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Exact solutions for nonlinear partial differential equations via a fusion of classical methods and innovative approaches

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This paper presents a new approach for finding exact solutions to certain classes of nonlinear partial differential equations (NLPDEs) by combining the variation of parameters method with classical techniques such as the method of characteristics. Our primary focus is on NLPDEs of the form $u_{tt} + a(x,t)u_{xt} + b(t)u_t = \alpha(x,t) + G(u)(u_t + a(x,t)u_x)e^{-\int b(t)dt}$ and $u_t^m(u_{tt} + a(x,t)u_{xt}) + b(t)u_t^{m+1} = e^{-(m+1)\int b(t)dt}(u_t + a(x,t)u_x)F(u, u_t e^{\int b(t)dt})$. We provide numerical validation through several examples to ensure accuracy and reliability. Our approach enhances the applicability of analytical solution methods for a broader range of NLPDEs.

Keywords Partial differential equations, Nonlinear partial differential equations, Variation of parameters, Method of characteristics, Mathematica

Nonlinear partial differential equations are prevalent in many physical problems, such as solid mechanics, fluid dynamics, acoustics, nonlinear optics, plasma physics, and quantum field theory. They also find applications in chemical and biological systems and formulate the fundamental laws of nature. Within this broad spectrum, a particularly intriguing class of nonlinear partial differential equations known as soliton equations gives rise to physically attractive solutions known as solitons. These solitons have significantly contributed to the field of applied sciences. For a comprehensive understanding of these phenomena, refer to[1–11] and the references therein, which offer detailed insights from both the theoretical and experimental perspectives.

There are important works related to the recent development in partial differential equations and their applications, including nonlinear pseudo hyperbolic partial differential equations[12–15] and third-order fractional partial differential equations[16–18]. Pursuing suitable analytical methods to solve nonlinear partial differential equations is a central focus. Among the most widely adopted techniques are the variational iteration method[19], the inverse scattering method[20], the integral transform method[21], the truncated expansion method[22], the extended tanh-function method[23], Jacobi elliptic method[24], the Backlund transformations[25], F-expansion method[26,27], the sine-cosine function method[28], the (G'/G)-expansion method[29], and various extensions.

One of the valuable tools for solving certain types of PDEs is the method of characteristics[1,9–11] and[30]. It involves transforming a PDE into a set of ordinary differential equations along characteristic curves. The characteristic curves represent the paths along which the solution of the PDE remains constant. The method of characteristics is a powerful technique for solving first-order partial differential equations (PDEs), including linear first-order PDEs such as the transport equation or the linear advection equation.

The well-known classical method usually refers to the variation of parameters[31–34] and[35]. The variation of parameters is primarily a technique used for linear differential equations, both ordinary and partial. It involves finding a particular solution to a non-homogeneous equation by introducing a new function to replace a constant in the homogeneous solution. Solving NLPDEs can pose considerably greater complexity and demand a problem-specific approach since nonlinear equations lack the superposition properties present in linear equations. The approach relies on the particular structure and characteristics of the NLPDE being addressed.

The variation of parameters method has been successfully applied to certain nonlinear differential equations. We can refer to[31] and[33] as interesting studies. Common examples of second-order equations that can be converted into first-order forms include various types of nonlinear wave equations, heat equations, and specific

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conservation laws. The exact procedure for this reduction may vary depending on the specific equation and the desired format for further analysis.

However, the method of characteristics and the variation of parameters are two distinct methods used in different contexts. While these two methods have distinct applications, this study shows that combining the classical techniques derives new solutions for NLPDEs with specific initial conditions.

Several analytical methods consistently solve classes of second-order differential equations by variation of parameters. In[31] and[33], some types of nonlinear differential equations have been reduced to first-order using suitable parameter variations. The resulting first-order differential equations are, in most cases, transformable to well-known integrable or solvable classical differential equations. However, these methods are not applicable when dealing with nonlinear partial differential equations. Certain types of NLPDEs remain unsolvable using variations of parameters independently. By leveraging the strengths of classical techniques, we demonstrate an expanded scope of solvable NLPDEs, thus increasing the applicability of analytical solution methods to a broader range of problems.

As an extension of a previous study[36], we introduced new solutions to NLPDEs. In this study, we consider the classes of nonlinear partial differential equations of the form:

$$u_{tt} + a(x,t)u_{xt} + b(t)u_t = \alpha(x,t) + G(u)(u_t + a(x,t)u_x)e^{-\int b(t)dt},$$

and

$$u_t^m(u_{tt} + a(x,t)u_{xt}) + b(t)u_t^{m+1} = e^{-(m+1)\int b(t)dt}(u_t + a(x,t)u_x)F(u, u_t e^{\int b(t)dt}).$$

Notably, some exceptional cases can arise. For example, we mention the nonlinear differential equations recorded in[31], where the functions were restricted to one variable.

The remainder of this paper is organized as follows. In "First class of reducible nonlinear partial differential equations", we apply our methodology to the first class of reducible second-order partial differential equations to determine the exact solutions of NLPDEs of the first type.

"Second class of reducible nonlinear partial differential equations" delves into the second class of reducible nonlinear partial differentiable equations. Based on these results, a new class of solutions was derived. We demonstrate the application of the proposed method using concrete examples to demonstrate its viability and efficiency. Using Mathematica algorithms, relevant numerical representations were exhibited in each example to show the pertinence of obtained analytical solutions. Finally, "Conclusion" concludes the paper.

First class of reducible nonlinear partial differential equations

Description of the method and construction of the general solutions

We consider the first class of nonlinear second-order partial differential equations compilable in the following general form:

$$u_{tt} + a(x,t)u_{xt} + b(t)u_t = \alpha(x,t) + G(u)(u_t + a(x,t)u_x)e^{-\int b(t)dt}, \tag{1}$$

where u denotes a function of $(x,t) \in \mathbb{R}^2$.

First, we solve the characteristic equation

$$\begin{cases} \frac{d}{dt}(x(t)) &= a(x(t),t) \\ x(0) &= x_0. \end{cases}$$

Then (1) can be rewritten as

$$u_{tt} + a(x(t),t)u_{xt} + b(t)u_t = \alpha(x(t),t) + G(u)(u_t + a(x(t),t)u_x)e^{-\int b(t)dt}. \tag{2}$$

Multiplying both sides of (2) by $e^{\int b(t)dt}$, we get

$$\frac{d}{dt}(u_t(x(t),t)e^{\int b(t)dt}) = \alpha(x(t),t)e^{\int b(t)dt} + G(u)(u_t + a(x(t),t)u_x). \tag{3}$$

The nonlinear second-order partial differential equation (1) can be solved easily if we assume that

$$u_t(x,t) = (H(t) + K(u))e^{-\int b(t)dt},$$

where H and K are differentiable functions of t and u respectively.

Then, we differentiate to obtain

$$\frac{d}{dt}(u_t(x(t),t)e^{\int b(t)dt}) = H'(t) + K'(u)(u_t + a(x(t),t)u_x). \tag{4}$$

Substituting (4) into (3), we find that:

$$H'(t) = \alpha(x(t),t)e^{\int b(t)dt}, \tag{5}$$

and

$$K'(u) = G(u). \tag{6}$$

We obtain the following result.

Proposition 1 *The second order partial differential equation (1) can be reduced to the first order differential equation*

$$u_t(x,t) = (H(t) + K(u))e^{-\int b(t)dt},$$

where the functions H and K are the general solutions of $H'(t) = \alpha(x(t),t)e^{\int b(t)dt}$, *and* $K'(u) = G(u)$.

Remark 1 Let $G = u^n$, where n is a non zero positive integer.

Then, the second order partial differential equation (1) becomes

$$u_{tt} + a(x,t)u_{xt} + b(t)u_t = \alpha(x,t) + u^n(u_t + a(x,t)u_x)e^{-\int b(t)dt}.$$

Applying Eqs. (5) and (6), we get an Abel equation of the form

$$u_t(x,t)) = (H(t) + \frac{u^{n+1}}{n+1})e^{-\int b(t)dt}.$$

A comprehensive compilation of integrable Abel equations can be found in[37–39] and[40].

Application

Example 1 **Let** $a(x,t) = x$**,** $\alpha(x,t) = xe^{-t}$**,** $b(x,t) = 1$ **and** $G(u) = 2u$**.**

$$u_{tt} + xu_{xt} = xe^{-t} + 2u(u_t + xu_x)e^{-t}. \tag{7}$$

with the initial conditions $u(x,0) = 1$ and $u_t(x,0) = x + 1$.

Solution:

We solve the characteristic equation

$$\begin{cases} \frac{dx(t)}{dt} &= x \\ x(0) &= x_0 \end{cases}$$

which leads to $x(t) = x_0e^t$.

The functions H and K are general solutions of

$$\begin{aligned} H'(t) &= x(t)e^{-t}e^t = x(t), \\ K'(u) &= \quad 2u. \end{aligned}$$

Then we get

$$H(t) = x + C_1,$$

and

$$K(u) = u^2 + C_2,$$

where C_1 and C_2 are arbitrary constants.

The second-order partial differential equation (7) is reduced to the first-order differential equation

$$u_t(x,t) = (u^2 + x)e^{-t}, \tag{8}$$

with initial condition $u(x,0) = 1$.

The first-order differential equation (8) is an Abel equation, which can be solved using various methods. For more details, refer to[37–39].

Using initial condition $u(x,0) = 1$ and $u_t(x,0) = x + 1$, we obtain explicit solutions of (7)

$$u(x,t) = \sqrt{x}\ \tan\left(\sqrt{x}\,\mathrm{e}^{-t}\left(-1 + \frac{e^t\left(\sqrt{x} - arcos\left(-\frac{\sqrt{x}}{\sqrt{1+x}}\right)\right)}{\sqrt{x}}\right)\right).$$

Visualizing the precise solutions obtained by Mathematica algorithms (Fig. 1) and plotting solution profiles at different values of t, we observe the characteristics of several solutions of (7) with initial conditions $u(x,0) = 1$ and $u_t(x,0) = x + 1$. As a result, these solutions develop singularities at certain values of x and t. Note that despite the smoothness of the initial data, the spontaneous singular behavior in the solutions must be due to the nonlinear term of the equation.

Figure 1 displays the 2D, 3D and contour plots of the solutions in (7) within $-10 \leq x \leq 10$ and $0 \leq t \leq 4$ for 3D and contour graphs, $t = 1$ for 2D graph.

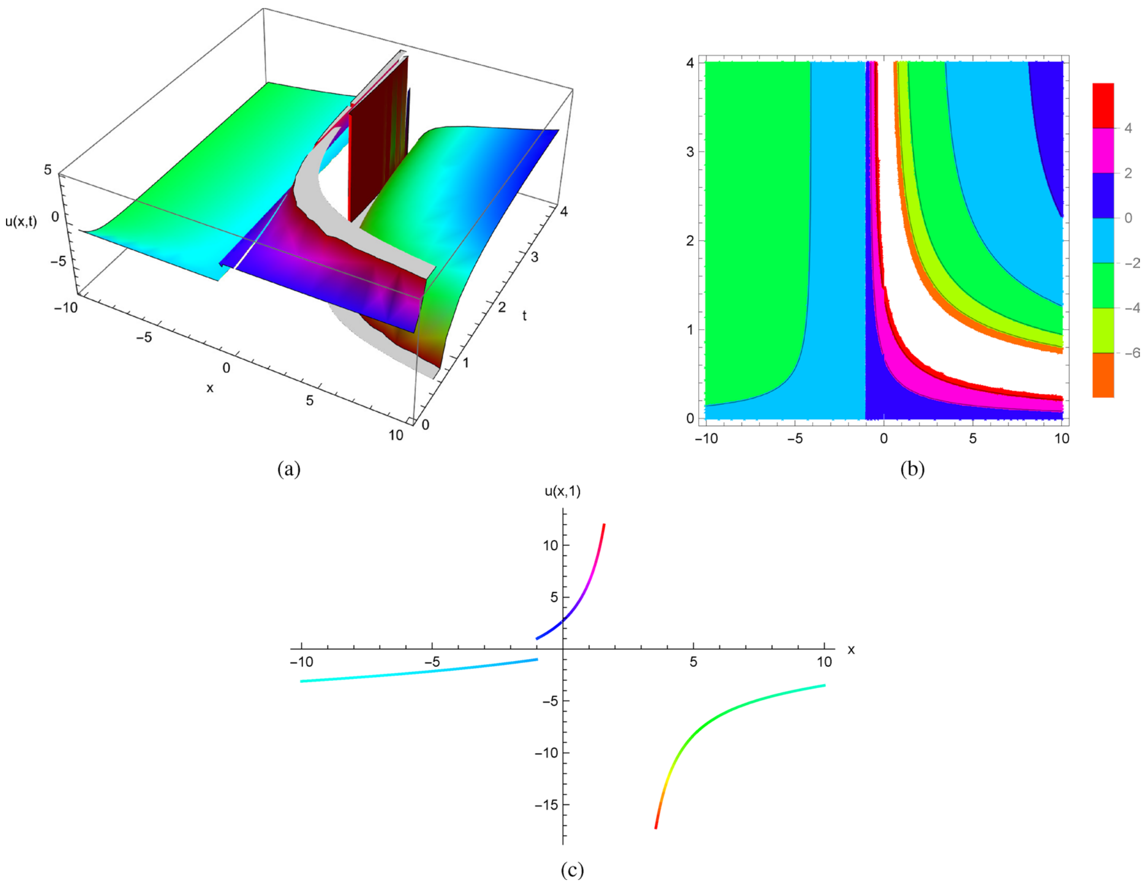

Figure 1. The profile of the solutions in (7) with $u(x,0) = x$ and $u_t(x,0) = x+1$: (**a**) and (**b**) 3D and Contour plots with $-10 \leq x \leq 10$ and $0 \leq t \leq 4$, (**c**) 2D plot at $t = 1$.

Second class of reducible nonlinear partial differential equations

Description of the method and construction of the general solutions

The second group of second-order partial differential equations is formulated as follows:

$$u_t^m(u_{tt} + a(x,t)u_{xt}) + b(t)u_t^{m+1} = e^{-(m+1)\int b(t)dt}(u_t + a(x,t)u_x)F(u, u_t e^{\int b(t)dt}). \tag{9}$$

As in the previous section, the second-order nonlinear partial differential equation (9) can be readily solved if we suppose that

$$u_t(x,t) = K(u)e^{-\int b(t)dt}. \tag{10}$$

where $u = u(x(t), t)$ and K is a differentiable function.

Then

$$(u_t(x,t))^{m+1} = K^{m+1}(u)e^{-(m+1)\int b(t)dt}. \tag{11}$$

Differentiate (11) to obtain

$$\begin{aligned}(m+1)u_t^m(u_{tt} + a(x,t)u_{xt}) =&(m+1)K^m(u)K'(u)(u_t + a(x,t)u_x)e^{-(m+1)\int b(t)dt}, \\ &- b(t)(m+1)K^{m+1}(u)e^{-(m+1)\int b(t)dt}.\end{aligned} \tag{12}$$

Then

$$\begin{aligned}&K^m(u)K'(u)(u_t + a(x,t)u_x)e^{-(m+1)\int b(t)dt} - b(t)K^{m+1}(u)e^{-(m+1)\int b(t)dt} + b(t)u_t^{m+1} \\ &= e^{-(m+1)\int b(t)dt}(u_t + a(x,t)u_x)F(u, u_t e^{\int b(t)dt}).\end{aligned} \tag{13}$$

Substituting Eqs. (10), (11) and (12) into (13), we find

$$K^m(u)K'(u) = F(u, K(u)).$$

Therefore, the following statement holds.

Proposition 2 *The second order nonlinear partial differential equation* (9) *can be reduced to the first order differential equation*

$$u_t(x,t) = K(u)e^{-\int b(t)dt}.$$

where the function K *is the general solution of* $K^m(u)K'(u) = F(u, K(u))$.

Applications

Example 2 **Let** $m = 0$**,** $b = 0$ **and** $a = 1$**. Suppose that function** F **satisfies** $F(s, w) = w$**.**

$$u_{tt} + u_{xt} = u_t^2 + u_x u_t, \tag{14}$$

with the initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = x^2$.

Solution:
Using the previous result, we find that the second-order nonlinear partial differential equation

$$u_{tt} + u_{xt} = u_t^2 + u_x u_t,$$

can be reduced to the first-order differential equation

$$u_t(x,t) = K(u), \tag{15}$$

where K is the general solution of $K'(u) = K(u)$. Taking $x(t) = t + x_0$, we obtain $K(u) = A(x_0)e^u$, where A is an arbitrary constant of integration.
The differential equation (15) takes the form

$$u_t(x,t) = A(x_0)e^u$$

and the exact solutions of the second-order nonlinear partial differential equation (14) are analytically determined and take the following form:

$$u(x,t) = -\ln(F(x-t) + G(x)),$$

where F and G are arbitrary functions.
It follows from the initial conditions at $(x_0, 0)$ given by $u(x, 0) = 0$ and $u_t(x, 0) = x^2$ that the exact solutions of (14) can be expressed explicitly as follows

$$u(x,t) = -\ln(1 - \frac{t^3}{3} + t^2x - tx^2).$$

Envisioning the precise solutions obtained by Mathematica (Fig. 2) and plotting solution profiles at different values of t, we have seen equations with smooth coefficients and initial data develop spontaneous singularities due to the nonlinearity of the equations. The solutions of (14) break down at some values of x and t, and no classical solution for the initial value problems exists beyond this point of breakdown.
Note that the nonlinear partial differential Eq. (14) yields a more straightforward solution than the initial value problem in the previous example.
Figure 2 displays the 2D, 3D and contour plots of the solutions in (14) within $-2 \le x \le 2$ and $0 \le t \le 2$ for 3D and contour graphs, $t = 2$ for 2D graph.

Example 3 Let $m = 0, b = 1$ and $a = 1$.
Suppose the function F satisfies $F(s, w) = w$.

$$u_{tt} + u_{xt} + u_t = (u_t + u_x)u_t, \tag{16}$$

with the initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = x^2$.

Solution:
Using the previous result, we find that the second-order nonlinear partial differential equation

$$u_{tt} + u_{xt} + u_t = (u_t + u_x)u_t,$$

can be reduced to the first-order differential equation

$$u_t(x,t) = e^{-t}K(u), \tag{17}$$

where K is the general solution of $K'(u) = K(u)$.
The differential equation (17) takes the form

$$u_t(x,t) = e^{-t}A(x_0)e^u,$$

and the exact solutions of the second order nonlinear partial differential equation (16) are analytically determined and take the following form:

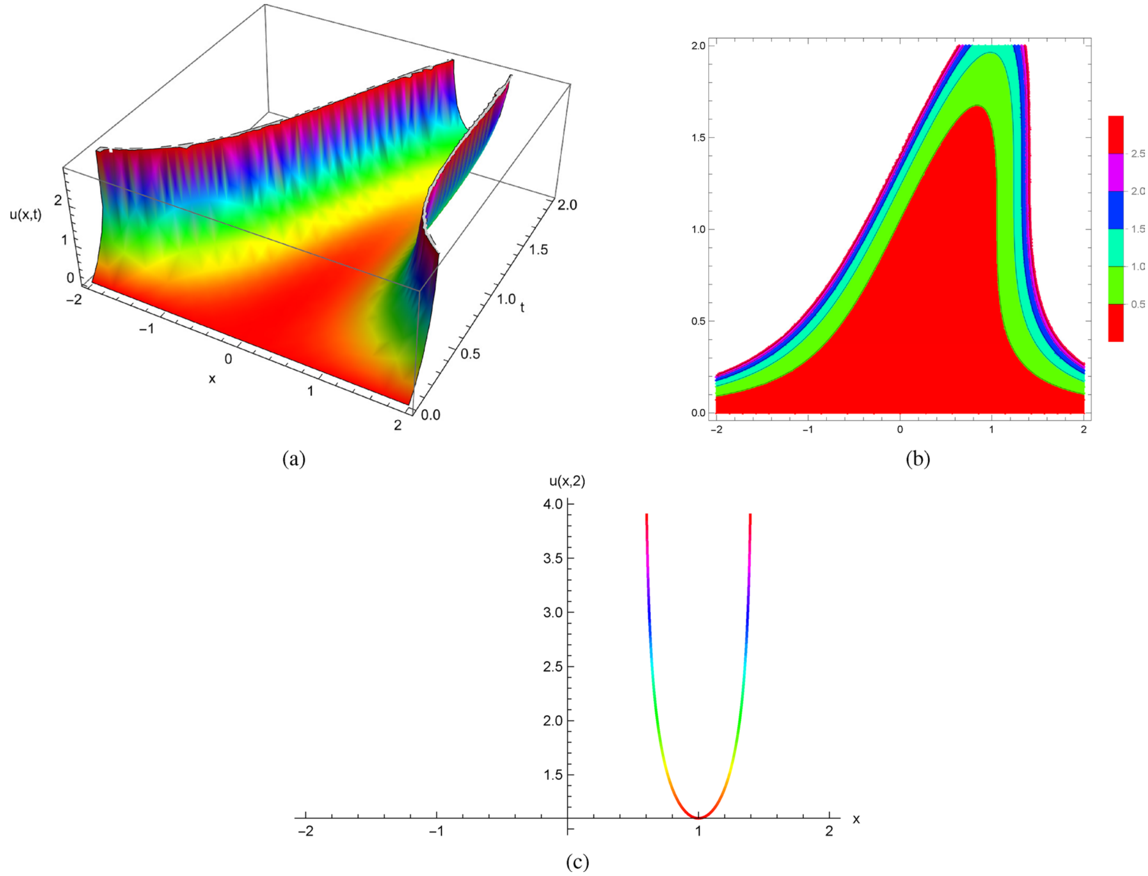

Figure 2. The profile of the solutions in (14) with $u(x, 0) = 0$ and $u_t(x, 0) = x^2$: (**a**) and (**b**) 3D and Contour plots with $-2 \leq x \leq 2$ and $0 \leq t \leq 2$, (**c**) 2D plot at $t = 2$.

$$u(x, t) = -\ln(e^{-t}F(x - t) + G(x))$$

where F and G are arbitrary functions.

It follows from the initial conditions at $(x_0, 0)$ given by $u(x, 0) = 0$ and $u_t(x, 0) = x^2$ that the exact solutions of (16) can be expressed explicitly as follows

$$u(x, t) = -\ln(e^{-t}t^2 + e^{-t}x^2 - 2e^{-t}tx - 2e^{-t}x + 2e^{-t}t + 2e^{-t} - x^2 + 2x - 1).$$

When we envision the exact solutions of (16) generated by Mathematica as depicted in Fig. 3 and create plots showing the solution profiles at various time points, we find that they deteriorate at specific values of both x and t. Beyond this point, a classical solution is no longer viable for the initial value problems.

Figure 3 displays the 2D, 3D and contour plots of the solutions in (16) within $-2 \leq x \leq 2$ and $0 \leq t \leq 2$ for 3D and contour graphs, $t = 2$ for 2D graph.

Example 4 Let $m = 0, b = 0$ and $a = 1$. Suppose that function F satisfies $F(s, w) = s$.

$$u_{tt} + u_{xt} = (u_t + u_x)u, \tag{18}$$

with the initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = x^2$.

Solution:

Using the previous result, we find that the second-order nonlinear partial differential equation

$$u_{tt} + u_{xt} = (u_t + u_x)u,$$

can be reduced to the first-order differential equation

$$u_t(x, t) = K(u), \tag{19}$$

where K is the general solution of $K'(u) = u$.

The differential equation (19) takes the form

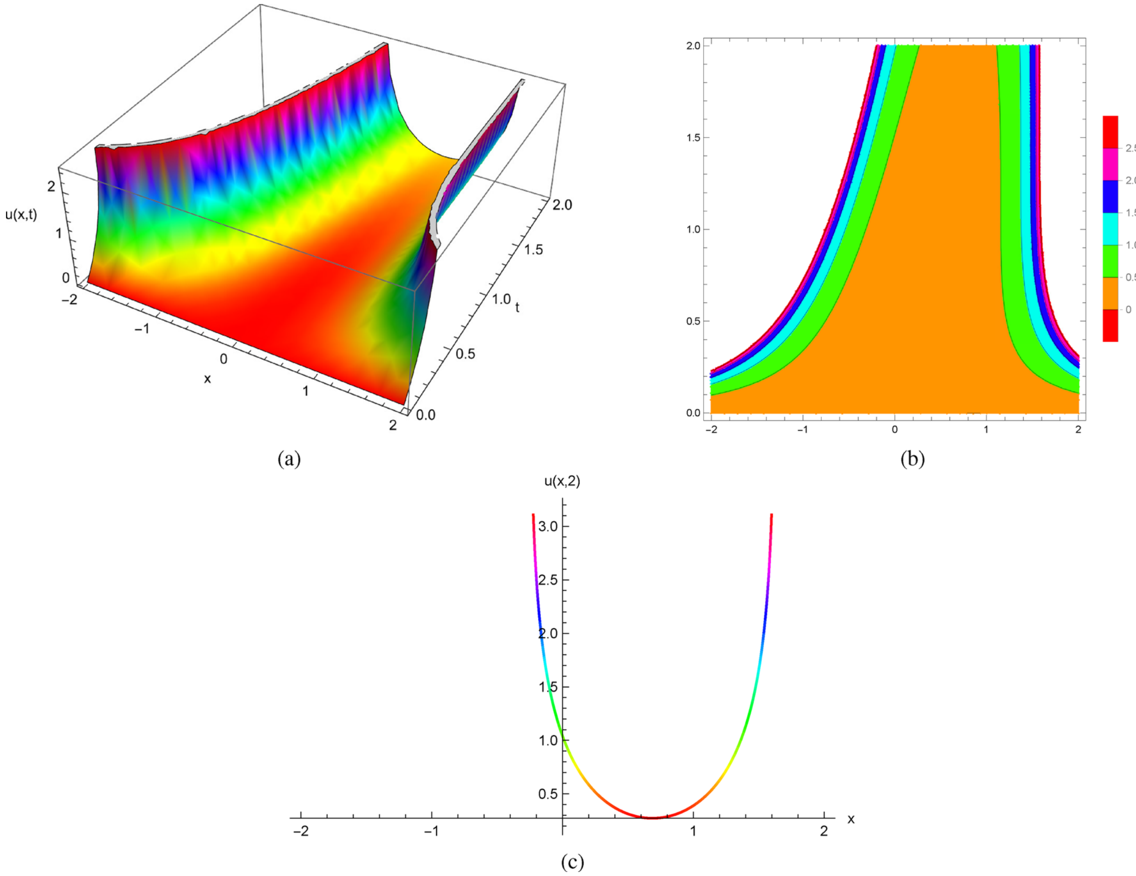

Figure 3. The profile of the solutions in (16) with $u(x,0)=0$ and $u_t(x,0)=x^2$: (**a**) and (**b**) 3D and Contour plots with $-2\leq x\leq 2$ and $0\leq t\leq 2$, (**c**) 2D plot at $t=2$.

$$u_t(x,t)=\frac{1}{2}u^2+f(x-t),$$

where f is an arbitrary function that leads to a Ricatti differential equation.

Taking in account the initial conditions $u(x,0)=0$ and $u_t(x,0)=x^2$, the differential equation (19) becomes

$$u_t(x,t)=\frac{1}{2}u^2+(x-t)^2.$$

The result was obtained using Mathematica code as a complicated function. As in the previous examples, the nonlinearity of the partial differential equations produces singular behavior in the solutions.

Figure 4 shows the 2D, 3D and contour plots of the solutions in (18) within $-2\leq x\leq 2$ and $0\leq t\leq 2$ for 3D and contour graphs, $t=2$ for 2D graph.

Example 5 Let $m=0, b=0$ and $a=x$.

Suppose that function F satisfies $F(s,w)=s^2$.

$$u_{tt}+xu_{xt}=(u_t+xu_x)u^2, \tag{20}$$

with the initial conditions $u(x,0)=x$ and $u_t(x,0)=\frac{x^3}{3}$.

Solution:

Using the previous result, we find that the second-order nonlinear partial differential equation

$$u_{tt}+xu_{xt}=(u_t+xu_x)u^2,$$

can be reduced to the first-order differential equation

$$u_t(x,t)=K(u), \tag{21}$$

where K is the general solution of $K'(u)=u^2$.

Differential equation (21) takes the form of an Abel equation

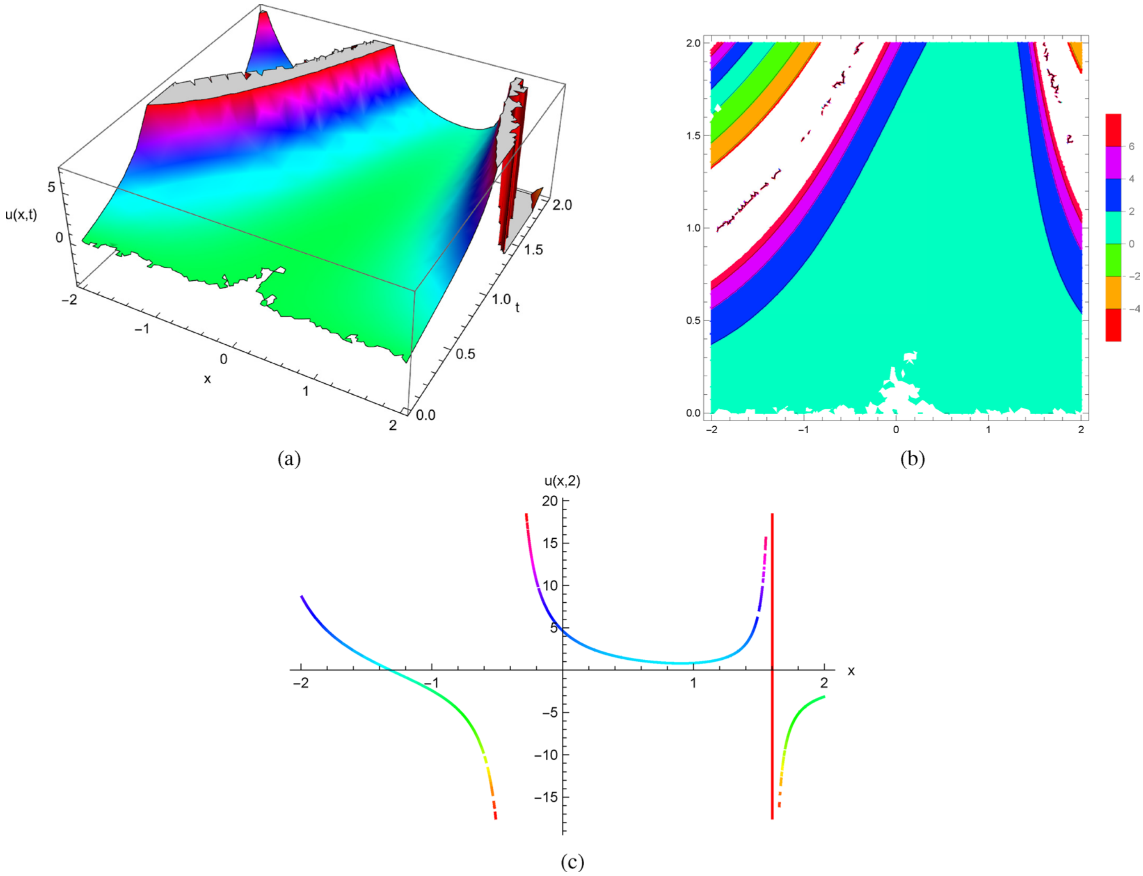

Figure 4. The profile of the solutions in (18) with $u(x,0)=0$ and $u_t(x,0)=x^2$: (**a**) and (**b**) 3D and Contour plots with $-2\le x\le 2$ and $0\le t\le 2$, (**c**) 2D plot at $t=2$.

$$u_t(x,t)=\frac{1}{3}u^3+f(xe^{-t}),$$

where f is an arbitrary function.

By applying the initial conditions $u(x,0)=x$ and $u_t(x,0)=\frac{x^3}{3}$, the exact solutions of (20) are implicitly obtained by generating the Mathematica codes.

Plotting the solution profiles for several values of t (as depicted in Fig. 5) shows that the solution breaks down at some points.

Figure 5 shows the 2D, 3D and contour plots of the solutions in (20) within $-1\le x\le 1$ and $0\le t\le 2$ for 3D and contour graphs, $t=1$ for 2D graph.

Example 6 Let $m=0, b(t)=\frac{1}{t}$.

Suppose that function F satisfies $F(s,w)=(\frac{w}{s})^2+2\frac{w}{s}$.

The second-order nonlinear partial differential equation (9) becomes

$$u_{tt}+a(x,t)u_{xt}+\frac{1}{t}u_t=\frac{1}{t}(u_t+a(x,t)u_x)((\frac{tu_t}{u})^2+2\frac{tu_t}{u}), \tag{22}$$

with the initial conditions $u(x,1)=1$ and $u_t(x,1)=x^2$.

Remark 2 Some interesting particular cases of (22) can be formed. As an example, we mention the equation

$$u''+\frac{1}{t}u'=tu'(\frac{u'}{u})^2+2\frac{u'^2}{u}, \tag{23}$$

recorded in[31] as Eq. (53). Equation (23) is obtained from (22) if we assume that u is only a function of t.

Solution:

Using our previous result, we find that (22) can be reduced to the first-order differential equation

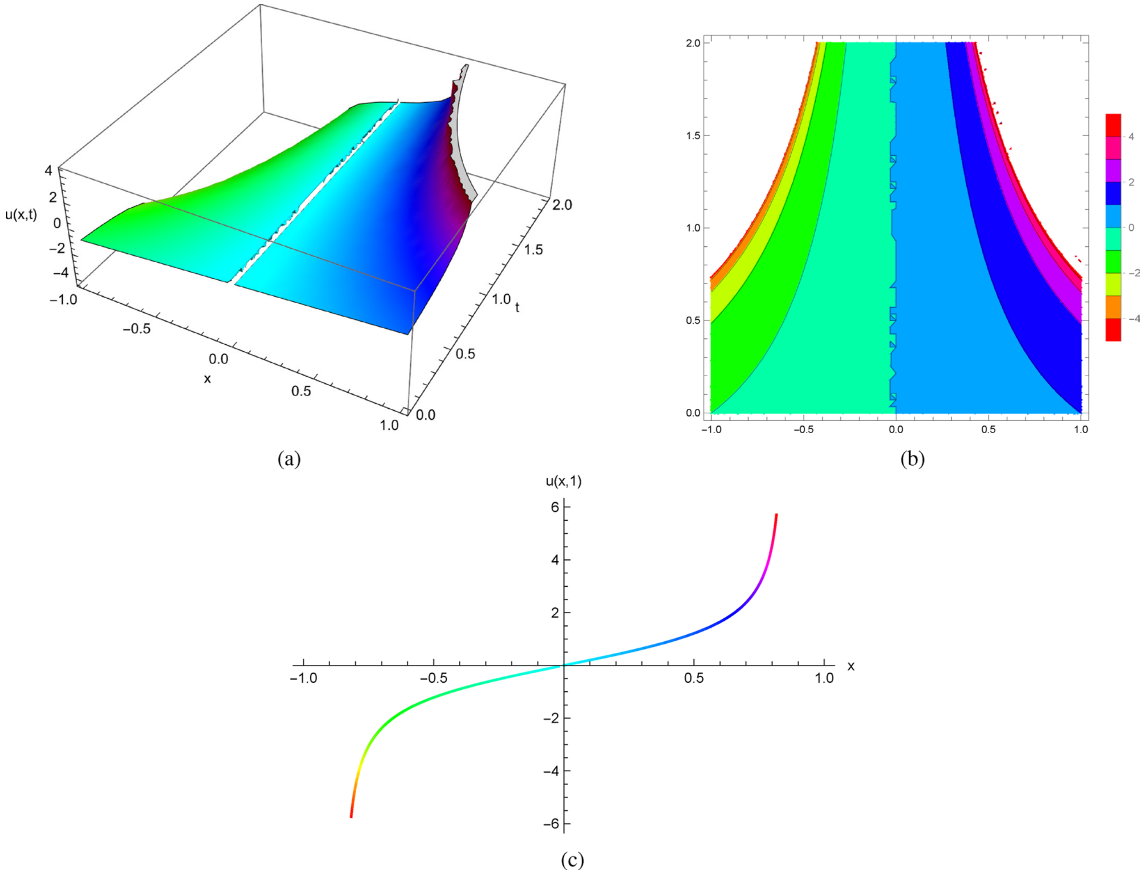

Figure 5. The profile of the solutions in (20) with $u(x,0)=x$ and $u_t(x,0)=\frac{x^3}{3}$: (**a**) and (**b**) 3D and Contour plots with $-1\leq x\leq 1$ and $0\leq t\leq 2$, (**c**) 2D plot at $t=1$.

$$u_t(x,t))=K(u)\frac{1}{t}, \tag{24}$$

where K is the general solution of $K'(u)=\frac{2}{u}K(u)+\frac{1}{u^2}K^2(u)$.

We get

$$K(u)=\frac{u^2}{A-u}.$$

For $a=1$, (22) takes the form

$$u_{tt}+u_{xt}+\frac{1}{t}u_t=\frac{1}{t}(u_t+u_x)((\frac{tu_t}{u})^2+2\frac{tu_t}{u}). \tag{25}$$

Then the general solutions of (25) are given by

$$u_t(x,t)=\frac{u^2}{f(x-t)-u}\frac{1}{t},$$

where f denotes an arbitrary function.

Remark 3 Let $u(x,t)=t^{-1}B(x)$ be a family of solutions to Eq. (25).

If $f(x,t)=A$, we get

$$-\frac{A}{u(x,t)}-\ln(u(x,t))=\ln(t)+B(x),$$

which is an implicit solution of (25) where A an arbitrary constant and B is a function of x.

We checked the implicit solutions of (25) by generating Mathematica codes and considering the initial conditions $u(x,1)=1$ and $u_t(x,1)=x^2$.

Figure 6 shows the 2D, 3D and contour plots of the solutions in (25) within $0 \leq x \leq 1$ and $0 \leq t \leq 2$ for 3D and contour graphs, $t = 2$ for 2D graph.

Example 7 Let $m = 0, b(t) = \frac{2}{t}$ and $a = 1$. Suppose that function F satisfies $F(s, w) = w + w^3$.

$$u_{tt} + u_{xt} + \frac{2}{t}u_t = (u_t + u_x)(u_t + t^4 u_t^3), \tag{26}$$

with the initial value conditions $u(x, 1) = x$ and $u_t(x, 1) = \frac{1}{\sqrt{2e^{-2x} - 1}}$.

Remark 4 In[31], the author studied special case of (26) where u is a single variable function of t.
In (26), if we suppose that u is only a function of t, we get

$$u'' + \frac{2}{t}u' = u'^2 + (tu')^4,$$

which is the differential equation (61) investigated by the authors in[31].

Solution:
(26) is reduced to the differential equation

$$u_t(x, t) = K(u)\frac{1}{t^2}, \tag{27}$$

where K is the general solution of

$$K'(u) = k(u) + k^3(u).$$
$$K(u) = \pm(Ae^{-2u} - 1)^{-\frac{1}{2}}.$$

Then the general solutions of 26 are given by

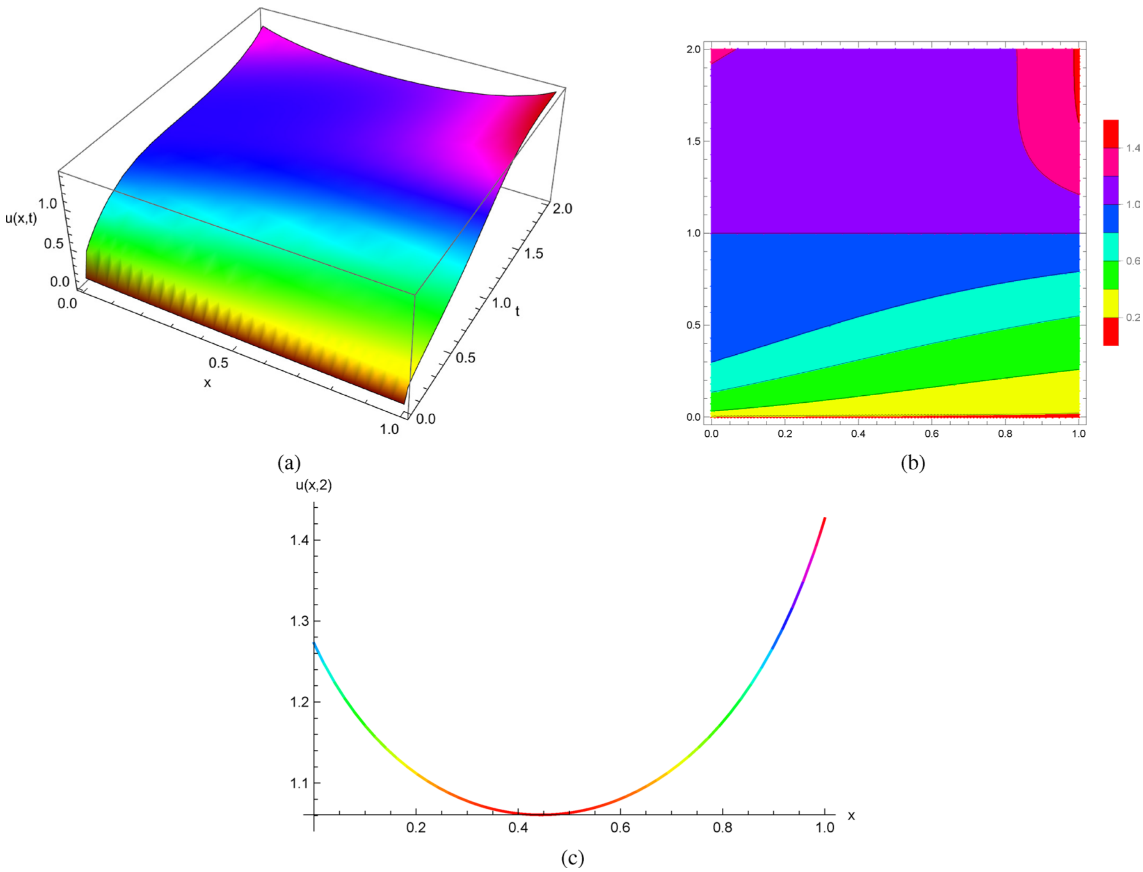

Figure 6. The profile of the solutions in (25) with $u(x, 1) = 1$ and $u_t(x, 1) = x^2$: (**a**) and (**b**) 3D and Contour plots with $0 \leq x \leq 1$ and $0 \leq t \leq 2$, (**c**) 2D plot at $t = 2$.

$$u_t(x,t) = t^{-2}(f(x-t)e^{-2u} - 1)^{-\frac{1}{2}},$$

where f is an arbitrary function.

By generating Mathematica codes, we obtain implicit solutions of (26)

$$\sqrt{2e^{-2u(x,t)}-1}\,t - \arctan\left(\sqrt{2e^{-2u(x,t)}-1}\right)t - t\sqrt{2e^{-2x}-1} + t\arctan\left(\sqrt{2e^{-2x}-1}\right) + t - 1 = 0.$$

Figure 7 shows the 2D, 3D and contour plots of the solutions in (26) within $-1 \leq x \leq 1$ and $0 \leq t \leq 2$ for 3D and contour graphs, $t = 2$ for 2D graph.

Example 8 Let $m = 0, b(t) = -\frac{1}{t}$. Suppose that function F satisfies $F(s,w) = 1 + 2\frac{s}{w}$.

$$u_{tt} + au_{xt} - \frac{1}{t}u_t = t(u_t + au_x)(1 + 2t\frac{u}{u_t}). \tag{28}$$

Remark 5 A special case of our findings was recorded in[31]. If we suppose that u is only a function of t in (28), we get

$$u'' - (\frac{1}{t} + t)u' - 2t^2u = 0,$$

which is exactly the differential equation (78) investigated by the authors of [31].

Solution:

(28) is reduced to the differential equation

$$u_t(x,t) = tK(u), \tag{29}$$

where K satisfies

$$K'(u)K(u) = K(u) + 2u. \tag{30}$$

Two particular solutions to (30) are given by $K_1(u) = 2u$ and $K_2(u) = -u$.

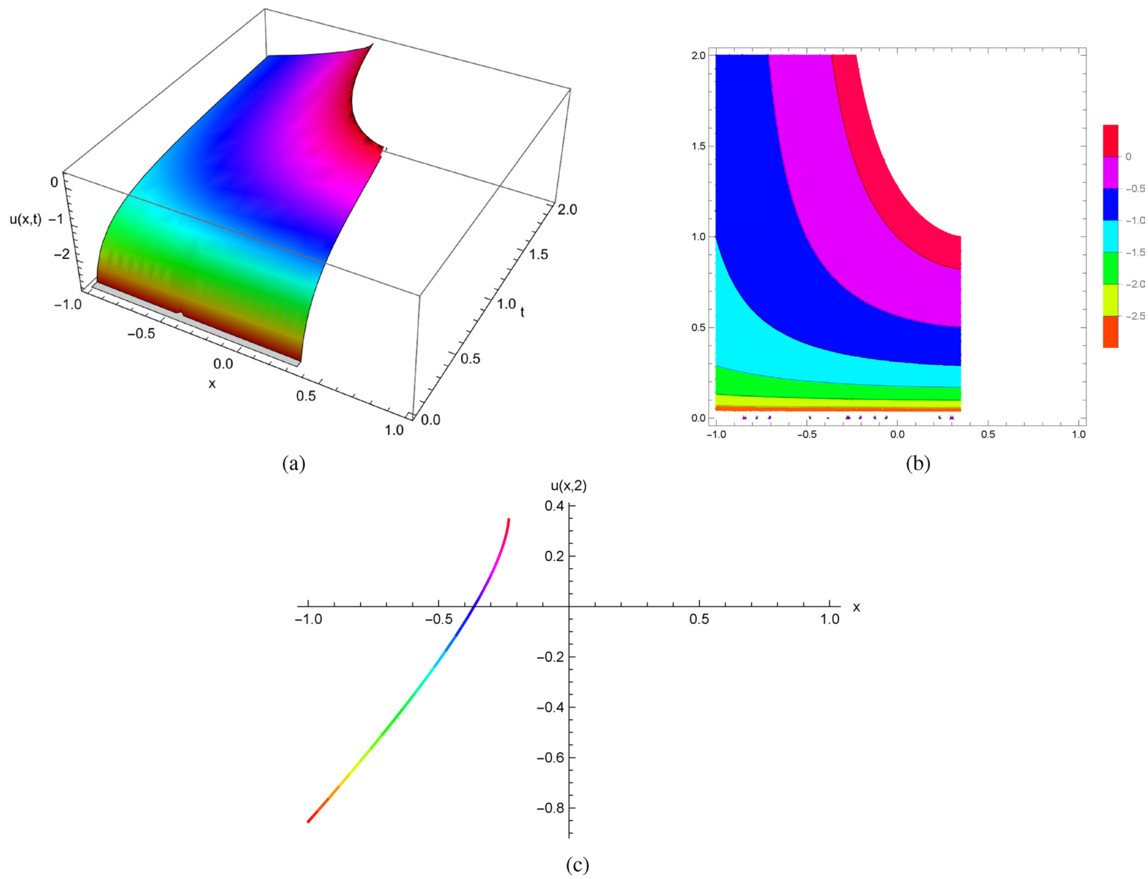

Figure 7. The profile of the solutions in (26) with $u(x,1) = x$ and $u_t(x,1) = \frac{1}{\sqrt{2e^{-2x}-1}}$: (**a**) and (**b**) 3D and Contour plots with $-1 \leq x \leq 1$ and $0 \leq t \leq 2$, (**c**) 2D plot at $t = 2$.

The general solutions of (30) satisfy the algebraic equation

$$(K(u) - 2u)^2(K(u) + u) = A, \tag{31}$$

where A is an arbitrary constant.

A real solution of (31) can be computed to yield

$$K(u) = \frac{\sqrt[3]{\sqrt{A^2 - 4Au^3} - 2u^3 + A}}{\sqrt[3]{2}} + \frac{\sqrt[3]{2}u^2}{\sqrt[3]{\sqrt{A^2 - 4Au^3} - 2u^3 + A}} + u.$$

If $a = 1$, $K(u) = \phi(u, x - t)$ where ϕ is an arbitrary function.

Hence, we obtain an implicit solution of (28) as

$$u_t(x, t) = t\phi(u, x - t).$$

We checked the implicit solutions of (28) by generating Mathematica codes and considering the initial conditions $u(x, 1) = 1$.

Figure 8 shows the 2D, 3D and contour plots of the solutions in (28) within $1 \leq x \leq 4$ and $0 \leq t \leq 4$ for 3D and contour graphs, $t = 2$ for 2D graph.

Conclusion

In this paper, we presented a new method, a combination of the variation of parameters and other techniques, such as the method of characteristics, to derive exact solutions of nonlinear partial differential equations alongside specific initial conditions, a framework extensively applied in mathematical physics. Illustrative examples were provided to demonstrate the applicability of this method. Problems that are non-trivial when approached with conventional methods now appear straightforward, as the resulting functions are univariate. Our research findings indicate that fusing established classical techniques with innovative approaches yields efficient analytical solutions.

However, the combined approach may require significant computational effort, especially for NLPDEs with complex boundary conditions or high-dimensional spaces. Additionally, instability or convergence problems may arise, especially if the initial parameter or condition estimations are inadequate.

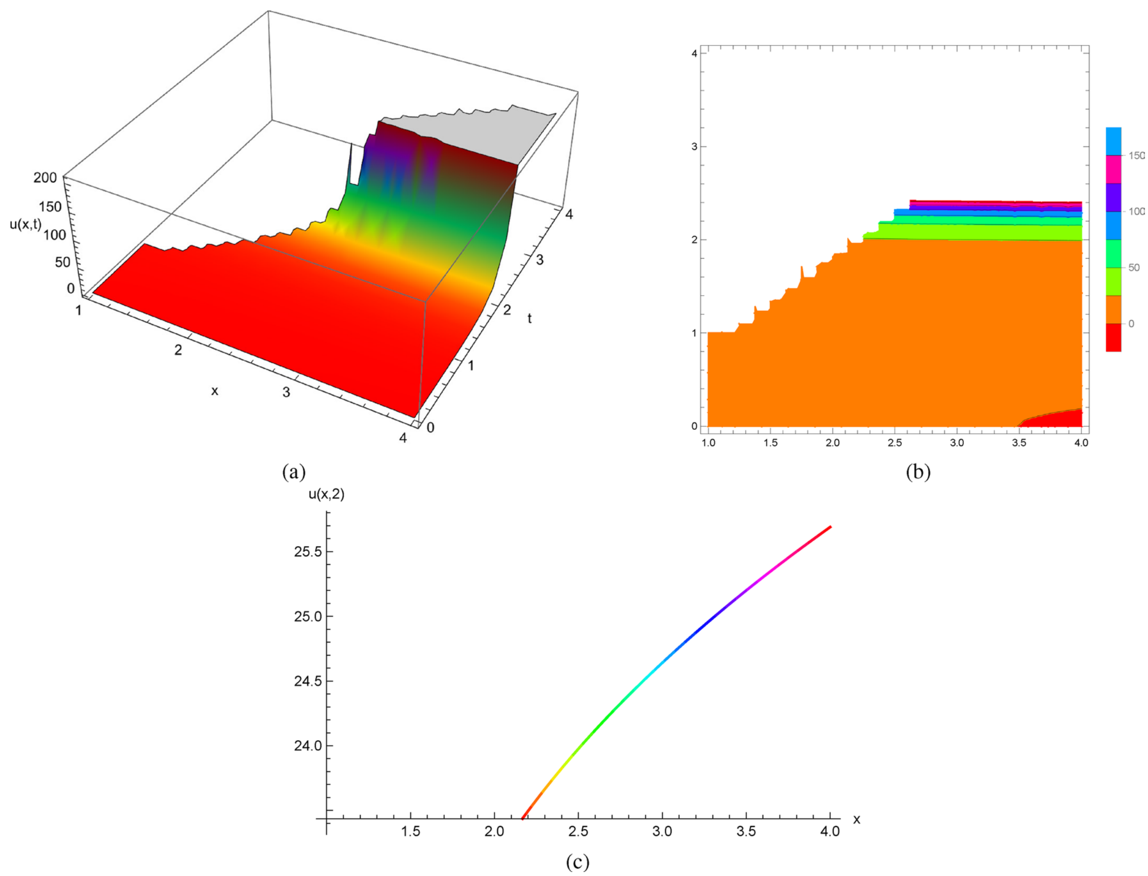

Figure 8. The profile of the solutions in (28) with $u(x, 1) = 1$ and $u_t(x, 1) = x^2$: (**a**) and (**b**) 3D and Contour plots with $1 \leq x \leq 4$ and $0 \leq t \leq 4$, (**c**) 2D plot at $t = 2$.

Data availability

The authors confirm that the data presented in this study are available within the manuscript.

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References

1. Polyanin, A. D. & Manzhirov, A. V. *Handbook of exact solutions for ordinary differential equations* 2nd edn. (Chapman and Hall/CRC Press, Boca Raton, 2003).
2. Faddeev, L. D. & Takhtanjan, L. A. *Hamiltonian methods in the theory of soliton* (Springer Series in Soviet Mathematcs. Springer-Verlag, Berlin, 1987).
3. Gana, S. & Mhadhbi, N. Pseudospectra of the complex harmonic oscillator. *Glob. J. Pure Appl. Math.* **2**(3), (2007).
4. Gana, S. Numerical computation of spectral solutions for Sturm-Liouville eigenvalue problems. *Int. J. Anal. Appl.* https://doi.org/10.28924/2291-8639-21-2023-86 (2023).
5. Mhadhbi Kaidi, N. & Rouleux, M. Quasi-invariant tori and semi-excited states for schrodinger operators. i. asymptotics. *Commun. Part. Differ. Equ.* **27**(9–10), 1695–1750. https://doi.org/10.1081/PDE-120016126 (2002).
6. Mhadhbi, N., Gana, S. & Alharbi, H. Exact solutions of classes of second order nonlinear partial differential equations reducible to first order. *Int. J. Adv. Appl. Sci.* **10**(10), 78–85. https://doi.org/10.21833/ijaas.2023.10.009 (2023).
7. Mhadhbi Kaidi, N., & Kerdelhué, P. Forme normale de birkhoff et résonances. *Asymptotic Analysis. IOS Press*. vol. 23, no. 1, pp. 1–21, (2000).
8. Mâagli, H., Mhadhbi, N. & Zeddini, N. Existence and exact asymptotic behavior of positive solutions for a fractional boundary value problem. *Abstr. Appl. Anal.* **2013**, 1–6. https://doi.org/10.1155/2013/420514 (2013).
9. Myint, U. & Debnath, L. *Linear partial differential equations for scientists and engineers* 4th edn. (Birkhauser, 2007).
10. Zachmanoglou, E. C. & Thoe, D. W. *Introduction to partial differential equations with applications* (Dover Publications, 1986).
11. Aris, R., Rhee, H. K. & Amundson, N. R. *First-order partial differential equations* Vol. 1 (Prentice Hall, 1986).
12. Modanli, M., Abdulazeez, S. T., & Hussein, A. M. Solutions to nonlinear pseudo hyperbolic partial differential equations with nonlocal conditions by using residual power series method. *Sigma J. Eng. Nat. Sci.* **41**(3), 488–492. https://doi.org/10.14744/sigma.2023.00055 (2023).
13. Modanli, M., Abdulazeez, S. T. & Hussein, A. M. Numerical scheme methods for solving nonlinear pseudo-hyperbolic partial differential equations. *J. Appl. Math. Comput. Mech.* **21**(4), 5–15. https://doi.org/10.17512/jamcm.2022.4.01 (2022).
14. Abdulazeez, S. T., Modanli, M., & Hussein, A. M. A residual power series method for solving pseudo hyperbolic partial differential equations with nonlocal conditions. *Num. Methods Part. Differ. Equ* **37**(3), 2235–2243. https://doi.org/10.1002/num.22683 (2021).
15. Abdulazeez, S. T. & Modanli, M. Solutions of fractional order pseudo-hyperbolic telegraph partial differential equations using finite difference method. *Alex. Eng. J.* **61**(12), 12443–12451. https://doi.org/10.1016/j.aej.2022.06.027 (2022).
16. Abdulazeez, S. T., Abdulla, S. O. & Modanli, M. Comparison of third-order fractional partial differential equation based on the fractional operators using the explicit finite difference method. *Alex. Eng. J.* **70**, 37–44. https://doi.org/10.1016/j.aej.2023.02.032 (2023).
17. Karadag, K., Modanli, M., & Abdulazeez, S. T. Solutions of the mobile–immobile advection–dispersion model based on the fractional operators using the crank–nicholson difference scheme. *Chaos Solitons Fract.* **167**, 113114. https://doi.org/10.1016/j.chaos.2023.113114 (2023).
18. Modanli, M., Rasheed, S. K. & Abdulazeez, S. T. Stability analysis and numerical implementation of the third-order fractional partial differential equation based on the caputo fractional derivative. *J. Appl. Math. Comput. Mech.* **22**(3), 33–42. https://doi.org/10.17512/jamcm.2023.3.03 (2023).
19. Wu, G. C., He, J. H. & Austin, F. The variational iteration method which should be followed. *Nonlinear Sci. Lett. A.* **1**(1), 1–30 (2010).
20. Ablowitz, M. J. & Clarkson, P. A. *Solitons, nonlinear evolution equations and inverse scattering transform* (Cambridge University Press, 1990).
21. Sadiq Murad, M. A., Modanli, M., & Abdulazeez, S. T. A new computational method-based integral transform for solving time-fractional equation arises in electromagnetic waves. *Zeitschrift für angewandte Mathematik und Physik* **186** (2023). https://doi.org/10.1007/s00033-023-02076-9.
22. Kudryashov, N. A. & Loguinova, N. B. Extended simplest equation method for nonlinear differential equations. *Appl. Math. Comput.* **205**(1), 396–402. https://doi.org/10.1016/j.amc.2008.08.019 (2008).
23. Fan, E. Extended tanh-function method and its applications to nonlinear equations. *Phys. Lett. A* **277**(4–5), 212–218. https://doi.org/10.1016/S0375-9601(00)00725-8 (2000).
24. Shida, L., Shikuo, L., Zuntao, F. & Qiang, Z. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. *Phys. Lett. A* **289**(1–2), 69–74. https://doi.org/10.1016/S0375-9601(01)00580-1 (2001).
25. Hong, B., LU, D. & Tian, L. Bäcklund transformation and n-soliton-like solutions to the combined kdv-burgers equation with variable coefficients. *Int. J. Nonlinear Sci.* **2**, 3–10 (2006).
26. Zhang, D. New exact travelling wave solutions for some nonlinear evolution equations. *Chaos Solitons Fract.* **26**, 921–925 (2005).
27. Zhang, D. Doubly periodic solutions of modified kawahara equation 26, 1155–1160. *Chaos Solitons Fract.* **25**, 1155–1160 (2005).
28. Wazwaz, A. M. The sine-cosine method for handling nonlinear wave equations. *Math. Comput. Model.* **40**(2004), 499–508 (2004).
29. Wang, M., Li, X. & Zhang, J. The (g′/g) expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Phys. Lett. A* **372**(4), 417–423. https://doi.org/10.1016/j.physleta.2007.07.051 (2008).
30. Higgins, B. G. Introduction to the method of characteristics. *University of California, Davis CA95616*, (2019).
31. Hounkonnou Pascal, M., & Sielenou, A., T. classes of second order nonlinear differential equations reducible to first order ones by variation of parameters. arXiv:0902.4175 V1, (2009).
32. Polyanin, A. D. & Manzhirov, A. V. *Handbook of mathematics for engineers and scientists* (Chapman and Hall/CRC Press, 2006).
33. Kečkić, J. D. Additions to kamke's treatise: Variation of parameters for nonlinear second order differential equations. *Univ. Beograd. Pool, Elektrotehn. fak. Ser. Mat. Fiz.* No. 544 - No. 576, 31–36 (1946).
34. Olver, P. J. *Introduction to partial differential equations* (Springer, 2014).
35. Kevorkian, J. *Partial differential equations: Analytical solutions techniques* 2nd edn. (Springer, 1999).
36. Mhadhbi, N., Gana, S., & Alharbi, H. Classes of second order nonlinear partial differential equations reducible to first order. arXiv:2305.03128 (2023). https://doi.org/10.48550/arXiv.2305.03128.
37. Zwillinger, D. *Handbook of differential equations* 3rd edn. (Academic Press, 1998).
38. Murphy, G. M. *Ordinary differential equations and their solutions* (New york, 1960).
39. Kamke, E. *Differentialgleichungen: Losungsmethoden und losungen, i, gewohnliche differentialgleichungen* (B. G. Teubner, Leipzig, 1977).
40. Panayotounakos, D. E., & Zarmpoutis, T. I. Construction of exact parametric or closed form solutions of some unsolvable classes of nonlinear odes (abel's nonlinear odes of the first kind and relative degenerate equations). *Int. J. Math. Math. Sci.* (2011).

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N.M. conceptualization, methodology, formal analysis, investigation, writing—review and editing, validation, supervision. S.G. conceptualization, investigation, formal analysis, writing of the original draft, data curation, software, resources, writing—review and editing, validation. M.A. conceptualization, investigation, data curation. All authors read and approved the final manuscript.

Competing interests

The authors declare no competing interests.

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