrak{h}}^{\mathfrak{p}}}} \right) \right) \right) \right)^{\frac{1}{\mathfrak{p}}}, \\ \left( 1 - \left( 1 - \prod_{\Omega} \left( \left( 1 - \prod_{g=1}^{\mathfrak{p}} \left( 1 - m_{ig}^q \right)^{\mathfrak{q}_{ig}} \right) \right)^{\frac{1}{c_{\mathfrak{h}}^{\mathfrak{p}}}} \right)^{\frac{1}{\mathfrak{p}}} \right)^{\frac{1}{q}}, \\ \left( \left( 1 - \left( \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^{\mathfrak{p}} n_{ig}^{\mathfrak{q}_{ig}} \right) \right) \right)^{\frac{1}{c_{\mathfrak{h}}^{\mathfrak{p}}}} \right)^{\frac{1}{\mathfrak{p}}} \right)^{\frac{1}{q}} \end{array} \right) \right)^{\frac{1}{\mathfrak{p}}} \quad (34)$$

Hence, according to Definition 6, the scores  $\widehat{S}(\bowtie_i^{(1)})$  and  $\widehat{S}(\bowtie_i^{(2)})$  in Eqs. (33) and (34) can be determined, respectively. Additionally, we obtain the combined measure of the  $q$ -RF2LPWASPAS method  $\bowtie_i$  for each alternative by merging the measures of  $q$ -RF2L-PSM and  $q$ -RF2L-PPM. This result is calculated using the following formula:

$$\bowtie_i = \theta \bowtie_i^{(1)} + (1 - \theta) \bowtie_i^{(2)}, \quad (35)$$

where  $\theta$  represents the aggregating coefficient of decision precision, and  $\theta \in [0, 1]$ . Finally, we rank the alternatives in descending order based on the value of  $\bowtie_i$ .

### Decision making procedure

This section is devoted to outlining an approach based on utilizing the  $q$ -RF2LPMSM,  $q$ -RF2LPDMSM, and the WASPAS method. The key steps of this novel decision-making process are presented as follows:

- Step 1** Considering the real-world decision-making scenario, we establish the set of feasible alternatives denoted by  $\beta = \{\beta_1, \beta_2, \dots, \beta_m\}$  and the set of evaluation criteria represented by  $\epsilon = \{\epsilon_1, \epsilon_2, \dots, \epsilon_{\mathfrak{h}}\}$ , along with their respective priority information. Let  $\mathbb{C} = \{\mathbb{C}_1, \mathbb{C}_2, \dots, \mathbb{C}_p\}$  be the group of DMs, with a specific prioritization among them. Consequently, we generate the  $q$ -rung orthopair fuzzy 2-tuple linguistic decision matrix  $M^{(t)} = \partial_{ig}^{(t)}$ , provided by expert  $\mathbb{C}^{(t)}$ .
- Step 2** By the application of Eq. (28), we standardize the individual decision matrix  $M^{(t)}$  by  $\widetilde{M}^{(t)} = \widetilde{\partial}_{ig}^{(t)}$  ( $i = 1, 2, \dots, m; g = 1, 2, \dots, \mathfrak{h}$ ).
- Step 3** Utilizing Eq. (29), we determine the priority degrees among the DMs.
- Step 4** Once the value of  $\mathfrak{p}$  is given, we employ  $q$ -RF2LPMSM and Eq. (30) or  $q$ -RF2LPDMSM and Eq. (31) to aggregate the normalized individual decision matrices into a unified collective decision matrix. Then, we construct the collective decision matrix  $M = \left( \partial_{ig} \right)_{m \times \mathfrak{h}}$ ,  $i = 1, 2, \dots, m; g = 1, 2, \dots, \mathfrak{h}$ .
- Step 5** Based on Eq. (32), we calculate the priority degrees among the criteria.
- Step 6** Using Eq. (33), we compute the measures of  $q$ -RF2LPMSM  $\bowtie_i^{(1)}$  for each alternative  $\beta_i$ .
- Step 7** Using Eq. (34), we calculate the measures of  $q$ -RF2LPPM  $\bowtie_i^{(2)}$  for each alternative  $\beta_i$ .
- Step 8** Next, we apply Eq. (35) to obtain the aggregated measure of WASPAS  $\bowtie_i$  for each alternative  $\beta_i$ .
- Step 9** Finally, we rank all the alternatives in descending order based on the  $\bowtie_i$  values.

The developed MCGDM approach is visually illustrated in Fig. 1.

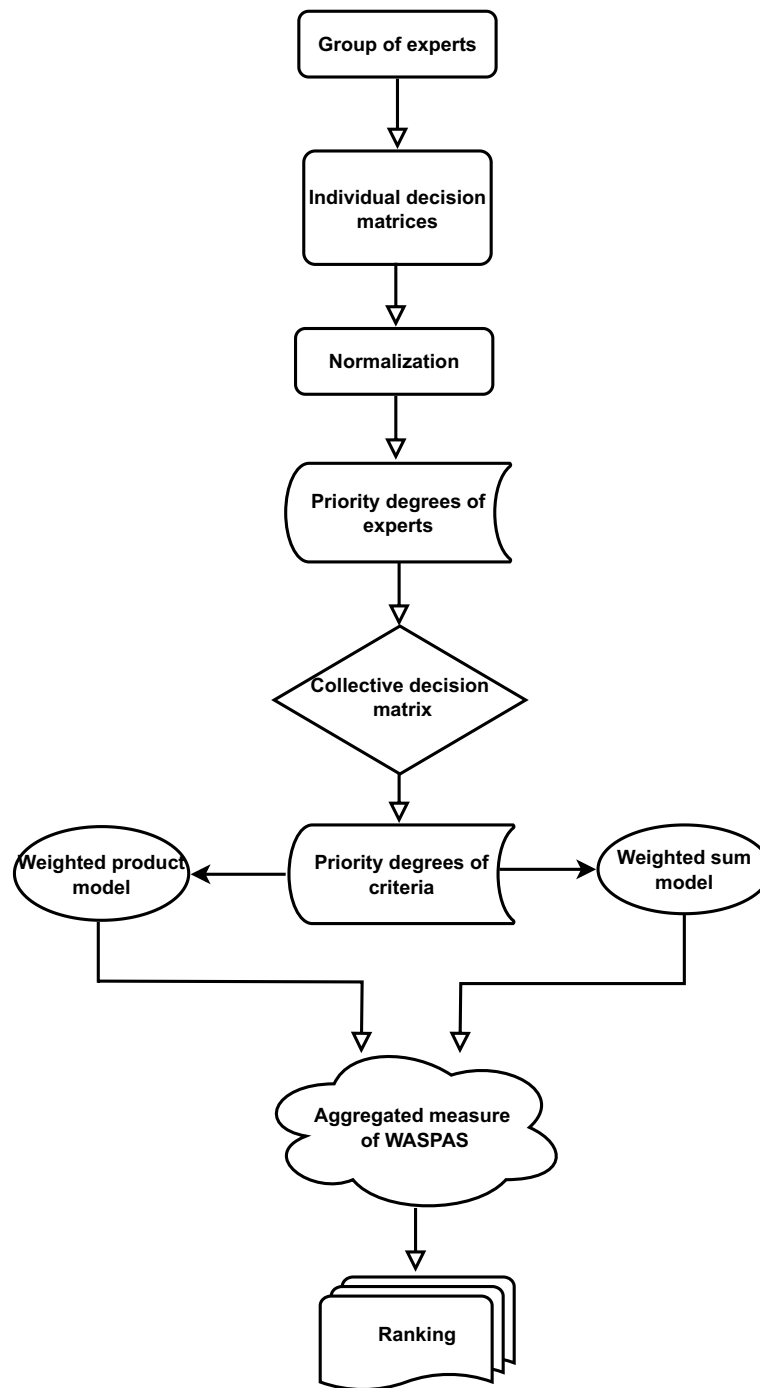
### Application

In this section, we implement the proposed method to address the prioritization problem of patients in a prominent teaching hospital in Ghana. Furthermore, we perform sensitivity and comparison analyses to showcase the validity and reliability of our proposed approach.

### Background

Ghana's growing population has led to an increase in health-related cases in recent years, posing significant challenges for healthcare institutions like the Komfo Anokye Teaching Hospital (KATH), especially with the limited availability of beds. To enhance the patient admission process, KATH established an inpatient Admission Service Unit (ASU) aimed at optimizing hospital admissions and ensuring efficient allocation of medical resources. This initiative is particularly focused on promptly admitting patients with severe illnesses. Previously, the hospital relied on an artificial screening process for patient admission, which was flawed due to human intervention in





**Figure 1.** Graphical illustration of the developed MCGDM approach.

bed allocation and the complexity of decision-making, as noted in<sup>2</sup>. The process involved balancing conflicting factors and managing uncertainty in information. To improve the efficiency of the ASU, we are implementing the  $q$ -rung orthopair fuzzy 2-tuple linguistic WASPAS method to prioritize patient admissions effectively. Three health experts from the ASU, denoted as  $\mathbb{E} = \{\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3\}$ , will assess patients post-initial examination by doctors, with a priority order of  $\mathbb{E}_1 > \mathbb{E}_2 > \mathbb{E}_3$ . The evaluation criteria adopted by the experts, based on studies from<sup>2,3,7</sup>, and relevant to Ghana's context, include disease severity, emergency degree, priority for special diseases, and type of medical insurance. These criteria are prioritized as  $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4$ . To manage the inherent uncertainty of information, the health experts will use  $q$ -RF2LTNs to provide their preference values for various alternatives  $\beta_i (i = 1, 2, 3, 4, 5)$  relative to the criteria  $\epsilon_g (g = 1, 2, 3, 4)$ . The resulting  $q$ -rung orthopair fuzzy 2-tuple linguistic decision matrices from these evaluations are presented in Tables 1, 2, and 3.

	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$
$\beta_1$	$(\mathbb{I}_2, 0), (0.6, 0.3)$	$(\mathbb{I}_3, 0), (0.4, 0.5)$	$(\mathbb{I}_5, 0), (0.5, 0.4)$	$(\mathbb{I}_4, 0), (0.6, 0.3)$
$\beta_2$	$(\mathbb{I}_1, 0), (0.5, 0.4)$	$(\mathbb{I}_3, 0), (0.6, 0.4)$	$(\mathbb{I}_2, 0), (0.7, 0.3)$	$(\mathbb{I}_4, 0), (0.8, 0.7)$
$\beta_3$	$(\mathbb{I}_2, 0), (0.7, 0.1)$	$(\mathbb{I}_1, 0), (0.8, 0.3)$	$(\mathbb{I}_4, 0), (0.4, 0.1)$	$(\mathbb{I}_5, 0), (0.6, 0.1)$
$\beta_4$	$(\mathbb{I}_1, 0), (0.3, 0.4)$	$(\mathbb{I}_3, 0), (0.7, 0.1)$	$(\mathbb{I}_4, 0), (0.5, 0.5)$	$(\mathbb{I}_2, 0), (0.5, 0.1)$
$\beta_5$	$(\mathbb{I}_4, 0), (0.5, 0.4)$	$(\mathbb{I}_2, 0), (0.4, 0.3)$	$(\mathbb{I}_1, 0), (0.6, 0.2)$	$(\mathbb{I}_3, 0), (0.7, 0.3)$

**Table 1.**  $q$ -RF2L decision matrix  $M^{(1)}$ .

	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$
$\beta_1$	$(\mathbb{I}_5, 0), (0.6, 0.5)$	$(\mathbb{I}_6, 0), (0.4, 0.3)$	$(\mathbb{I}_2, 0), (0.5, 0.5)$	$(\mathbb{I}_3, 0), (0.6, 0.3)$
$\beta_2$	$(\mathbb{I}_4, 0), (0.5, 0.4)$	$(\mathbb{I}_3, 0), (0.6, 0.3)$	$(\mathbb{I}_1, 0), (0.7, 0.3)$	$(\mathbb{I}_6, 0), (0.6, 0.3)$
$\beta_3$	$(\mathbb{I}_5, 0), (0.6, 0.4)$	$(\mathbb{I}_3, 0), (0.7, 0.5)$	$(\mathbb{I}_6, 0), (0.9, 0.1)$	$(\mathbb{I}_1, 0), (0.8, 0.2)$
$\beta_4$	$(\mathbb{I}_2, 0), (0.4, 0.3)$	$(\mathbb{I}_1, 0), (0.6, 0.5)$	$(\mathbb{I}_3, 0), (0.7, 0.2)$	$(\mathbb{I}_5, 0), (0.8, 0.1)$
$\beta_5$	$(\mathbb{I}_5, 0), (0.5, 0.3)$	$(\mathbb{I}_2, 0), (0.6, 0.3)$	$(\mathbb{I}_6, 0), (0.8, 0.1)$	$(\mathbb{I}_5, 0), (0.8, 0.2)$

**Table 2.**  $q$ -RF2L decision matrix  $M^{(2)}$ .

	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$
$\beta_1$	$(\mathbb{I}_1, 0), (0.5, 0.4)$	$(\mathbb{I}_3, 0), (0.6, 0.5)$	$(\mathbb{I}_4, 0), (0.7, 0.1)$	$(\mathbb{I}_5, 0), (0.8, 0.4)$
$\beta_2$	$(\mathbb{I}_2, 0), (0.6, 0.1)$	$(\mathbb{I}_1, 0), (0.6, 0.1)$	$(\mathbb{I}_3, 0), (0.5, 0.4)$	$(\mathbb{I}_4, 0), (0.9, 0.1)$
$\beta_3$	$(\mathbb{I}_4, 0), (0.7, 0.4)$	$(\mathbb{I}_5, 0), (0.8, 0.6)$	$(\mathbb{I}_1, 0), (0.1, 0.2)$	$(\mathbb{I}_6, 0), (0.4, 0.1)$
$\beta_4$	$(\mathbb{I}_5, 0), (0.8, 0.3)$	$(\mathbb{I}_1, 0), (0.9, 0.1)$	$(\mathbb{I}_2, 0), (0.6, 0.2)$	$(\mathbb{I}_3, 0), (0.5, 0.5)$
$\beta_5$	$(\mathbb{I}_6, 0), (0.8, 0.5)$	$(\mathbb{I}_2, 0), (0.7, 0.4)$	$(\mathbb{I}_1, 0), (0.6, 0.5)$	$(\mathbb{I}_4, 0), (0.6, 0.6)$

**Table 3.**  $q$ -RF2L decision matrix  $M^{(3)}$ .

**Method implementation process**

*Step 1* The initial evaluation matrices provided by three DMs  $\epsilon_1, \epsilon_2$  and  $\epsilon_3$  are listed in Tables 1, 2, and 3, respectively.

*Step 2* Referring to Eq. (28), it is required to normalize the individual decision matrices  $M^{(1)}$  to  $M^{(3)}$ . However, since all the criteria are benefit criteria, normalization is unnecessary.

*Step 3* Utilizing the information provided in Tables 1, 2 and 3 and Eq. (29), we compute the priority degrees  $\mathcal{T}_{ig}^{(i)}$  among the DMs, and the results are given below:

$$\mathcal{T}_{ig}^{(1)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \mathcal{T}_{ig}^{(2)} = \begin{pmatrix} 0.170 & 0.201 & 0.379 & 0.340 \\ 0.076 & 0.247 & 0.188 & 0.334 \\ 0.192 & 0.106 & 0.304 & 0.434 \\ 0.069 & 0.288 & 0.286 & 0.161 \\ 0.303 & 0.148 & 0.086 & 0.282 \end{pmatrix}, \mathcal{T}_{ig}^{(3)} = \begin{pmatrix} 0.066 & 0.089 & 0.054 & 0.087 \\ 0.023 & 0.063 & 0.018 & 0.17 \\ 0.079 & 0.028 & 0.225 & 0.047 \\ 0.01 & 0.022 & 0.082 & 0.087 \\ 0.120 & 0.025 & 0.056 & 0.151 \end{pmatrix}.$$

Next, we will use the  $q$ -RF2LPMSM and  $q$ -RF2LPDMSM operators for further analysis. In what follows, we begin by addressing the problem using the  $q$ -RF2LPMSM operator, followed by applying the  $q$ -RF2LPDMSM operator.

*Solving problem via the WASPAS method with PMSM operator*

In this part, we utilize the  $q$ -RF2L PMSM operator to integrate information from individual decision matrices ranging from  $M^{(1)}$  to  $M^{(3)}$ .

*Step 4:* Taking  $p = 2$ , we use Eq. (30) to determine the overall preference value of alternative  $\beta_i$  in relation to  $\epsilon_g (i = 1, 2, 3, 4, 5; g = 1, 2, 3, 4)$  to form the collective decision matrix depicted in Table 4.

	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$
$\beta_1$	$(\llbracket_1, 0.3168 \rrbracket, (0.8796, 0.2415))$	$(\llbracket_2, -0.4661 \rrbracket, (0.8171, 0.2991))$	$(\llbracket_2, -0.3446 \rrbracket, (0.8326, 0.2924))$	$(\llbracket_2, -0.3901 \rrbracket, (0.8735, 0.2093))$
$\beta_2$	$(\llbracket_1, 0.0381 \rrbracket, (0.8929, 0.2348))$	$(\llbracket_1, 0.4579 \rrbracket, (0.8749, 0.2295))$	$(\llbracket_1, 0.2364 \rrbracket, (0.9162, 0.1942))$	$(\llbracket_2, -0.3328 \rrbracket, (0.9185, 0.3314))$
$\beta_3$	$(\llbracket_1, 0.3495 \rrbracket, (0.9011, 0.1420))$	$(\llbracket_1, 0.0451 \rrbracket, (0.9455, 0.2347))$	$(\llbracket_2, -0.4277 \rrbracket, (0.8303, 0.0867))$	$(\llbracket_2, -0.4513 \rrbracket, (0.8946, 0.095))$
$\beta_4$	$(\llbracket_1, 0.0195 \rrbracket, (0.8822, 0.2127))$	$(\llbracket_1, 0.3674 \rrbracket, (0.8965, 0.1637))$	$(\llbracket_2, -0.4036 \rrbracket, (0.8709, 0.2378))$	$(\llbracket_1, 0.3364 \rrbracket, (0.8838, 0.1307))$
$\beta_5$	$(\llbracket_2, -0.3250 \rrbracket, (0.8453, 0.2623))$	$(\llbracket_1, 0.2793 \rrbracket, (0.9129, 0.1944))$	$(\llbracket_1, 0.0501 \rrbracket, (0.9153, 0.1633))$	$(\llbracket_2, -0.4619 \rrbracket, (0.9048, 0.246))$

**Table 4.** Aggregated decision matrix obtained via q-RF2L PMSM operator.

	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$
$\beta_1$	$(\llbracket_1, -0.2865 \rrbracket, (0.3898, 0.8256))$	$(\llbracket_1, 0.0642 \rrbracket, (0.2885, 0.8077))$	$(\llbracket_1, 0.0774 \rrbracket, (0.3526, 0.7835))$	$(\llbracket_1, 0.1164 \rrbracket, (0.4331, 0.7414))$
$\beta_2$	$(\llbracket_0, 0.3563 \rrbracket, (0.3106, 0.8493))$	$(\llbracket_1, -0.1986 \rrbracket, (0.4045, 0.7517))$	$(\llbracket_0, 0.4180 \rrbracket, (0.4625, 0.7943))$	$(\llbracket_1, 0.3896 \rrbracket, (0.5618, 0.7780))$
$\beta_3$	$(\llbracket_1, -0.1934 \rrbracket, (0.4527, 0.7893))$	$(\llbracket_0, 0.3735 \rrbracket, (0.4948, 0.8701))$	$(\llbracket_1, 0.2081 \rrbracket, (0.4136, 0.5964))$	$(\llbracket_1, -0.0810 \rrbracket, (0.4799, 0.6492))$
$\beta_4$	$(\llbracket_0, 0.2796 \rrbracket, (0.2220, 0.8658))$	$(\llbracket_1, -0.4433 \rrbracket, (0.4649, 0.7863))$	$(\llbracket_1, 0.0039 \rrbracket, (0.3963, 0.7482))$	$(\llbracket_1, -0.2407 \rrbracket, (0.4009, 0.7173))$
$\beta_5$	$(\llbracket_1, 0.3417 \rrbracket, (0.3833, 0.7711))$	$(\llbracket_1, -0.4962 \rrbracket, (0.3107, 0.8071))$	$(\llbracket_0, 0.4376 \rrbracket, (0.4253, 0.7780))$	$(\llbracket_1, 0.1233 \rrbracket, (0.5101, 0.7292))$

**Table 5.** Aggregated decision matrix obtained via q-RF2L PDMSM operator.

Step 5 Based on Table 4, we calculate the priority degrees  $\mathcal{T}_{ig}$  for the set of criteria. The outcomes are computed

as follows (Table 5):  $\mathcal{T}_{ig} = \begin{pmatrix} 1 & 0.1568 & 0.0261 & 0.0048 \\ 1 & 0.1264 & 0.0217 & 0.0034 \\ 1 & 0.1667 & 0.0228 & 0.0036 \\ 1 & 0.1220 & 0.0205 & 0.0038 \\ 1 & 0.1900 & 0.0303 & 0.0040 \end{pmatrix}$ .

Step 6 Using Eq. (33), along with  $\mathcal{T}_{ig}$ , we calculate the prioritized sum model for each alternative  $\beta_i$  ( $i = 1, 2, 3, 4, 5$ ), as shown in Table 6.

Step 7 Similarly, applying Eq. (34),  $\mathcal{T}_{ig}$ , we compute the prioritized product model for each alternative, also presented in Table 6.

Step 8 To obtain the combined measure  $\triangleright_i$  for each alternative, we utilize Eq. (35) and display the outcomes in Table 6.

Step 9 Based on the values of  $\triangleright_i$  ( $i = 1, 2, 3, 4, 5$ ), we establish the following ranking for the alternatives:  $\beta_5 > \beta_3 > \beta_1 > \beta_4 > \beta_2$ . As a result, the most significant patient is  $\beta_5$ .

*Solving problem via the WASPAS method with PDMSM operator*

In the present part, we consider the q-RF2L PDMSM operator to derive the integrated information from individual decision matrices ranging from  $M^{(1)}$  to  $M^{(3)}$ .

Step 4 Fixing  $p = 2$ , we apply Eq. (31) to compute the overall preference value of each alternative  $\beta_i$  concerning  $\epsilon_g$  ( $i = 1, 2, 3, 4, 5; g = 1, 2, 3, 4$ ). These preference values are then consolidated to form the collective decision matrix, illustrated in Table 5.

Step 5 In view of Table 5 and Eq. (32), we derive the priority degrees for the criteria set. The outcomes are

obtained as follows:  $\mathcal{T}_{ig} = \begin{pmatrix} 1 & 0.0253 & 0.001 & 0.00004 \\ 1 & 0.0106 & 0.0004 & 0.000007 \\ 1 & 0.0346 & 0.0004 & 0.00003 \\ 1 & 0.0072 & 0.0002 & 0.000008 \\ 1 & 0.0573 & 0.001 & 0.00002 \end{pmatrix}$ .

	$\triangleright_i^{(1)}$	$\triangleright_i^{(2)}$	$\triangleright_i$
$\beta_1$	0.1497	0.0056	0.0776
$\beta_2$	0.1472	0.0047	0.0737
$\beta_3$	0.1505	0.0060	0.0782
$\beta_4$	0.1419	0.0045	0.0732
$\beta_5$	0.1580	0.0071	0.0826

**Table 6.** Measures for q-RF2LPMSM-based WASPAS.

	$\succcurlyeq_i^{(1)}$	$\succcurlyeq_i^{(2)}$	$\succcurlyeq_i$
$\beta_1$	0.1249	0.00008	0.0625
$\beta_2$	0.1139	0.00002	0.0569
$\beta_3$	0.1288	0.000006	0.0645
$\beta_4$	0.1113	0.00001	0.0556
$\beta_5$	0.1418	0.0001	0.0710

**Table 7.** Measures for  $q$ -RF2LPDMSM-based WASPAS.

*Step 6* Referring to Eq. (33), along with  $\mathcal{T}_{ig}$ , we compute the evaluations for each alternative  $\beta_i (i = 1, 2, 3, 4, 5)$  using the Prioritized Sum Model  $\succcurlyeq_i^{(1)}$ . The outcomes are presented in Table 7.

*Step 7* Following Eq. (34),  $\mathcal{T}_{ig}$ , we determine the evaluations for each alternative  $\beta_i$  using the prioritized product Model  $\succcurlyeq_i^{(2)}$  as tabulated in Table 7.

*Step 8* By applying Eq. (35), we combine the overall preference value  $\succcurlyeq_i$  for each alternative, as displayed in Table 7.

*Step 9* Considering the values of  $\succcurlyeq_i (i = 1, 2, 3, 4, 5)$ , we arrange the alternatives in the following order:  $\beta_5 > \beta_3 > \beta_1 > \beta_4 > \beta_2$ . As a result, the most significant patient is  $\beta_5$ .

### Parameter analysis

This section presents an analysis of how sensitive the proposed model is to changes in the parameters  $\theta$  and  $q$ , using the numerical example provided in “Background” section.

#### Sensitivity analysis with respect to $\theta$

The subsequent analysis focuses on understanding the impact of coefficient  $\theta$  variations on our results. We employ the  $q$ -RF2LMSM-based WASPAS method to assess different scenarios in patient prioritization, considering a range of  $\theta$  values from 0 to 1. The outcomes for each decision alternative  $\beta_i (i = 1, 2, 3, 4, 5)$  at varying  $\theta$  levels are detailed in Table 8. An examination of Table 8 and Fig. 2 reveals a trend where the values of  $\succcurlyeq_i (i = 1, 2, 3, 4, 5)$  rise with an increase in  $\theta$  from 0 to 1. Notably, alternative  $\beta_5$  consistently emerges as the top choice, indicating the robustness of the proposed method against changes in the strategy values set by experts.

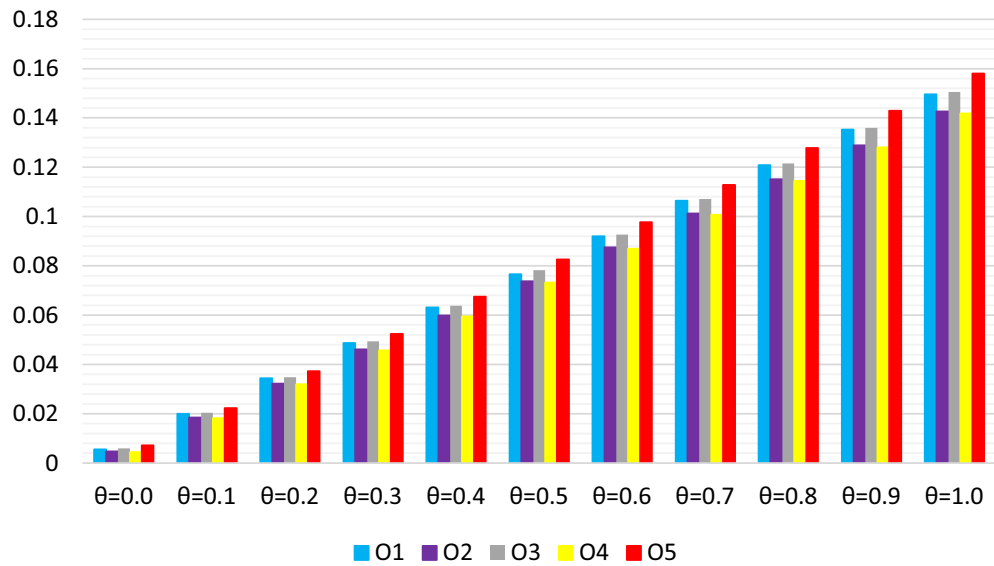
Next, we apply the  $q$ -RF2LDMSM-based WASPAS method to re-examine the patient prioritization decision-making process, varying  $\theta$  within the 0 to 1 range. The outcomes for each option  $\beta_i (i = 1, 2, 3, 4, 5)$  under these different  $\theta$  values are shown in Table 9 and Fig. 3. Table 9 shows a similar pattern of increasing  $\succcurlyeq_i (i = 1, 2, 3, 4, 5)$  values as  $\theta$  escalates from 0 to 1. Interestingly, alternative  $\beta_5$  is again identified as the preferred option in all scenarios, affirming the stability of our method. The rankings of the other alternatives remain largely consistent, except the scenario where  $\theta$  equals 0, further underscoring the resilience of the proposed approach amidst substantial modifications in expert strategy values.

#### Sensitivity analysis with respect to $q$

This section examines the impact of the parameter  $q$  on the results. Initially, we alter the  $q$  values within the  $q$ -RF2LPMSM-based WASPAS framework, keeping the other parameter,  $\theta$ , constant at 0.5. The results, summarized in Table 10 and Fig. 4, show a distinct trend: as  $q$  increases, the scores for all alternatives decrease. However, the overall ranking of alternatives remains unaffected by changes in  $q$ .

$\theta$	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$	$\beta_5$	Ranking relation
$\theta = 0.0$	0.00555	0.00469	0.00597	0.00450	0.00715	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.1$	0.01996	0.01849	0.02042	0.01824	0.02224	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.2$	0.03437	0.03229	0.03488	0.03198	0.03733	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.3$	0.04878	0.04610	0.04933	0.04571	0.05242	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.4$	0.06319	0.05990	0.06378	0.05942	0.06751	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.5$	0.07660	0.07370	0.07820	0.07320	0.08259	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.6$	0.09201	0.08750	0.09269	0.08693	0.09769	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.7$	0.10642	0.10131	0.10714	0.10067	0.11277	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.8$	0.12083	0.11511	0.12159	0.11440	0.12787	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.9$	0.13524	0.12891	0.13604	0.12814	0.14296	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 1.0$	0.14965	0.14271	0.15050	0.14188	0.15804	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$

**Table 8.** Ranking results of  $q$ -RF2LPMSM-based WASPAS with varying  $\theta$  values.



**Figure 2.** Ranking results via  $q$ -RF2LPMSM operator with different values of  $\theta$ .

$\theta$	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$	$\beta_5$	Ranking relation
$\theta = 0.0$	0.00008	0.00002	0.00006	0.00001	0.0001	$\beta_5 > \beta_1 > \beta_3 > \beta_2 > \beta_4$
$\theta = 0.1$	0.01256	0.01141	0.01293	0.01114	0.01427	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.2$	0.02504	0.02279	0.02581	0.02227	0.02844	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.3$	0.03753	0.03418	0.03868	0.03334	0.04261	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.4$	0.05001	0.04557	0.05155	0.04452	0.05678	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.5$	0.06249	0.05696	0.06443	0.05565	0.07095	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.6$	0.07497	0.06835	0.07730	0.06678	0.08512	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.7$	0.08745	0.07973	0.09018	0.07791	0.09290	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.8$	0.09999	0.09112	0.10305	0.08904	0.11346	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 0.9$	0.11242	0.10251	0.11593	0.10017	0.12760	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
$\theta = 1.0$	0.12490	0.11390	0.12880	0.11130	0.14180	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$

**Table 9.** Ranking results of  $q$ -RF2LPDMSM-based WASPAS with varying  $\theta$  values.

Similarly, a comparable pattern emerges when adjusting  $q$  within the  $q$ -RF2LPDMSM-based WASPAS method: scores fluctuate with changes in  $q$ . Yet, the ranking order of alternatives stays consistent (see Table 11 and Fig. 5). This consistency across different  $q$  values highlights the robustness and stability of the proposed approach about its parameters.

### Validation of results

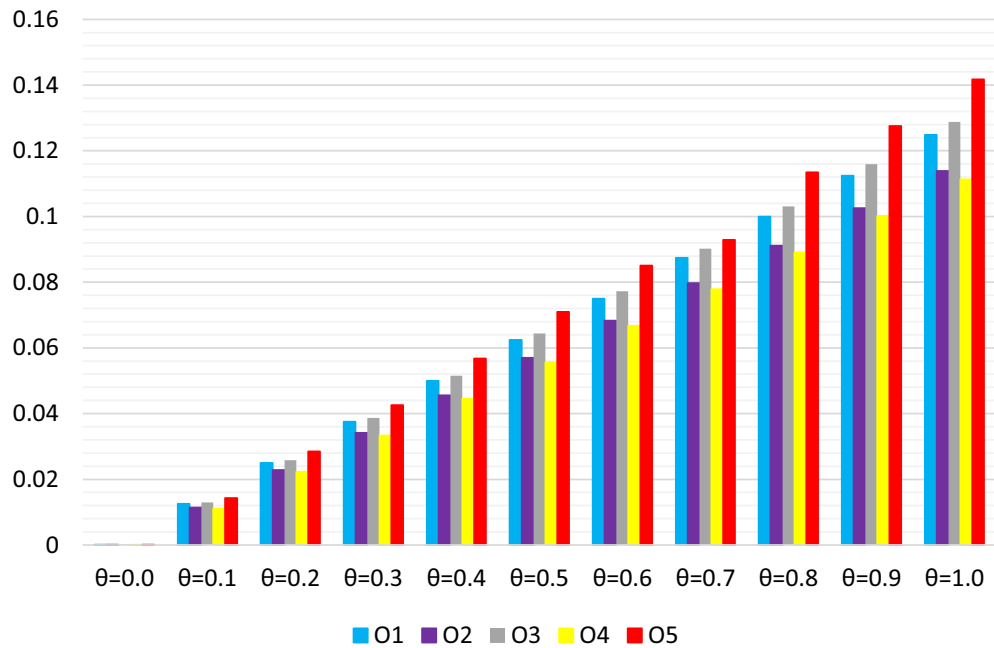
This section evaluates the proposed model’s comparability with existing methodologies, highlighting its efficiency and superiority. Given the limited number of studies that utilize multi-criteria modeling methods like WASPAS in conjunction with prioritized operators, our options for comparison with MCGDM approaches are somewhat restricted. Consequently, we undertake a step-by-step comparison of the proposed approach against each existing method.

### $q$ -RF2L PROMETHEE II method

In this section, we contrast our developed approach with the  $q$ -RF2L preference ranking organization method for enrichment evaluation (PROMETHEE) II technique, as introduced by Li et al.<sup>48</sup>. This comparison aims to assess the logical validity of our method, beginning with the aggregated decision matrix presented in Table 4.

Based on Eq. (36), the deviations between the two assessment values are calculated in Table 11.

$$d_{ig}(\beta_i, \beta_r) = \begin{cases} d_H(\partial_{ig}, \partial_{rg}), & \text{if } \partial_{ig} > \partial_{rg} \\ 0, & \text{if } \partial_{ig} < \partial_{rg}. \end{cases} \tag{36}$$



**Figure 3.** Ranking results via  $q$ -RF2LPDMSM operator with different values of  $\theta$ .

$q$	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$	$\beta_5$	Ranking relation
3	0.0776	0.0737	0.0782	0.0732	0.0826	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
5	0.0767	0.0732	0.0774	0.0728	0.0821	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
7	0.0758	0.0726	0.0768	0.0721	0.0814	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
9	0.0750	0.0720	0.0762	0.0715	0.0809	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
11	0.0743	0.0716	0.0758	0.0711	0.0804	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
13	0.0737	0.0711	0.0754	0.0706	0.0801	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
15	0.0732	0.0708	0.0750	0.0704	0.0800	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$

**Table 10.** Ranking results of  $q$ -RF2LPDMSM-based WASPAS with varying  $q$  values.

where  $d_H(\partial_{ig}, \partial_{rg})$  is the Hamming distance between  $\partial_{ig}$  and  $\partial_{rg}$ <sup>48</sup>.

$$\Pi(\beta_i, \beta_r) = \sum_{g=1}^n w_g P_g(\beta_i, \beta_r), \tag{37}$$

where  $P_g(\beta_i, \beta_r) = f(d_g(\beta_i, \beta_r))$ . Here,  $f(\cdot)$  is the preference function.

Following Eq. (37), the global preference of each patient is conducted shown follows:

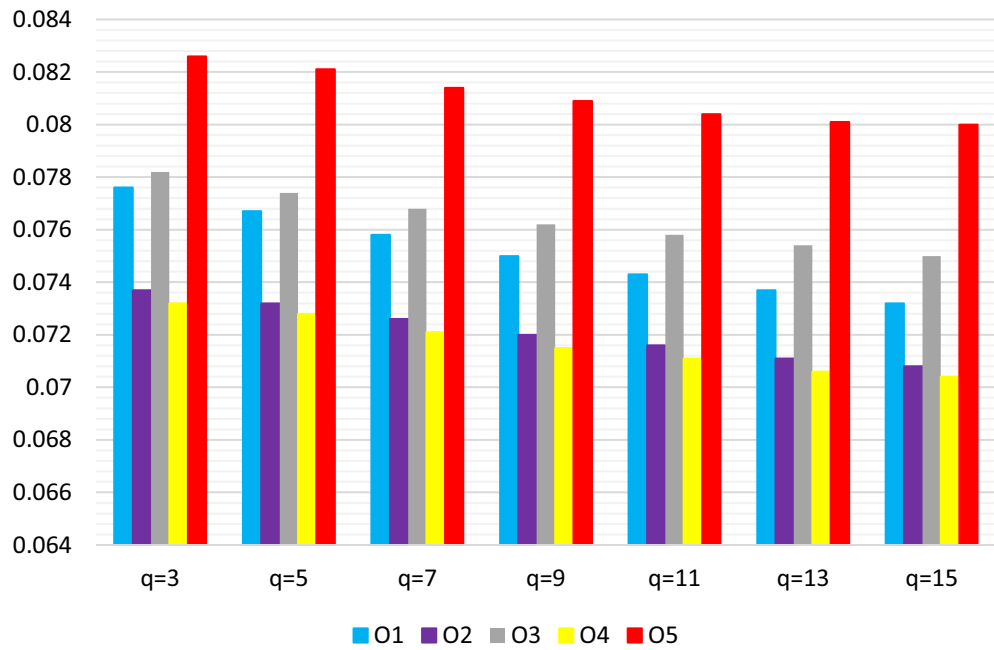
$$\varphi(\beta_i, \beta_g) = \begin{pmatrix} 0.0000 & 0.4381 & 0.2278 & 0.3981 & 0.3075 \\ 0.0990 & 0.0000 & 0.2799 & 0.2153 & 0.2926 \\ 0.0812 & 0.4724 & 0.0000 & 0.4443 & 0.2665 \\ 0.0351 & 0.2109 & 0.2596 & 0.0000 & 0.3111 \\ 0.2708 & 0.3367 & 0.3312 & 0.4435 & 0.0000 \end{pmatrix}$$

$$\psi^+(\beta_i) = \frac{1}{m-1} \sum \Pi(\beta_i, \beta_r), \tag{38}$$

$$\psi^-(\beta_i) = \frac{1}{m-1} \sum \Pi(\beta_r, \beta_i). \tag{39}$$

According to Eqs. (38) and (39), the positive flow  $\Psi^+(\beta_i)$  and negative flow  $\Psi^-(\beta_i)$  of each option are calculated, respectively, as given below:

$$\begin{aligned} \Psi^+(\beta_1) &= 0.3429, \Psi^+(\beta_2) = 0.2217, \Psi^+(\beta_3) = 0.3161, \Psi^+(\beta_4) = 0.2042, \Psi^+(\beta_5) = 0.3456. \\ \Psi^-(\beta_1) &= 0.1641, \Psi^-(\beta_2) = 0.3645, \Psi^-(\beta_3) = 0.2746, \Psi^-(\beta_4) = 0.3753, \Psi^-(\beta_5) = 0.2944. \end{aligned}$$



**Figure 4.** Ranking results via  $q$ -RF2LPMSM operator with different values of  $q$ .

$q$	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$	$\beta_5$	Ranking relation
3	0.0625	0.0569	0.0645	0.0555	0.0710	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
5	0.0617	0.0566	0.0637	0.0556	0.0699	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
7	0.0607	0.0562	0.0632	0.0553	0.0688	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
9	0.0602	0.0559	0.0624	0.0551	0.0680	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
11	0.0598	0.0557	0.0621	0.0548	0.0674	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
13	0.0592	0.0553	0.0618	0.0546	0.0669	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$
15	0.0588	0.0551	0.0616	0.0545	0.0664	$\beta_5 > \beta_3 > \beta_1 > \beta_2 > \beta_4$

**Table 11.** Ranking results of  $q$ -RF2LPDMSM-based WASPAS with varying  $q$  values.

Using the values of  $\Psi^+(\beta_i)$  and  $\Psi^-(\beta_i)$ , the net flows  $\Psi(\beta_i) = \psi^+(\beta_i) - \psi^-(\beta_i)$  for each alternative are determined as follows:

$$\Psi(\beta_1) = 0.1789, \Psi(\beta_2) = -0.1428, \Psi(\beta_3) = 0.0415, \Psi(\beta_4) = -0.1711, \Psi(\beta_5) = 0.0512.$$

Thus, the ranking with  $q$ -RF2L PROMETHEE II method<sup>48</sup> is  $\beta_1 > \beta_5 > \beta_3 > \beta_2 > \beta_4$ .

### $q$ -RFOL Muirhead mean aggregation operators-based method

This section focuses on obtaining results using the  $q$ -RF2L Muirhead mean ( $q$ -RF2LMM) AOs-based method, as detailed in<sup>34</sup>. The process is outlined through the following steps:

To aggregate  $q$ -RF2L assessment values  $\partial_{ig}$  for each alternative  $\beta_i$  across all criteria  $\in_g (g = 1, 2, 3, 4)$  into a single overall assessment value  $\partial_i$ , as shown in Table 4, we utilize the  $q$ -RF2LMM operator. This operator is detailed in Eq. (40), and is applied with the parameter  $P = (1, 1, 2, 1)$  to aggregate the values for each alternative  $\beta_i (i = 1, 2, 3, 4, 5)$  (Table 12).

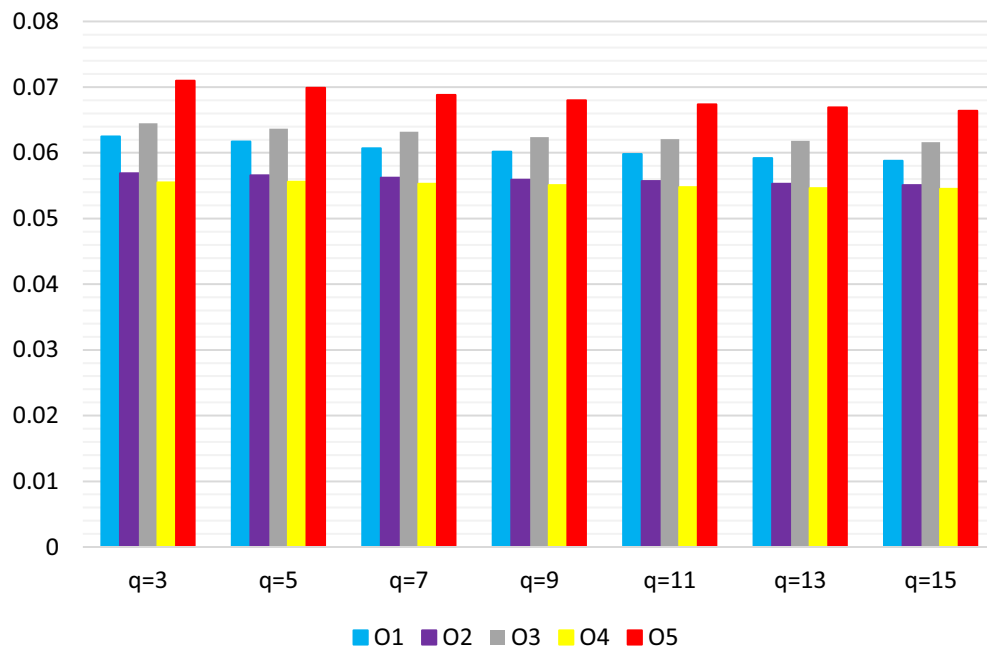


Figure 5. Ranking results via  $q$ -RF2DLMSM operator with different values of  $q$ .

	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$
$d_q(\beta_1, \beta_2)$	1.5124	0.0000	1.3771	0.0000
$d_q(\beta_1, \beta_3)$	0.0000	1.4568	0.3521	0.0420
$d_q(\beta_1, \beta_4)$	1.7063	0.0000	0.0000	1.4499
$d_q(\beta_1, \beta_5)$	0.0000	0.3028	2.5221	0.0528
$d_q(\beta_2, \beta_1)$	0.0000	0.3069	0.0000	0.8057
$d_q(\beta_2, \beta_3)$	0.0000	1.7637	0.0000	0.8478
$d_q(\beta_2, \beta_4)$	0.1939	0.2438	0.0000	2.2557
$d_q(\beta_2, \beta_5)$	0.0000	0.6097	1.1449	0.8585
$d_q(\beta_3, \beta_1)$	0.4821	0.0000	0.0000	0.0000
$d_q(\beta_3, \beta_2)$	1.9945	0.0000	1.0251	0.0000
$d_q(\beta_3, \beta_4)$	2.1884	0.0000	0.0000	1.4079
$d_q(\beta_3, \beta_5)$	0.0000	0.0000	2.1699	0.0000
$d_q(\beta_4, \beta_1)$	0.0000	0.0631	0.2053	0.0000
$d_q(\beta_4, \beta_2)$	0.0000	0.0000	1.5826	0.0000
$d_q(\beta_4, \beta_3)$	0.0000	1.5199	0.5574	0.0000
$d_q(\beta_4, \beta_5)$	0.0000	0.3659	2.7274	0.0000
$d_q(\beta_5, \beta_1)$	1.6088	0.0000	0.0000	0.0000
$d_q(\beta_5, \beta_2)$	3.1212	0.0000	0.0000	0.0000
$d_q(\beta_5, \beta_3)$	1.1267	1.1540	0.0000	0.0108
$d_q(\beta_5, \beta_4)$	3.3151	0.0000	0.0000	1.3971

Table 12. Deviations of pairwise evaluations.



$$q - RF2LMM(\partial_1, \partial_2, \dots, \partial_{\hbar}) = \left\langle \left( \begin{array}{l} \wedge \left( \left( \frac{1}{\hbar!} \sum_{\vartheta \in \mathbb{I}_{\hbar}} \prod_{g=1}^{\hbar} (\vee (s_{\vartheta g}, \kappa_{\vartheta g}))^{P_g} \right)^{\frac{1}{\sum_{g=1}^{\hbar} P_g}} \right), \right. \\ \left. \left( \left( 1 - \left( \prod_{\vartheta \in \mathbb{I}_{\hbar}} \left( 1 - \prod_{g=1}^{\hbar} m_{\vartheta g}^{q \cdot P_g} \right) \right)^{\frac{1}{\hbar!}} \right)^{\frac{1}{q}} \right)^{\frac{1}{\sum_{g=1}^{\hbar} P_g}} \right), \right. \\ \left. \left( \left( 1 - \left( 1 - \left( \prod_{\vartheta \in \mathbb{I}_{\hbar}} \left( 1 - \prod_{g=1}^{\hbar} (1 - n_{\vartheta g}^q) \right) \right)^{\frac{1}{\hbar!}} \right)^{\frac{1}{q}} \right)^{\frac{1}{\sum_{g=1}^{\hbar} P_g}} \right) \right) \right\rangle, \tag{40}$$

where  $P = (P_1, P_2, \dots, P_{\hbar})$  is a vector of parameters and  $\vartheta_g (g = 1, 2, \dots, \hbar)$  is a permutation of  $(1, 2, \dots, \hbar)$  and  $S_{\hbar}$  is a set of all permutations of  $(1, 2, \dots, \hbar)$ .

After the implication of Eq. (41) the overall assessment values of alternatives  $\beta_i (i = 1, 2, 3, 4, 5)$  are obtained as:

$$\begin{aligned} \partial_1 &= (\mathbb{I}_2, -0.4676), \langle 0.8509, 0.2654 \rangle, \partial_2 = (\mathbb{I}_1, 0.3420), \langle 0.9007, 0.2574 \rangle, \\ \partial_3 &= (\mathbb{I}_1, 0.3719), \langle 0.8925, 0.1619 \rangle, \partial_4 = (\mathbb{I}_1, 0.3198), \langle 0.8835, 0.1949 \rangle, \\ \partial_5 &= (\mathbb{I}_1, 0.3749), \langle 0.8944, 0.2230 \rangle. \end{aligned}$$

According to Eq. (6), the score values of overall assessment value  $\partial_i (i = 1, 2, 3, 4, 5)$  are determined as given below:

$$\widehat{S}(\partial_1) = 0.1748, \widehat{S}(\partial_2) = 0.1643, \widehat{S}(\partial_3) = 0.1672, \widehat{S}(\partial_4) = 0.1586, \widehat{S}(\partial_5) = 0.1674.$$

Based on the above score values, the ranking of all patients is obtained:  $\beta_1 > \beta_5 > \beta_3 > \beta_2 > \beta_4$ .

Based on the outcome provided above, it is evident that there exists a minor distinction in the ranking of the two approaches, i.e., the alternative  $\beta_1$  only changes the position while the remaining alternatives have the same rank.

**q-Rung orthopair fuzzy Macularin symmetric mean operators-based method**

The effectiveness of the proposed method is further substantiated through a comparative analysis with the q-rung orthopair Maclaurin symmetric mean (q-ROFMSM) operators-based method created by Wei et al.<sup>49</sup>. In order to align the Wei et al. method with the specific issue at hand, we have excluded the linguistic terms and their corresponding symbolic translations, as detailed in the data presented in Table 4. This modification enables us to derive the subsequent calculated results, which are comprehensively displayed in Table 13.

$$q - ROFMSM(a_1, a_2, \dots, a_n) = \left\langle \left( \left( 1 - \prod_{\Omega} \left( 1 - \left( \prod_{g=1}^p (m_{i_g}) \right)^{\frac{1}{c_n}} \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \right), \left( 1 - \left( 1 - \left( \prod_{\Omega} \left( 1 - \prod_{g=1}^p (1 - n_{i_g}^q) \right) \right)^{\frac{1}{c_n}} \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \right\rangle, \tag{41}$$

where  $a_i = \langle m_i, n_i \rangle (i = 1, 2, \dots, \hbar)$  represent family of q-ROFNs,  $p = 1, 2, \dots, \hbar$  and  $\Omega = 1 \leq i_1 < \dots < i_p \leq \hbar$ .

By employing the q-ROFMSM operator (see Eq. (41)), the overall assessment values for the alternatives  $\beta_i (i = 1, 2, 3, 4, 5)$  are calculated as follows:

	c <sub>1</sub>	c <sub>2</sub>	c <sub>3</sub>	c <sub>4</sub>
$\beta_1$	(0.8796, 0.2415)	(0.8171, 0.2991)	(0.8326, 0.2924)	(0.8735, 0.2093)
$\beta_2$	(0.8729, 0.2348)	(0.8749, 0.2295)	(0.9162, 0.1942)	(0.9185, 0.3314)
$\beta_3$	(0.9011, 0.1420)	(0.9455, 0.2347)	(0.8303, 0.0876)	(0.8946, 0.0951)
$\beta_4$	(0.8822, 0.2127)	(0.8964, 0.1647)	(0.8709, 0.2378)	(0.8838, 0.1307)
$\beta_5$	(0.8453, 0.2623)	(0.9129, 0.1944)	(0.9153, 0.1633)	(0.9048, 0.2460)

**Table 13.** q-rung orthopair fuzzy group decision matrix.

Method	Ranking results
$q$ -RFL PROMETHEE II method <sup>48</sup>	$\beta_1 > \beta_5 > \beta_3 > \beta_2 > \beta_4$
$q$ -RF2LMM AOs-based method <sup>44</sup>	$\beta_1 > \beta_5 > \beta_3 > \beta_2 > \beta_4$
$q$ -ROFMSM operators-based method <sup>49</sup>	$\beta_3 > \beta_5 > \beta_2 > \beta_4 > \beta_1$
Proposed method	$\beta_5 > \beta_3 > \beta_1 > \beta_4 > \beta_2$

**Table 14.** Decision results obtained by using different methods.

$$\alpha_1 = (0.8519, 0.2632), \alpha_2 = (0.9014, 0.2513), \alpha_3 = (0.8957, 0.1462), \alpha_4 = (0.8836, 0.1909), \alpha_5 = (0.8959, 0.2199).$$

Based on the score function of  $q$ -ROFNs<sup>19</sup>, the score values of overall assessment value  $\alpha_i$  ( $i = 1, 2, 3, 4, 5$ ) are determined as outlined below:

$$S(\alpha_1) = 0.8000, S(\alpha_2) = 0.8502, S(\alpha_3) = 0.8578, S(\alpha_4) = 0.8415, S(\alpha_5) = 0.8543.$$

Based on the above score values, the ranking of all patients is obtained:  $\beta_3 > \beta_5 > \beta_2 > \beta_4 > \beta_1$ .

From Table 12, it can be seen that all the ranking results of the two approaches are different except  $\beta_4$ . The difference is due to the absence of linguistic terms in the approach of Wei et al.<sup>49</sup>. It is basically developed on  $q$ -ROFNs, which only process the uncertain data in quantitative form. This technique is unsuitable for modeling decision information when DMs prefer utilizing linguistic values.

## Discussion

- (i) Analysis of Table 14 reveals that alternative  $\beta_5$ , initially deemed optimal, ranks second when evaluated through existing methodologies. Specifically, in the method proposed by Lie et al.<sup>48</sup>, the  $q$ -RF2LWA operator is utilized for the aggregation phase in decision-making. A notable limitation of this operator is its failure to account for the interrelationships among multiple input arguments. Additionally, the Lie et al. method lacks mechanisms to prioritize criteria and experts. Although it includes weight determination strategies, these are not sufficient for practical problems where prioritization of both criteria and DMs is crucial. In such contexts, our proposed approach demonstrates superior adaptability and effectiveness.
- (ii) When compared with the Ju et al. method<sup>34</sup>, it's evident that the overall rankings diverge from those obtained using our proposed approach. This discrepancy primarily arises because the existing approach focuses solely on the interrelationship among input arguments, as evident from Eq. (40). It also overlooks the crucial aspect of modeling the prioritization relationships among experts and the set criteria. In contrast, our proposed methods, which leverage the  $q$ -RF2LPMSM and  $q$ -RF2LPDMSM, adeptly handle the interrelationships among multiple input arguments and duly consider the prioritization among both DMs and criteria. Additionally, our method integrates the  $q$ -RF2LPMSM and  $q$ -RF2LPDMSM operators within the WASPAS method framework. These distinctive features render our proposed methods notably more practical and adaptable.
- (iii) From Table 14, we can see that most of the alternative ranking positions obtained via the Wei et al.<sup>49</sup> approach is inconsistent with that obtained through our proposed method. This discrepancy primarily stems from the fact that the operator in Eq. (41) evaluates alternatives using  $q$ -rung orthopair fuzzy data but disregards linguistic terms and their symbolic translations. In contrast, the  $q$ -RF2LPMSM and  $q$ -RF2LPDMSM operators introduced in our study focus on  $q$ -RF2L data and effectively incorporate these linguistic aspects. When the linguistic terms and their symbolic translations are omitted, these operators essentially become the  $q$ -rung orthopair fuzzy prioritized MSM and DMSM operators, respectively. The inclusion of linguistic terms and symbolic translations, along with the prioritization feature, enables our operators to mirror the subjective perspectives of DMs more accurately. The omission of these aspects in Wei et al.'s approach results in a loss of crucial information, potentially leading to erroneous outcomes. Furthermore, the approach of Wei et al. is limited to MCDM problems and does not adequately address DM scenarios involving multiple experts.

From the preceding discussion, the key advantages of our initiated approach are summarized as follows:

1. The foremost benefit of our proposed approach lies in the design of the formulated operators, which adeptly incorporate interdependencies among arguments. Additionally, these operators facilitate prioritization among criteria and DMs within the DM process.
2. The proposed method demonstrates robust stability, even with significant adjustments made to the involved parameters,  $\theta$  and  $q$ . This stability is evidenced by the consistent ranking order, as detailed in "5.3" section.
3. The AOs developed in our study (Eqs. (13) and (21)) are advanced generalizations of numerous existing AOs. For details and insights into their extended capabilities, please refer to "3.1" and "3.2" sections.
4. Unlike the existent methods<sup>34,49</sup>, our developed approach is not solely reliant on AOs. It innovatively integrates the formulated operators,  $q$ -RF2LPMSM and  $q$ -RF2LPDMSM, with the WASPAS method within

a q-RF2L environment. This integration significantly enhances the adaptability and applicability of our methodology, setting it apart from existing methods.

In addition to the aforementioned advantages, it is noteworthy that the proposed prioritized AOs exhibit certain limitations. Specifically, these AOs lack the property of idempotency, wherein the weighted average value of equivalent fuzzy numbers fluctuates with their prioritized weights. Moreover, the designed AOs do not conform to the MSM/DMSM when their prioritized weights are equal, thus indicating a deficiency in reducibility property.

### Concluding remarks

The pressing need for efficient allocation of limited medical resources has led many healthcare facilities to prioritize patient hospitalization. In response, this paper introduced a novel MCGDM technique to address patient prioritization effectively. This technique, which acknowledges the importance of prioritization relationships, integrates the MSM and the DMSM operators with the PA operator, leading to the development of the q-RF2LPMSM and q-RF2LPDMSM operators. We have thoroughly verified the key characteristics of these newly proposed prioritized aggregation operators and examined various special cases. Building on these operators, we have enhanced the WASPAS method to create an innovative ranking mechanism tailored for q-RF2L MCGDM challenges. The q-RF2LPMSM and q-RF2LPDMSM operators are not only adept at managing the interdependence among multiple input parameters but also effectively consider the prioritization among experts and criteria, reflecting the experts' attitudinal perspectives. The created method is highly flexible, which makes it suitable for resolving similar prioritization challenges that utilize q-RF2L information for evaluating alternatives. However, this study has a limitation in that it does not consider the weights of DMs and the absence of a consensus-building model among them. Future research should aim to improve information aggregation by determining the weights of DMs and refining the formulations of prioritized operators to ensure compliance with the properties of idempotency and reducibility.

### Data availability

All data generated or analysed during this study are included in this published article.

Received: 1 January 2024; Accepted: 18 March 2024

Published online: 09 May 2024

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## Author contributions

All the authors have contributed equally in this paper. All authors appeared in this paper agreed to publication in this journal.

## Competing interests

The authors declare no competing interests.

## Additional information

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