



OPEN

Bound entanglement is not Lorentz invariant

Paweł Caban^{1✉} & Beatrix C. Hiesmayr²

Bound entanglement, in contrast to free entanglement, cannot be distilled into maximally entangled states by two local observers applying measurements and utilizing classical communication. In this paper we ask whether a relativistic observer classifies states according to being separable, bound or free entangled in the same manner as an unboosted observer. Surprisingly, this turns out not to be the case. And that even if the system in a given inertial frame of reference is separable with respect to the partition momenta versus spins. In detail, we show that if the spin state is initially bound entangled, some boosted observers observe their spin states to be either bound entangled, separable or free entangled. This also explains why a general measure of the entanglement property is difficult to find.

Detecting entanglement, even given the full information of the physical state, namely the density matrix is a NP-hard problem¹ (NP-hard stands for non-polynomial-time hard), because of the existence of bound or PPT (positive partial transposition) entangled states². Those states cannot be detected by taking the partial transpose in one subsystem and finding at least one eigenvalue negative, in which case we are dealing with *free* entanglement. This mathematical property of the density matrix has crucial physical implementation, i.e. an ensemble of free entangled states can always be distilled to maximally entangled states by local operations and classical communication (LOCC), in strong contrast to bound entangled states. Those states can be generated by maximally entangled states, Bell states, but this entanglement is then bounded, i.e. cannot be distilled. This aspect of entanglement gave rise to a lot of speculations why Nature provides us with that kind of entanglement and for what it could be useful^{3,4}. Entanglement and other aspects of quantum information theory in the relativistic setting were discussed in many papers, see, e.g.,^{5–15} and references therein. However, up to our best knowledge, the behavior of bound entanglement under Lorentz boosts was not analyzed up to now. One of the reasons is that bound entanglement is difficult to detect. However, recent works^{16–18} have shown some new insights on the structure of bound entangled states in the Hilbert space for the lowest dimensional cases of two qutrits or two ququarts. We use those results in our present work.

In this paper we analyze how bound entanglement changes under Lorentz boosts. To this end we consider a system of two massive spin-1 particles. In a one inertial frame of reference this system is prepared in a state that is separable with respect to the partition momenta versus spins and the spin part of this state is bound entangled. We show that there exist such states and boosts that the boosted state is also bound entangled or separable or even free entangled. This is also visualized in Fig. 1.

Methods

Two-qutrit bound entangled states. Entanglement is a genuine quantum feature of multipartite systems. However, even in a bipartite case complete characterization of entanglement can be given only for systems with dimensions $2 \otimes 2$ and $2 \otimes 3$. In these dimensions one can fully characterize entanglement with the help of the Peres–Horodecki Positive Partial Transpose (PPT) criterion^{19,20}. This criterion says that if $\rho_{AB}^{T_B}$ (where the superscript T_B denotes partial transposition with respect to the system B) is not positive semidefinite then the state ρ_{AB} is entangled. For $2 \otimes 2$ and $2 \otimes 3$ dimensional systems also the inverse is true. However, for higher dimensional systems there exist entangled states ρ_{AB} such that $\rho_{AB}^{T_B}$ is positive semidefinite². Such states are called bound entangled, in opposition to free entangled states for which $\rho_{AB}^{T_B}$ possesses at least one negative eigenvalue. These names—free and bound entanglement—come from the fact that free entanglement can be distilled while it is impossible for bound entanglement.

The detection of bound entanglement is not an easy task—given method can certify entanglement of a certain family of states while it can be useless for other¹⁶. One of the most useful and easiest in application methods is the realignment or computable cross-norm criterion^{21,22}, on which we focus firstly. In the last section we show

¹Department of Theoretical Physics, University of Lodz, Pomorska 149/153, 90-236 Łódź, Poland. ²Faculty of Physics, University of Vienna, Boltzmanngasse 5, 1090 Vienna, Austria. ✉email: Pawel.Caban@uni.lodz.pl

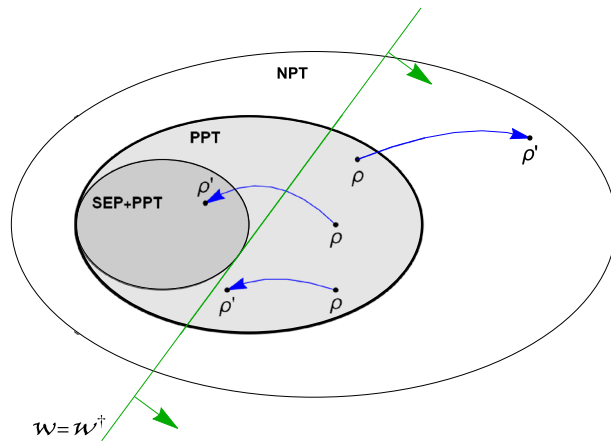


Figure 1. This visualizes the state space and the effect of a relativistic boost on bound entangled state. An entanglement witness \mathcal{W} is visualized as a line separating some entangled states from separable states.

how those properties of states can be transferred via so called entanglement witnesses, i.e. hermitian observables, to an experimental realization. Let us denote

$$\text{Realignment}(\rho_{AB}) = \log_2 \left(\sum_i \sigma_i(\tilde{\rho}_{AB}) \right), \tag{1}$$

where $\sum_i \sigma_i(\tilde{\rho}_{AB})$ is the sum of all the singular values of the realigned matrix $\tilde{\rho}_{AB}$, where $[\tilde{\rho}_{AB}]_{ij,ab} = [\rho_{AB}]_{ia,jb}$. The realignment criterion says that if $\text{Realignment}(\rho_{AB}) > 0$ then the state ρ_{AB} is entangled. The above statement cannot be inverted—there exist entangled states ρ for which $\text{Realignment}(\rho) < 0$. Moreover, this criterion can detect bound entanglement of certain classes of states.

Bound entanglement can be also revealed by other methods like the quasispin criterion²³ or with the help of entanglement witnesses (here the witnesses constructed with the help of mutually unbiased bases, so called MUB witnesses^{24–26} are useful), which we discuss in the last section.

In this paper we use bound entangled states from the so called magic simplex for which recently a classification into separable, bound entangled and free entangled states was possible with a success probability of 95% for qutrits¹⁷ and for ququarts of 75%¹⁸. In the $3 \otimes 3$ dimensional case this simplex has the form^{27,28}

$$\mathcal{M}_3 \equiv \left\{ \rho = \sum_{k,l=0}^2 c_{k,l} P_{k,l} \mid \sum_{k,l=0}^2 c_{k,l} = 1, c_{k,l} \geq 0 \right\}, \tag{2}$$

where $P_{k,l} = |\Omega_{k,l}\rangle\langle\Omega_{k,l}|$ and the Bell states $|\Omega_{k,l}\rangle$ can be generated from $|\Omega_{0,0}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ via the relation $|\Omega_{k,l}\rangle = W_{k,l} \otimes \mathbb{I}_3 |\Omega_{0,0}\rangle$. In the last equation $W_{k,l}$ are the unitary Weyl operators $W_{k,l} \equiv \sum_{j=0}^2 \omega^{j \cdot k} |j\rangle\langle j+l|$ with $\omega = e^{\frac{2\pi i}{3}}$ being the root of unity.

In particular we will use an interesting one parameter state¹⁶

$$\rho_b(x) = \sum_{k,l=0}^2 d_{k,l} |\Omega_{k,l}\rangle\langle\Omega_{k,l}| \tag{3}$$

with

$$\begin{pmatrix} d_{0,0} & d_{0,1} & d_{0,2} \\ d_{1,0} & d_{1,1} & d_{1,2} \\ d_{2,0} & d_{2,1} & d_{2,2} \end{pmatrix} = \begin{pmatrix} 2x & 0 & \frac{1}{3} - x \\ 0 & x & \frac{1}{3} - x \\ 0 & 0 & \frac{1}{3} - x \end{pmatrix} \tag{4}$$

and $0 \leq x \leq \frac{1}{3}$. This state is PPT for $x \in [0, \frac{2}{15}]$, however, entangled for all values x except $x = 0$. This means that for $x \in \{0, \frac{2}{15}\}$ it is bound entangled, which is detected by the realignment criterion (1) as well as by the later introduced MUB-witness^{24–26}.

Action of Lorentz boosts on quantum states. Bound entanglement can be observed in a two-particle system with at least $3 \otimes 3$ dimensions, where in turn the PPT criterion is only necessary but not sufficient for entanglement. Thus, to analyze the behavior of bound entanglement under Lorentz boost, we consider here a system of two massive, relativistic spin-1 particles. We identify the Hilbert space of states of such a particle, \mathcal{H} , with the carrier space of the irreducible, unitary massive representation of the Poincaré group for spin 1. The space \mathcal{H}

is spanned by the eigenvectors of the four-momentum operator $|k, \sigma\rangle$, where $k = (k^0, \mathbf{k})$, $k^2 = k^{02} - \mathbf{k}^2 = m^2$, denotes the four-momentum of the particle and $\sigma = -1, 0, 1$ its spin component along z -axis.

We denote here space-time coordinates with Greek indices running from 0 to 3, four-vectors by plain letters, spacial vectors by bold letters, e.g., $k = (k^0, \mathbf{k})$. The Minkowski tensor is assumed to be $\eta = \text{diag}(1, -1, -1, -1)$. We also use natural units with $c = \hbar = 1$.

We use the Lorentz-covariant normalization

$$\langle k, \sigma | k', \sigma' \rangle = 2k^0 \delta^3(\mathbf{k} - \mathbf{k}') \cdot \delta_{\sigma\sigma'}. \tag{5}$$

The vectors $|k, \sigma\rangle$ can be generated from the standard vector $|\tilde{k}, \sigma\rangle$, where $\tilde{k} = m(1, 0, 0, 0)$ is the four-momentum of the particle in its rest frame. We have $|k, \sigma\rangle = U(L_k)|\tilde{k}, \sigma\rangle$, where the standard Lorentz boost L_k is defined by relations $k = L_k \tilde{k}$, $L_k^{-1} = \mathbb{I}_4$. The explicit form of the boost L_k reads

$$L_k = \frac{1}{m} \begin{pmatrix} k^0 & \mathbf{k}^T \\ \mathbf{k} & m\mathbb{I}_3 + \frac{\mathbf{k} \otimes \mathbf{k}^T}{m+k^0} \end{pmatrix}. \tag{6}$$

With the help of the standard Wigner procedure²⁹⁻³¹ we get

$$U(\Lambda)|k, \sigma\rangle = \mathcal{D}_{\lambda\sigma}(R(\Lambda, k))|\Lambda k, \lambda\rangle, \tag{7}$$

where the Wigner rotation $R(\Lambda, k)$ is defined as $R(\Lambda, k) = L_{\Lambda k}^{-1} \Lambda L_k$ and \mathcal{D} is a three dimensional, unitary, irreducible representation of the rotation group. It is well known that in each dimension there exists, up to unitary equivalence, only one unitary, irreducible representation of the rotation group. Therefore, the representation $\mathcal{D}(R)$ is unitary equivalent to R

$$\mathcal{D}(R) = VRV^\dagger, \quad V^\dagger V = \mathbb{I}_3, \tag{8}$$

and the explicit form of the matrix V is the following:

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}. \tag{9}$$

For more details on spin-1 irreducible, unitary representation of the Poincare group see, e.g., Refs.²⁹⁻³¹.

For our computations we identify spin projection values $-1, 0, 1$ with indices of computational basis vectors $0, 1, 2$ in the following way: $-1 \leftrightarrow 0, 0 \leftrightarrow 1, 1 \leftrightarrow 2$.

Results

Bound entanglement under Lorentz boosts. Now, let us consider two inertial frames, \mathcal{O} and \mathcal{O}' , and let the frame \mathcal{O}' move with the velocity \mathbf{v} with respect to the frame \mathcal{O} . In the frame \mathcal{O} we prepare a two-particle state ρ . For simplicity we treat momentum degrees of freedom as discrete, i.e. we assume that momenta of the particles are chosen from the finite set $\{k_1, \dots, k_N\}$. To prove all of our results it is enough to limit to only two momenta $\{k_1, k_2\}$ with

$$k_{1,2} = (k^0, \pm|\mathbf{k}|, 0, 0) \tag{10}$$

Consequently, our total state under interest acts in $(2 \otimes 2)_{\text{mom}} \otimes (3 \otimes 3)_{\text{spin}} = 36$ dimensional space, i.e. in \mathbb{C}^{36} . Of course, also higher dimensions in the momentum space are possible, but not necessary to obtain our results. Moreover, without loss of generality we can take

$$m = 1, \quad k^0 = 1 + E, \quad |\mathbf{k}| = \sqrt{E(2 + E)}, \tag{11}$$

where E is a kinetic energy of the particle. Thus, the most general two-particle state we consider is of the form

$$\rho = \sum \rho_{mn,m'n'}^{\sigma\lambda,\sigma'\lambda'} |k_m, k_n; \sigma, \lambda\rangle \langle k_{m'}, k_{n'}; \sigma', \lambda'|, \tag{12}$$

where $m, n, m', n' = 1, 2$, $\sigma, \lambda, \sigma', \lambda' = 0, 1, 2$ and $\rho_{mn,m'n'}^{\sigma\lambda,\sigma'\lambda'}$ fulfill all necessary conditions to guarantee that ρ is a valid density matrix. We have also reordered the products of momentum and spin components, i.e. $|k, p; \sigma, \lambda\rangle = |k, p\rangle \otimes |\sigma, \lambda\rangle \equiv |k, \sigma\rangle \otimes |p, \lambda\rangle$.

The state ρ as seen from the frame \mathcal{O}' has the following form

$$\rho' = [U(\Lambda(\mathbf{e}, \xi)) \otimes U(\Lambda(\mathbf{e}, \xi))] \rho [U(\Lambda(\mathbf{e}, \xi)) \otimes U(\Lambda(\mathbf{e}, \xi))]^\dagger. \tag{13}$$

Here $\Lambda(\mathbf{e}, \xi)$ is the Lorentz boost in the direction $\mathbf{e} = \mathbf{v}/|\mathbf{v}|$ with rapidity ξ , $\tanh \xi = -|\mathbf{v}|$, joining frames \mathcal{O} and \mathcal{O}' . Its explicit form is the following:

$$\Lambda(\mathbf{e}, \xi) = \begin{pmatrix} \cosh \xi & \mathbf{e}^T \sinh \xi \\ \mathbf{e} \sinh \xi & \mathbb{I}_3 + (\cosh \xi - 1) \mathbf{e} \otimes \mathbf{e}^T \end{pmatrix}. \tag{14}$$

The action of $U(\Lambda)$ on basis states of the space \mathcal{H} is given in Eq. (7).

The spin parts of the states ρ and ρ' (i.e. ρ_{spin} and ρ'_{spin} , respectively) we obtain by tracing out momentum degrees of freedom and normalizing the result since in covariant normalization (5) basis vectors are orthogonal but not orthonormal. Of course in the frame \mathcal{O}' momenta of the particles belong to the set $\{k'_1, k'_2\}$, where $k'_i = \Lambda(\mathbf{e}, \xi)k_i, i = 1, 2$.

Pure momentum part of the state. As the first case we consider the simple situation when in the frame \mathcal{O} we prepare a two-particle state

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad p_i \geq 0 \quad \text{with} \quad \sum_i p_i = 1, \tag{15}$$

where

$$|\psi_i\rangle = |\psi_i^{\text{mom}}\rangle \otimes |\varphi_i^{\text{spin}}\rangle, \quad |\psi_i\rangle \in \mathcal{H}^{\text{mom}} \otimes \mathcal{H}^{\text{spin}}, \tag{16}$$

and $|\psi_i^{\text{mom}}\rangle, |\varphi_i^{\text{spin}}\rangle$ are momentum and spin parts of the state $|\psi_i\rangle$, respectively. Thus, we assume that the momentum parts of all of the states $|\psi_i\rangle$ are identical. Notice that the full state ρ defined in Eqs. (15, 16) is separable with respect to the partition: momenta versus spins (although it is not the most general separable state, the general separable state we consider in the next section). Therefore, we can write (15) as

$$\rho = |\psi^{\text{mom}}\rangle\langle\psi^{\text{mom}}| \otimes \left(\sum_i p_i |\varphi_i^{\text{spin}}\rangle\langle\varphi_i^{\text{spin}}| \right) \equiv |\psi^{\text{mom}}\rangle\langle\psi^{\text{mom}}| \otimes \rho_{\text{spin}}, \tag{17}$$

and of course $\rho_{\text{spin}} = \text{Tr}^{\text{mom}}(\rho)$. We are interested in the situation when ρ_{spin} is a bound entangled state.

The most general form of $|\psi^{\text{mom}}\rangle$ in our case reads

$$|\psi^{\text{mom}}\rangle = \sum_{i,j=1}^2 a_{ij} |k_i, k_j\rangle, \quad \sum_{i,j=1}^2 |a_{ij}|^2 = 1. \tag{18}$$

Now, we boost the state (17) and the spin part of the boosted state has the form

$$\rho'_{\text{spin}} = \text{Tr}^{\text{mom}}(\rho') = \sum_{i,j=1}^2 |a_{ij}| \left[\mathcal{D}^T(\Lambda, k_i) \otimes \mathcal{D}^T(\Lambda, k_j) \right] \rho_{\text{spin}} \left[\mathcal{D}^*(\Lambda, k_i) \otimes \mathcal{D}^*(\Lambda, k_j) \right]. \tag{19}$$

Following² and using the property

$$[(A \otimes B)\rho(C \otimes D)]^{T_B} = (A \otimes D^T)\rho^{T_B}(C \otimes B^T) \tag{20}$$

we easily see that if ρ_{spin} is a PPT state then also ρ'_{spin} is also PPT. Thus, if the spin state of the two-particle state (17) is bound entangled then for all other inertial observers this state is PPT, i.e. also bound entangled or separable. We now show that both of these cases can be realized. To this end we assume that

$$|\psi^{\text{mom}}\rangle = |\psi_1^{\text{mom}}\rangle = \frac{1}{\sqrt{2}}(|k_1, k_2\rangle + |k_2, k_1\rangle). \tag{21}$$

where k_1, k_2 are given in Eq. (10). Moreover, as a spin part of the state (17) we take the state ρ_b defined in Eq. (3). Thus, the state chosen in the frame \mathcal{O} has the form

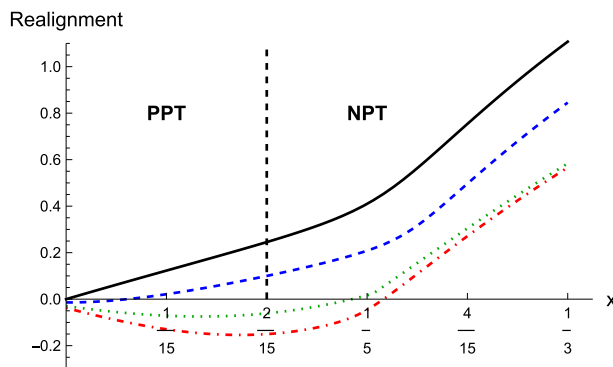


Figure 2. Value of the base-2 log of the sum of all singular values of the realigned spin density matrix for unboosted state (black, solid line) and states boosted with rapidity $\xi = 0.5$ (blue, dashed line), $\xi = 0.8$ (green, dotted line), $\xi = 1$ (red, dashed-dotted line).

$$\rho_1(x) = |\psi_1^{\text{mom}}\rangle\langle\psi_1^{\text{mom}}| \otimes \rho_b(x). \tag{22}$$

We further choose the boost direction $\mathbf{e} = (0, 0, 1)$ (compare with Eq. (14)) and kinetic energy of a particle $E = 1$ (compare with Eq. (11)).

In Fig. 2 we plotted the value of the base-2 log of the sum of all singular values of the realigned spin density matrix versus x for the unboosted and boosted state (22). According to the realignment criterion²¹, if this value is positive the state is entangled. We can see that for the unboosted state the realignment criterion detects entanglement for all values of x except 0, in contrast to the boosted state $\rho'_{\text{spin}}(x, \xi)$ where it detects entanglement only for $x \in (x_0, 1/3]$. Thus, in the considered case there always exist such a x and ξ that the bound entangled state $\rho_{\text{spin}} = \rho_b(x)$ after boost is also bound entangled.

In Fig. 3 we plotted the purity $\text{Tr}(\rho^2)$ versus realignment of 156600 magic bound entangled states on a grid in the magic simplex¹⁶, where only some are detected by the realignment criterion. The other ones are detected by different criteria. After the boost with $\xi = 0.8$ the purity decreases and fewer states are still detected by the realignment criterion. Thus the interesting question is: Does a boost of a bound entangled state result sometimes also in separable states?

To find that out we can try to apply other entanglement criteria such as the quasispin criterion²³ or the MUB witness²⁴ which failed. Thus, for e.g. $x = \frac{1}{15}, \xi = 0.8$ we tried to show that ρ'_{spin} is separable and were successful. We performed this by numerically minimizing the Hilbert-Schmidt distance $[\text{Tr}((\sum_{i=1}^k p_i^{\text{sep}} |\psi_i^{\text{sep}}\rangle\langle\psi_i^{\text{sep}}| - \rho'_{\text{spin}}(\frac{1}{15}, \frac{4}{5}))^2)]^{1/2}$, where $p_i^{\text{sep}} > 0, \sum_{i=1}^k p_i^{\text{sep}} = 1$, and $|\psi_i^{\text{sep}}\rangle$ represent separable states, for some large enough k . For $k = 10$ we found probabilities p_i^{sep} and vectors $|\psi_i^{\text{sep}}\rangle$ such that

$$\left[\text{Tr} \left(\left(\sum_{i=1}^{10} p_i^{\text{sep}} |\psi_i^{\text{sep}}\rangle\langle\psi_i^{\text{sep}}| - \rho'_{\text{spin}}(\frac{1}{15}, \frac{4}{5}) \right)^2 \right) \right]^{1/2} \approx 7 \times 10^{-8}, \tag{23}$$

i.e. practically equal to zero. The explicit form of probabilities p_i^{sep} and vectors $|\psi_i^{\text{sep}}\rangle$ we give in Supplementary Material. We were also successful for other ξ values and other x values like $x = \frac{1}{10}$, but not for $x > \frac{2}{15}$, the NPT area.

In summary, we have found at least some states which are after the boost separable, which is apparently against the intuition if it depends on the boost of the observer whether a state is classified as bound entangled or separable.

General separable state momenta versus spins. Now, let us assume that in the frame \mathcal{O} we prepare the most general state that is separable in the partition momenta versus spins, i.e.

$$\rho = \sum_i p_i \rho_{\text{mom}}^i \otimes \rho_{\text{spin}}^i, \tag{24}$$

where $p_i > 0, \sum_i p_i = 1$. The spin part of this state has the form

$$\rho_{\text{spin}} = \text{Tr mom}(\rho) = \sum_i p_i \rho_{\text{spin}}^i. \tag{25}$$

As previously, we are interested in the situation when the whole state ρ_{spin} is bound entangled.

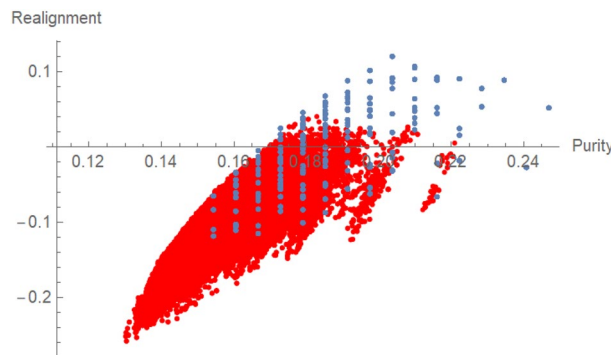


Figure 3. This picture visualizes the purity $\text{Tr} \rho^2$ versus the realignment $\sum_i \sigma_i(\tilde{\rho}_{AB}) - 1$ of 156600 bound entangled magic simplex states¹⁶. Those states result in 2228 (blue) points and if boosted with $\xi = 0.8$ they result in 87116 (red) points for our particular choice of boost. This shows that the relativistic boost changes the amount of entanglement (in this case measured by the realignment criterion) not in dependence of the initial amount of entanglement. This is also the case if other entanglement measures or purity measures are chosen.

Now, we boost the state (24) and calculate the spin part of the boosted state as $\rho'_{\text{spin}} = \text{Tr}^{\text{mom}}(\rho')$ and we obtain

$$\rho'_{\text{spin}} = \sum_{i,m,n} p_i c_{mn,mn}^i \left[\mathcal{D}^T(\Lambda, k_m) \otimes \mathcal{D}^T(\Lambda, k_n) \right] \rho_{\text{spin}}^i \left[\mathcal{D}^*(\Lambda, k_m) \otimes \mathcal{D}^*(\Lambda, k_n) \right], \tag{26}$$

where

$$\rho_{\text{mom}}^i = \sum_{m,n,m',n'} c_{mn,m'n'}^i |k_m, k_n\rangle \langle k_{m'}, k_{n'}|, \tag{27}$$

i.e. $c_{mn,mn}^i > 0, \sum_{m,n} c_{mn,mn}^i = 1$.

Now, we ask whether the transformation (26), the relativistic boost, preserves PPT. However, applying Eq. (20) we have to take into account that the whole state $\sum_i p_i \rho_{\text{spin}}^i$ is PPT but some of the states ρ_{spin}^i can be NPT. Next, factors $c_{mn,mn}^i$ change relative weights in the sum (26). Thus, we cannot conclude that (26) preserves PPT. In fact, we can find such states and boosts that (25) is bound entangled while (26) is free entangled. For example, let us consider the following state:

$$\rho_0(p, x) = p \rho_1(x) + (1 - p) \rho_2, \tag{28}$$

where $0 \leq p \leq 1, \rho_1(x)$ is given in Eq. (22) and

$$\rho_2 = |k_1, k_2\rangle \langle k_1, k_2| \otimes |\varphi_{\text{spin}}\rangle \langle \varphi_{\text{spin}}|, \tag{29}$$

with

$$|\varphi_{\text{spin}}\rangle = \sum_{k,l=0}^2 a_{k,l} |\Omega_{k,l}\rangle, \tag{30}$$

and

$$\begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ a_{1,0} & a_{1,1} & a_{1,2} \\ a_{2,0} & a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} 0 & 2/9 & 2/9 \\ 0 & 2/9 & 1/18 \\ 5/18 & 0 & 0 \end{pmatrix}. \tag{31}$$

The spin part of the state $\rho_0(0.04, \frac{7}{60}), \rho_0^{\text{spin}}(0.04, \frac{7}{60})$ is PPT and

$$\text{Realignment} \left(\rho_0^{\text{spin}}(0.04, \frac{7}{60}) \right) = 0.183 > 0, \tag{32}$$

thus it is bound entangled.

Now, if we boost this state in the direction $\mathbf{e} = (0, 0, 1)$ with $\xi = 0.95$ we obtain the state $\rho'_0(0.04, \frac{7}{60})$. The spin part $\rho'^{\text{spin}}_0(0.04, \frac{7}{60})$ is not PPT, thus it is free entangled and obviously no longer bound entangled.

In summary, we have shown that starting from a bound entangled state, we can boost to a separable, bound entangled or an free entangled state. In the next section we discuss how those counter-intuitive classifications of inertial observers relates to the principle that the physics observed, i.e. probabilities and expectation values, should be Lorentz invariant.

Discussion

Obviously, the physics for every observer should be the same. In our case this means that the values that a boosted or not boosted observer obtains by computing $\text{Tr}(\mathcal{O} \rho)$ for some observable $\mathcal{O} = \mathcal{O}^\dagger$ are identical. An observable \mathcal{W} for which

$$\min_{\rho_{\text{sep}}} (\text{Tr}(\mathcal{W} \rho_{\text{sep}})) \leq \text{Tr}(\mathcal{W} \rho) \leq \max_{\rho_{\text{sep}}} (\text{Tr}(\mathcal{W} \rho_{\text{sep}})) \tag{33}$$

does not hold for all states ρ is called an entanglement witness. The upper and lower bounds define the so called separability window of the witness and can be obtained by mirroring the witness³². A particular witness, decomposable into mutually unbiased bases (MUBs), is capable to detect bound entanglement²⁴⁻²⁶ and as it contains a recipe how to realize it experimentally, it gave raise to the first experiment detecting bound entanglement in bipartite systems³³. We will use this witness to show how we can solve the apparent puzzle. Obviously, we have

$$\text{Tr}(\mathcal{W} \rho) = \text{Tr}(\mathcal{W}^{\text{boosted}} \rho^{\text{boosted}}) \tag{34}$$

for any chosen boost on the *total* space. But this does not mean that this is the case also for the subsystem, i.e. an partition into first and second momentum versus first and second spin or first momentum and first spin versus second momentum second momentum nor first momentum and second spin versus second momentum and first spin nor in particle A (first momentum/first spin) versus particle B (second momentum/first spin) as we show in the following (for more details on the behavior of entanglement under different partitions see, e.g., Ref.¹⁰).

The MUB-witness for two qudits that is capable of detecting bound entanglement by the upper bound is defined by

$$\text{Tr}(\mathcal{W}\rho) = \sum_{i=0}^{d-1} \langle i_1 | \otimes \langle (i_1 + 1)^* | \rho | i_1 \rangle \otimes | (i_1 + 1)^* \rangle + \sum_{k=2}^{d+1} \sum_{i=0}^{d-1} \langle i_k | \otimes \langle i_k^* | \rho | i_k \rangle \otimes | i_k^* \rangle, \quad (35)$$

for which the orthonormal basis $\{|i_k\rangle_{k=1}^{d+1}\}$ are mutually unbiased $|\langle i_k | j_l \rangle|^2 = \delta_{k,l} \delta_{i,j} + (1 - \delta_{k,l}) \frac{1}{d}$. Thus we can boost each MUB vector and by that achieve the observable $\mathcal{W}^{\text{boosted}}$ that is seen by a boosted observer. On the total space we find for our particular example (optimization of the lower and upper bound via the composite parameterization³⁴)

$$-\frac{3}{4} \leq \text{Tr}(\mathcal{W}\rho) = \text{Tr}(\mathcal{W}^{\text{boosted}} \rho^{\text{boosted}}) = \frac{1}{2} + \frac{x}{4} \leq \frac{3}{4} \quad (36)$$

for every boost value ξ . This inequality is obviously not violated since our total state was chosen to be separable (momentum versus spin).

Obviously, the situation changes when we consider the spin or momentum subspaces, which we achieve by partial tracing over the respective subsystems. For the spin part we find for the unboosted case

$$\frac{2}{3} \leq \text{Tr}(\text{Tr}_{\text{mom}} W \text{Tr}_{\text{mom}} \rho) = 2 + x \leq 2 \quad (37)$$

which is obviously violated if $x \neq 0$. And e.g. for $\xi = 0.8$ we have

$$0.763 \leq \text{Tr}(\text{Tr}_{\text{mom}} W^{\text{boosted}} \text{Tr}_{\text{mom}} \rho^{\text{boosted}}) = 1.694 + 0.641x \leq 1.985, \quad (38)$$

i.e. no violation for any x . Consequently, if one ignores in the boosted case the momentum degrees of freedom, one does not consider the full experimental situation and this explains why an unboosted bound state becomes a separable, bound entangled or free entangled one when ignoring part of the systems, though, against our intuition, the momentum part of the witness is just the unity operator for any boost.

In this paper we analyzed the behavior of the bound entangled states under Lorentz boosts. To this end we considered a system of two spin-1, massive particles. In a given inertial frame of reference this system is prepared in a state which is separable with respect to the partition momenta/spins and such that its spin part is bound entangled. Next, we boost this state to a different inertial frame of reference and analyze the entanglement of its spin part. We showed that the boosted state can be also bound entangled. However, we were also able to find such states and boosts that the boosted state is separable or even free entangled. Thus, surprisingly, we found that Lorentz boosts can activate bound entanglement.

In the last section we explained how such a counter-intuitive classification of different inertial observers relates to the principle that measurement outcomes are invariant under Lorentz boosts. Boosting a state means that one boosts also the observable which means taking the trace gives the same result. But ignoring the momentum part, the situation changes drastically and leads to all three separability possibilities as we have shown by exemplary boosts and states.

From the purely practical point of view, we have presented a method to systematically produce different bound entangled states by application of different relativistic boosts, which may further be explored to reveal the very Nature of bound entanglement.

Data availability

Numerical data necessary to produce figures were taken from published works. The data are available from the corresponding author on reasonable request. No other data were created during the preparation of the paper.

Received: 8 March 2023; Accepted: 5 July 2023

Published online: 11 July 2023

References

- Gurvits, L. Classical deterministic complexity of Edmonds' problem and quantum entanglement. In *Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing*, STOC'03, 10–19 (Association for Computing Machinery, 2003).
- Horodecki, M., Horodecki, P. & Horodecki, R. Mixed-state entanglement and distillation: Is there a "bound" entanglement in Nature?. *Phys. Rev. Lett.* **80**, 5239–5242 (1998).
- Horodecki, K., Horodecki, M., Horodecki, P. & Oppenheim, J. Secure key from bound entanglement. *Phys. Rev. Lett.* **94**, 160502 (2005).
- Vértesi, T. & Brunner, N. Disproving the Peres conjecture by showing Bell nonlocality from bound entanglement. *Nat. Commun.* **5**, 5297 (2014).
- Czachor, M. Einstein–Podolsky–Rosen–Bohm experiment with relativistic massive particles. *Phys. Rev. A* **55**, 72–77 (1997).
- Peres, A., Scudo, P. F. & Terno, D. R. Quantum entropy and special relativity. *Phys. Rev. Lett.* **88**, 230402 (2002).
- Caban, P. & Rembieliński, J. Lorentz-covariant reduced spin density matrix and Einstein–Podolsky–Rosen–Bohm correlations. *Phys. Rev. A* **72**, 012103 (2005).
- Caban, P. Einstein–Podolsky–Rosen correlations of photons: quantum field theory approach. *Phys. Rev. A* **76**, 052102 (2007).
- Caban, P., Rembieliński, J. & Włodarczyk, M. Strange behavior of the relativistic Einstein–Podolsky–Rosen correlations. *Phys. Rev. A* **79**, 014102 (2009).
- Friis, N., Bertlmann, R. A., Huber, M. & Hiesmayr, B. C. Relativistic entanglement of two massive particles. *Phys. Rev. A* **81**, 042114 (2010).
- Palge, V., Vedral, V. & Dunningham, J. A. Behavior of entanglement and Cooper pairs under relativistic boosts. *Phys. Rev. A* **84**, 044303 (2011).
- Taillebois, E. R. F. & Avelar, A. T. Spin-reduced density matrices for relativistic particles. *Phys. Rev. A* **88**, 060302(R) (2013).

13. Saldanha, P. L. & Vedral, V. Wigner rotations and an apparent paradox in relativistic quantum information. *Phys. Rev. A* **87**, 042102 (2013).
14. Ondra, M. & Hiesmayr, B. C. Single particle entanglement in the mid- and ultra-relativistic regime. *J. Phys. A Math. Theor.* **54**, 435301 (2021).
15. Lee, H. Relativistic massive particle with spin-1/2: a vector bundle point of view. *J. Math. Phys.* **63**, 012201 (2022).
16. Hiesmayr, B. C. Free versus bound entanglement, a NP-hard problem tackled by machine learning. *Sci. Rep.* **11**, 19739 (2021).
17. Popp, C. & Hiesmayr, B. C. Almost complete solution for the NP-hard separability problem of Bell diagonal qutrits. *Sci. Rep.* **12**, 12472 (2022).
18. Popp, C. & Hiesmayr, B. C. Comparing bound entanglement of Bell diagonal pairs of qutrits and ququarts. *Sci. Rep.* **13**, 2037 (2023).
19. Peres, A. Separability criterion for density matrices. *Phys. Rev. Lett.* **77**, 1413–1415 (1996).
20. Horodecki, M., Horodecki, P. & Horodecki, R. Separability of mixed states: Necessary and sufficient conditions. *Phys. Lett. A* **223**, 1–8 (1996).
21. Chen, K. & Wu, L.-A. A matrix realignment method for recognizing entanglement. *Q. Inf. Comput.* **3**, 193–202 (2003).
22. Rudolph, O. Some properties of the computable cross-norm criterion for separability. *Phys. Rev. A* **67**, 032312 (2003).
23. Bae, J. *et al.* Detection and typicality of bound entangled states. *Phys. Rev. A* **80**, 022317 (2009).
24. Spengler, C., Huber, M., Briery, S., Adaktylos, T. & Hiesmayr, B. C. Entanglement detection via mutually unbiased bases. *Phys. Rev. A* **86**, 022311 (2012).
25. Hiesmayr, B. C. *et al.* Detecting entanglement can be more effective with inequivalent mutually unbiased bases. *New J. Phys.* **23**, 093018 (2021).
26. Bae, J., Bera, A., Chruściński, D., Hiesmayr, B. C. & McNulty, D. How many mutually unbiased bases are needed to detect bound entangled states?. *J. Phys. A Math. Theor.* **55**, 505303 (2022).
27. Baumgartner, B., Hiesmayr, B. C. & Narnhofer, H. State space for two qutrits has a phase space structure in its core. *Phys. Rev. A* **74**, 032327 (2006).
28. Baumgartner, B., Hiesmayr, B. & Narnhofer, H. The geometry of bipartite qutrits including bound entanglement. *Phys. Lett. A* **372**, 2190–2195 (2008).
29. Caban, P., Rembieliński, J. & Włodarczyk, M. Einstein–Podolsky–Rosen correlations of vector bosons. *Phys. Rev. A* **77**, 012103 (2008).
30. Caban, P. Helicity correlations of vector bosons. *Phys. Rev. A* **77**, 062101 (2008).
31. Barut, A. O. & Rączka, R. *Theory of Group Representations and Applications* (PWN, 1977).
32. Bae, J., Chruściński, D. & Hiesmayr, B. Mirrored entanglement witnesses. *npj Quantum Inf.* **6**, 15 (2020).
33. Hiesmayr, B. C. & Löffler, W. Complementarity reveals bound entanglement of two twisted photons. *New J. Phys.* **15**, 083036 (2013).
34. Spengler, C., Huber, M. & Hiesmayr, B. C. Composite parameterization and Haar measure for all unitary and special unitary groups. *J. Math. Phys.* **53**, 013501 (2012).

Acknowledgements

PC is supported by University of Lodz under the IDUB project. BCH acknowledges gratefully that this research was funded in whole, or in part, by the Austrian Science Fund (FWF) project P36102-N. For the purpose of open access, the author has applied a CC BY public copyright license to any Author Accepted Manuscript version arising from this submission.

Author contributions

P.C. and B.H. have equally contributed to the main result, the examples and the writing.

Competing interests

The authors declare no competing interests.

Additional information

Supplementary Information The online version contains supplementary material available at <https://doi.org/10.1038/s41598-023-38217-3>.

Correspondence and requests for materials should be addressed to P.C.

Reprints and permissions information is available at www.nature.com/reprints.

Publisher's note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

© The Author(s) 2023