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Finite-time complete periodic synchronization of memristive neural networks with mixed delays

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In this paper we study the oscillatory behavior of a new class of memristor based neural networks with mixed delays and we prove the existence and uniqueness of the periodic solution of the system based on the concept of Filippov solutions of the differential equation with discontinuous right-hand side. In addition, some assumptions are determined to guarantee the globally exponentially stability of the solution. Then, we study the adaptive finite-time complete periodic synchronization problem and by applying Lyapunov–Krasovskii functional approach, a new adaptive controller and adaptive update rule have been developed. A useful finite-time complete synchronization condition is established in terms of linear matrix inequalities. Finally, an illustrative simulation is given to substantiate the main results.

Recurrent neural network (RNN) is a deep learning model characterized by a topology that allows all connections between neurons, and by the fact each neuron usually has a complicated structure because of a large number of parallel connections with a diversity of axon lengths[1,2]. In addition, RNNs are well known for their capacity of classification, detecting regularities and their ability to learn. They can play the role of memory through feedback and are perfectly able to receive sensory data from our future agent[3,4]. In particular, continuous time RNNs (CTRNNs) are RNNs modeled by dynamical systems in the form of differential equation; they combine machine learning and physical modeling[5–7]. In fact, CTRNNs are mathematically easier to analyze, and continuous formulation offers more flexibility in adapting the system to the problem and adding constraints. Actually, researchers are attracted to the mathematical properties of RNNs, namely, the nature of solutions, stability and the oscillation properties[8,9].

Indeed, the dynamic properties of RNNs have been deeply discussed and several important works have been provided[10–13]. In particular, RNNs which exhibit periodic oscillation have been used to encode information in the vibration phase and to model many systems in many domains such as celestial mechanics, nonlinear vibration, electromagnetic theory, engineering, robotics[14–19]. In addition, the synchronization problem consists of analyzing the behavior between two systems: driver (or master) and responder (or slave) and could be seen in different real systems such as secure communication, information science, chaos generators and signal generators design, image processing, biological systems[20,21]. In fact, neuronal synchronization becomes one of the most attractive subjects in neuroscience, it indicates that the specific states of all the neurons in the neural networks converge to a common value[22–25].

To make these oscillating neurons, researchers are interested in the memristor component that is a combination of memory and resistor[26,27]. Chua pointed out that the behaviour of memristor is somewhat similar to the synapses in the human brain[28], and it can potentially offer both high connectivity and high density needed for efficient computing compared to other storage. A memristive neural networks (MNN) is described in Fig. 1. In addition, memristor studies show that MNN exhibits the feature of pinched hysteresis which means that a lag occurs between the application and the removal of a field and its subsequent effect, just as the neurons in the human brain have[29–31]. Some studies have been discussed to analyze the dynamic behaviour of MNN and a lot of researches were released[32–35].

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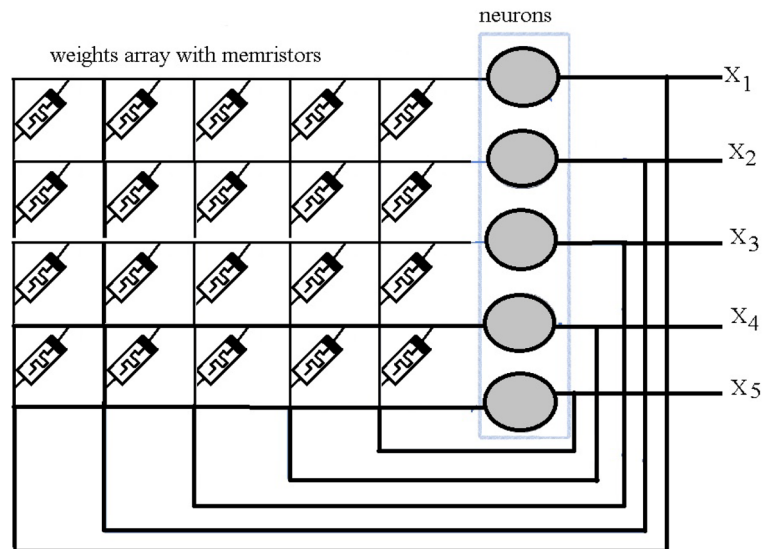
Figure 1. Memristive neural network with 5 neurons.

Hence, one can ask what is the impact of the delays (time-varying and distributed delay) for the stability and the synchronization of the periodic solution of MNNs. In Ref.[10], authors investigate whether periodic solutions exist, are unique and stable for a large class of memristor-based neural networks with time-varying delays. Moreover, a novel and useful finite-time complete synchronization condition is obtained in terms of linear matrix inequalities to ensure the synchronization goal in Ref.[36].

In this work, we extend these studies and the mathematical model of MNN with mixed delays. In fact, we analyse the stability of equilibrium points with executing significant results of the period behavior of the system. After that, we study the phenomena of synchronization from the point of view of the theory and control. In the considered system, the weights are discontinuous; the classical definition of the solution for differential equations cannot apply here. Therefore we shall propose the Filippov solution to handle this problem. Filippov developed a solution to the differential equation with a discontinuous right-hand side[37]. Based on this definition, a differential equation with a discontinuous right-hand side has the same solution set as a differential inclusion. Our contribution consists to investigate the existence and exponential stability of the periodic solutions for memristor-based neural networks with time-varying delays in the leakage term. The stability properties of this model are then analyzed and we show that the solutions of this new linear system converge to a periodic and stable limit cycle. The main novelty of our contribution lies in solving the problems of stability and synchronization and we demonstrate the impact of the mixed delays. Also results enhance and extend earlier studies on neural network dynamical systems with a continuous or discontinuous right-hand side that are memristor-based or conventional.

The rest of this paper is organized as follows. A delayed memristor-based neural networks is presented and some necessary definitions are given in "Model description and preliminaries" section. In "Uniqueness and global exponential stability" section, we introduce the Filippov's solution for our system and the existence of periodic solutions of the system. Our approaches are based on the differential inclusions and topological degree theories in set-valued analysis. In "Finite-time periodic synchronization" section, we shall study the uniqueness and global exponential stability of the w-periodic solution for the system. Especially, when the system is autonomous, we will give the sufficient conditions, uniqueness and global exponential stability of equilibrium point of the proposed system. Moreover, we designed novel control algorithms for the finite-time periodic synchronization to select neurons to pin the designed controller. In "Conclusions" section, a numerical example is obtained to show the effectiveness of the theoretical results given in the previous sections. It should be mentioned that the main results of this paper are Theorems 1–5.

Model description and preliminaries
In this paper, we shall investigate the following memristive neural networks with time-varying delay:

$$\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(x_j(t))f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(x_j(t-\tau_{ij}(t)))g_j(x_j(t-\tau_{ij}(t)))$$

$$\times \sum_{j=1}^{n} p_{ij}(x_j(t)) \int_{-\infty}^{t} k_{ij}(t-s)h_j(x_j(s))ds + J_i(t),$$

(1)

where $n \geq 1$ represents the number of neurons in the network, $x_i(t)$ correpond to the potential membrane of the neuron i at time t, the a_i is a positive constant rate with which the i th neuron will reset its potential to the resting state in isolation when it is disconnected. In addition, f_j, g_j, h_j and ϕ_j denote the activation functions of

jth neuron at time t, $b_{ij}(t)$, $c_{ij}(t)$, $p_{ij}(t)$ are the synaptic connection weight of the unit j on the unit i at time t, J_i is the input unit i and $\tau_{ij}(t)$ corresponds to the transmission delay of the ith unit along the axon of the jth unit at time t and is continuously differentiable function satisfying

$$0 \leq \tau_{ij}(t) \leq \tau, \tag{2}$$

where $\tau = max_{1 \leq i,j \leq n}\{max_{t \in [0,\omega]}, \tau_{ij}(t)\}$, τ is a nonnegative constant, $b_{ij}(t)$, $c_{ij}(t - \tau_{ij}(t))$ and $p_{ij}(t)$ are memristive synaptic weights. Basing on memristor feature and the current-voltage characteristic, we write:

$$b_{ij}(x_i(t)) = \begin{cases} \overline{b}_{ij}, & |x_j(t)| > T_j \\ \underline{b}_{ij}, & |x_j(t)| < T_j \end{cases}, \tag{3}$$

$$c_{ij}(x_i(t)) = \begin{cases} \overline{c}_{ij}, & |x_j(t)| > T_j \\ \underline{c}_{ij}, & |x_j(t)| < T_j \end{cases}, \tag{4}$$

$$p_{ij}(x_i(t)) = \begin{cases} \overline{p}_{ij}, & |x_j(t)| > T_j \\ \underline{p}_{ij}, & |x_j(t)| < T_j \end{cases}, \tag{5}$$

for $i, j = 1, 2, \ldots, n$; $t \in R$, where $T_j > 0$ is a switching jumps and let $\overline{a}_i > 0$, $\underline{a}_i > 0$, $\overline{b}_{ij}$, $\underline{b}_{ij}$, $\overline{c}_{ij}$, $\underline{c}_{ij}$, $\overline{p}_{ij}$, $\underline{p}_{ij}$ for $i, j = 1, 2, \ldots, n$ are all constants.

Definition 1 *(Periodic solution).* A solution $x(t)$ of system (1) on $[0, +\infty[$ is a ω-periodic solution with period ω if $x(t + \omega) = x(t)$, for all $t \geq 0$.

Throughout this paper, we always assume the following hypothesis:

(H1) $a_i(.)$, $\overline{b}_{ij}(.)$, $\underline{b}_{ij}(.)$, $\overline{c}_{ij}(.)$, $\underline{c}_{ij}(.)$, $\overline{p}_{ij}(.)$, $\underline{p}_{ij}(.)$, $\tau_{ij}(.)$ and $J_i(.)$ are continuous and w-periodic functions.

(H2) The neuron activation functions $f_j(.)$, $g_j(.)$ and $h_j(.)$ are Lipschitz-continuous on $\mathbb{R}$ with Lipschitz constants $F_j > 0$, $G_j > 0$ and $H_j > 0$ respectively, i.e.,

$|f_j(x) - f_j(y)| < F_j|x - y|, |g_j(x) - g_j(y)| < G_j|x - y|, |h_j(x) - h_j(y)| < H_j|x - y|$, for all $x, y \in \mathbb{R}$ and for all $j = 1, 2, .., n$.

(H3) $\exists M, \alpha \in \mathbb{R}_+$ such that $|k_{ij}(t - s)| = \begin{cases} = 0, & s \leq 0 \\ \leq Me^{-\alpha(t-s)}, & 0 \leq s \leq t \end{cases}$.

Definition 2 We say that a square matrix is an M-matrix if it has all nonpositive elements outside the diagonal and all positive successive principal minors[38].

Lemma 1 [39] *Given matrix $M = (m_{ij})_{n \times n}$ with nonpositive off-diagonal elements is a nonsingular M-matrix if and only if one of the following conditions holds:*

(1) *There exist n positive constants $\alpha_1, \alpha_2, \ldots \alpha_n$ such that $\alpha_i m_{ii} + \sum_{j=1,j \neq i}^{n} \alpha_j m_{ji} > 0$, $i = 1, \ldots, n$.*

(2) *There exist n positive constants $\beta_1, \beta_2, \ldots \beta_n$ such that $\beta_i m_{ii} + \sum_{j=1,j \neq i}^{n} \beta_j m_{ij} > 0$, $i = 1, \ldots, n$.*

Definition 3 *(Clarke Regular[40])* $V(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be regular, if for each $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$

(1) there exists the usual right directional derivative $D^+V(x, v) = \lim_{h \longrightarrow 0^+} \frac{V(x+hv)-V(x)}{h}$,

(2) the generalized directional derivative of V at x in the direction $v \in \mathbb{R}^n$ is defined as $D^{++}V(x, v) = \lim_{y \to x, h \longrightarrow 0^+} \frac{V(y+hv)-V(y)}{h}$, then $D^+V(x, v) = D^{++}V(x, v)$.

Definition 4 Consider the column vector $x = (x_1, x_2, \ldots, x_n)^T$, where T denotes the transpose of a vector, $\|x\|$ denotes any vector norm in $\mathbb{R}^n$. $\|x\|_1 = \sum_{j=1}^{n} |x_i|$, $\|x\|_2 = \left[\sum_{j=1}^{n} x_i^2\right]^{\frac{1}{2}}$.

Let the set $A \in \mathbb{R}^n$, $\overline{co}[A]$ denotes the closure of the convex hull of A, $\mu(A)$ is the Lebesgue measure of A, and ∂A is the boundary of A.

Definition 5 For a locally Lipschitz function $V(x) : \mathbb{R}^n \to \mathbb{R}$, we can define Clarke's generalized gradient of V at point x, as follows

$$\partial V(x) = \overline{co}[\lim_{k \leftarrow \infty} \nabla V(x_k) : x_k \to x, x_k \notin N, x_k \notin \Omega],$$

where $\Omega \subset \mathbb{R}^n$ is the set of points where V is not differentiable and $N \subset \mathbb{R}^n$ is an arbitrary set with measure zero[41]. In the following, for a continuous ω-periodic function $f(t)$ defined on $\mathbb{R}$, we define

$$\bar{f} = \frac{1}{\omega} \int_0^{\omega} f(t)dt, f^u = \sup_{t \in [o,\omega]} |f(t)|, f^l = \inf_{t \in [o,\omega]} |f(t)|.$$

Given $C_\tau := C([-\tau, 0])$ defines a Banach space of all continuous functions $e : [-\tau, 0] \to \mathbb{R}$.

For $x \in \mathbb{R}^n$, we can write $x \in C_\tau$ means $x(s) \equiv x$ in $[-\tau, 0]$. Given $e \in C_\tau$, let $\|e\|_c = \sup|e(s)|$.

The initial states proposed for system (1) are as follow

$$x_i(s) = e_i(s), s \in [-\tau, 0], i = 1, 2, \ldots, n. \tag{6}$$

Consider $x_t \in C([-\tau, 0], \mathbb{R}^n)$ described by $x_t(s) = x(t + s), -\tau \le s \le 0$, and the initial states (10) can be rewritten as

$$x_0 = e \in C_\tau := C([-\tau, 0], \mathbb{R}^n). \tag{7}$$

Suppose that $A \subset \mathbb{R}^n$, then $x \to \phi(x)$ is said a set-valued map from A to $\mathbb{R}^n$, if for every point $x \in A$, there exists a nonempty set $\phi(x) \subset \mathbb{R}^n$. We call a set-valued map ϕ with nonempty values, an upper semicontinuous at $x_0 \in A$, if for every open set N containing $\phi(x_0)$, there exists a neighborhood M of x_0 such that $\phi(M) \subset N$. The map $\phi(x)$ is said to have a closed (convex, compact) image if for each $x \in A$, $\phi(x)$ is closed (convex, compact).

Existence of periodic solution

In the rest of this section we will investigate the existence of periodic solutions of the generalized memristor system.

By the differential equation system (1), we consider the set-valued maps as follow: for $t \in \mathbb{R}$ and $i, j = 1, 2, \ldots, n$,

$$K[b_{ij}(x_j(t))] = \begin{cases} \bar{b}_{ij}, & |x_j(t)| > T_j \\ \overline{co}\{\bar{b}_{ij}, \underline{b}_{ij}\}, & |x_i(t)| = T_i \\ \underline{b}_{ij}, & |x_j(t)| < T_j \end{cases} \tag{8}$$

$$K[c_{ij}(x_j(t))] = \begin{cases} \bar{c}_{ij}, & |x_j(t)| > T_j \\ \overline{co}\{\bar{c}_{ij}, \underline{c}_{ij}\}, & |x_j(t)| = T_j \\ \underline{c}_{ij}, & |x_j(t)| < T_j \end{cases} \tag{9}$$

$$K[p_{ij}(x_j(t))] = \begin{cases} \bar{p}_{ij}, & |x_j(t)| > T_j \\ \overline{co}\{\bar{p}_{ij}, \underline{p}_{ij}\}, & |x_j(t)| = T_j \\ \underline{p}_{ij}, & |x_i(t)| < T_i \end{cases} \tag{10}$$

It is clear that $K[b_{ij}(x_i(t))], K[c_{ij}(x_i(t))]$ and $K[p_{ij}(x_i(t))]$ are all closed, convex and compact for every $t \in \mathbb{R}$ and $i, j = 1, \ldots, n$.

A function $x(t)$ is said to be a solution of system (1) on $[0, T)$ with initial condition (7) or (8), if $x(t)$ is absolutely continuous on any compact interval of $[0, T]$ and satisfies differential inclusions:

$$\frac{dx_i(t)}{dt} \in -a_i(t)x_i(t) + \sum_{j=1}^n K[b_{ij}(x_j(t))]f_j(x_j(t))$$
$$+ \sum_{j=1}^n K[c_{ij}(x_j(t))]g_j(x_j(t - \tau_{ij}(t))) + \sum_{j=1}^n K[p_{ij}(x_j(t))] \int_{-\infty}^t k_{ij}(t-s)h_j(x_j(s))ds + J_i(t), \tag{11}$$

or there exist $\gamma_{ij}(t) \in K[b_{ij}(x_i(t))], \eta_{ij}(t) \in K[c_{ij}(x_i(t))]$ and $v_{ij}(t) \in K[p_{ij}(x_i(t))]$ satisfy

$$\frac{dx_i(t)}{dt} \in -a_i(t)x_i(t) + \sum_{j=1}^n \gamma_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n \eta_{ij}(t)g_j(x_j(t - \tau)) + \sum_{j=1}^n v_{ij}(t) \int_{-\infty}^t k_{ij}(t-s)h_j(x_j)ds + J_i, \tag{12}$$

for a. a. $t \in [0, T)$, $i = 1, 2, \ldots, n$.

In the following, we discuss dynamical behavior of system (1) using the following set-valued version of the Mawhin coincidence theorem.

Lemma 2 (Mawhin coincidence theorem[42]) *Suppose that* $\phi : \mathbb{R} \times \mathbb{R}^n \to K_v(\mathbb{R}^n)$ *is upper semicontinuous and* ω *-periodic in t. If the following conditions hold*:

 4

(1) *There exists a bounded open set $\Delta \subset C_\omega$, the set of all continuous, ω-periodic functions: $\mathbb{R} \to \mathbb{R}^n$, such that for any $\lambda \in (0, 1)$, each ω-periodic function $x(t)$ of the inclusion*

$$\frac{dx}{dt} \in \lambda \phi(t, x), \tag{13}$$

satisfies $x \notin \partial \Delta$ if it exists.

(2) *Each solution $z \in \mathbb{R}^n$ of the inclusion $0 \in \frac{1}{\omega} \int_0^\omega \phi(t, z)dt = g_0(z)$ satisfies $z \notin \partial \Delta \cap \mathbb{R}^n$;*

(3) *$deg(g_0, \Delta \cap \mathbb{R}^n, 0) \neq 0$, then differential inclusion (13) has at least one ω-periodic solution $x(t)$ with $x \in \bar{\Delta}$. If a matrix $O \geq 0$ then all elements of O are greater than or equal to 0, and similarly we can describe $O > 0$. It follows that for given vectors or matrices O and P, $O \geq P$ (or $O > P$) and similarly that $O - P \geq 0$ (or $O - P > 0$). After that, we give sufficient conditions to guarantee the existence of periodic solutions for the memristive neural network.*

Theorem 1 *We consider $I \to S$ an M-matrix, where I is the identity matrix of dimension n, $S = (s_{ij})_{n \times n}$ and*

(H4) $s_{ij} = \frac{1}{a_i^l} \left(b_{ij}^u F_j + \frac{c_{ij}^u G_j}{\sqrt{1 - \tau_{ij}^D}} + \frac{M}{\alpha} p_{ij}^u F_j \right)$, $i, j = 1, 2, .., n$,

where $b_{ij}^u = max\left\{\overline{b}_{ij}^u, \underline{b}_{ij}^u\right\}$, $c_{ij}^u = max\left\{\overline{c}_{ij}^u, \underline{c}_{ij}^u\right\}$ and $p_{ij}^u = max\left\{\overline{p}_{ij}^u, \underline{p}_{ij}^u\right\}$.

Then there exists at least one ω-periodic solution of system (1).

Proof Define $E_\omega = \{x(t) \in C(\mathbb{R}, \mathbb{R}^n) : x(t + \omega) = x(t)\}$, and for $x(t) \in E_\omega$

$\|x(t)\|_{C_\omega} = \sum\limits_{i=1}^n \max\limits_{t \in [0, \omega]} |x_i(t)|.$

Then E_ω is a Banach space equipped with the norm $\|.\|_{E_\omega}$.
Let for $x(t) \in E_\omega$,

$$\phi(t, x) = (\phi_1(t, x), \phi_2(t, x), \ldots, \phi_n(t, x))^T,$$

where

$$\phi_i(t, x) = -a_i(t)x_i(t) + \sum_{j=1}^n K[b_{ij}(x_j(t))]f_j(x_j(t)) + \sum_{j=1}^n K[c_{ij}(x_j(t))]f_j(x_j(t - \tau))$$

$$+ \sum_{j=1}^n K[p_{ij}(x_j(t))] \int_{-\infty}^t f_j(x_j(s))ds + J_i(t),$$

$i = 1, 2, .., n$.

Let assumption (H4) holds, $\phi(t, x)$ is an upper semicontinuous set-valued map with nonempty compact convex values under H4. Here we need to find an appropriate open, bounded subset Δ, in order to apply Mawhin-Like Coincidence Theorem (Lemma 2),

From the differential inclusion (13), we obtain

$$\frac{dx_i(t)}{dt} \in \lambda[-a_i x_i(t) + \sum_{j=1}^n K[b_{ij}(x_j(t))]f_j(x_j(t)) + \sum_{j=1}^n K[c_{ij}(x_j(t))]f_j(x_j(t - \tau))$$

$$+ \sum_{j=1}^n K[p_{ij}(x_j(t))] \int_{-\infty}^t f_j(x_j(s))ds + J_i]. \tag{14}$$

Given $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ an arbitrary ω-periodic solution of the differential inclusion (14) for a certain $\lambda \in (0, 1)$. There exist $\gamma_{ij}(t) \in K[b_{ij}(x_j(t))]$ and $\eta_{ij}(t) \in K[c_{ij}(x_j(t))]$ $v_{ij}(t) \in K[p_{ij}(x_j(t))]$ satisfy

$$\frac{dx_i(t)}{dt} = \lambda[-a_i(t)x_i(t) + \sum_{j=1}^n \gamma_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n \eta_{ij}(t)f_j(x_j(t - \tau)) + \sum_{j=1}^n v_{ij}(t) \int_{-\infty}^t f_j(x_j(s))ds + J_i], \tag{15}$$

for a $[0, T)$, $i = 1, 2, .., n$.

Multiplying both sides of (15) by $x_i(t)$ and integrating over the interval $[0, \omega]$, we get

$$\int_0^\omega a_i(t)x_i^2(t)dt = \int_0^\omega x_i(t)[\sum_{j=1}^n \gamma_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n \eta_{ij}(t)g_j(x_j(t - \tau_{ij}))$$

$$+ \sum_{j=1}^n v_{ij}(t) \int_{-\infty}^t k_{ij}(t - s)h_j(x_j(s))ds + J_i(t)]dt. \tag{16}$$

From (H2), (8), (9), (10) and (11), it is clear to see that

$$
\begin{cases}
|\gamma_{ij}(t)| & \leq max\left\{|\overline{b}_{ij}|, |\underline{b}_{ij}|\right\} \leq b_{ij}^u \\
|\eta_{ij}(t)| & \leq max\left\{|\overline{c}_{ij}|, |\underline{c}_{ij}|\right\} \leq c_{ij}^u \\
|v_{ij}(t)| & \leq max\left\{|\overline{p}_{ij}|, |\underline{p}_{ij}|\right\} \leq p_{ij}^u \\
|f_j(x_j(t))| & \leq F_j|x_j(t)| + |f_j(0)|, \\
|g_j(x_j(t - \tau_{ij}))| & \leq G_j|x_j(t - \tau_{ij})| + |g_j(0)| \\
|h_j(x_j(t))| & \leq H_j|x_j(t)| + |h_j(0)|
\end{cases} \tag{17}
$$

From (16) and Cauchy–Schwarz inequality, we obtain:

$$
\begin{aligned}
a_i^l \int_0^\omega x_i^2(t)dt \leq & \sum_{j=1}^n b_{ij}^u F_j \int_0^\omega |x_i(t)||x_j(t)|dt + \sum_{j=1}^n c_{ij}^u G_j \int_0^\omega |x_i(t)||x_j(t - \tau_{ij})|dt \\
& + \sum_{j=1}^n p_{ij}^u H_j \int_0^\omega \left(\int_{-\infty}^t k_{ij}(t - s)|x_i(s)||x_j(s)|ds\right)dt \\
& + \left(\sum_{j=1}^n b_{ij}^u|f_j(0)| + c_{ij}^u|g_j(0)| + \frac{M}{\alpha}p_{ij}^u|h_j(0)| + J_i^I\right)\int_0^\omega |x_i(t)|dt.
\end{aligned} \tag{18}
$$

Noticing that

$$
\begin{aligned}
\int_0^\omega |x_j^2(t - \tau_{ij}(t))|dt &= \int_{-\tau_{ij}(0)}^{\omega - \tau_{ij}(\omega)} \frac{|x_j^2(t)|}{1 - \dot{\tau}_{ij}(\varphi_{ij}^{-1}(t))}dt = \int_{-\tau_{ij}(0)}^{\omega - \tau_{ij}(0)} \frac{|x_j^2(t)|}{1 - \dot{\tau}_{ij}(\varphi_{ij}^{-1}(t))}dt = \int_0^\omega \frac{|x_j^2(t)|}{1 - \dot{\tau}_{ij}(\varphi_{ij}^{-1}(t))}dt \\
&= \frac{1}{1 - \tau_{ij}^D}\int_0^\omega |x_j^2(t)|dt, \ j = 1, 2, .., n,
\end{aligned} \tag{19}
$$

where φ_{ij}^{-1} is the inverse function of $\varphi_{ij} = t - \tau_{ij}(t)$, $i, j = 1, 2, .., n$.

Then, for i=1,2,..,n, we obtain

$$
\begin{aligned}
a_i^l \int_0^\omega x_i^2(t)dt \leq & \sum_{j=1}^n \left(b_{ij}^u F_j + \frac{c_{ij}^u G_j}{\sqrt{1 - \tau_{ij}^D}} + \frac{M}{\alpha}p_{ij}^u H_j\right)\left(\int_0^\omega |x_j^2(t)|dt\right)^{\frac{1}{2}} + \sqrt{\omega}\left(\sum_{j=1}^n b_{ij}^u|f_j(0)| + c_{ij}^u|g_j(0)| \right. \\
& \left. + \frac{M}{\alpha}p_{ij}^u|h_j(0)| + J_i^I\right),
\end{aligned} \tag{20}
$$

which implies

$$
\begin{aligned}
\int_0^\omega x_i^2(t)dt \leq & \sum_{j=1}^n \frac{1}{a_i^l}\left(b_{ij}^u F_j + \frac{c_{ij}^u G_j}{\sqrt{1 - \tau_{ij}^D}} + \frac{M}{\alpha}p_{ij}^u H_j\right)\left(\int_0^\omega |x_j^2(t)|dt\right)^{\frac{1}{2}} \\
& + \frac{\sqrt{\omega}}{a_i^l}\left(\sum_{j=1}^n b_{ij}^u|f_j(0)| + c_{ij}^u|g_j(0)| + \frac{M}{\alpha}p_{ij}^u|h_j(0)| + J_i^I\right)
\end{aligned} \tag{21}
$$

$$
\int_0^\omega x_i^2(t)dt \triangleq q_{ij}\sum_{j=1}^n \left(\int_0^\omega x_j^2(t)dt\right)^{\frac{1}{2}} + \sqrt{\omega}\theta_i \ i = 1, 2, .., n,
$$

where

$$
\theta_i = \frac{1}{a_i^l}\left(\sum_{j=1}^n b_{ij}^u|f_j(0)| + c_{ij}^u|g_j(0)| + \frac{M}{\alpha}p_{ij}^u|h_j(0)| + J_i^I\right), \ i = 1, 2, .., n.
$$

Let $\|x_i\|_2^\omega = \left(\int_0^\omega x_i^2(t)dt\right)^2$, $x_i \in C(\mathbb{R}, \mathbb{R})$, $i = 1, 2, .., n$.
Thus

$$
(I - S)\left(\|x_1\|_2^\omega, \|x_2\|_2^\omega, \ldots, \|x_n\|_2^\omega\right)^T \leq \sqrt{\omega}\alpha. \tag{22}
$$

Since $(I - S)$ is an M-matrix, assumption (H4) holds and Lemma 1, there exists a vector

$$\nu = \left(\nu_1^*, \nu_2^*, \ldots, \nu_n^*\right) = \nu(I - S) > (0, 0, \ldots, 0), \tag{23}$$

and from (22), we have

$$
\begin{aligned}
min\{\nu_1^*, \nu_2^*, \ldots, \nu_n^*\}\left(\|x_1\|_2^\omega, \|x_2\|_2^\omega, \ldots, \|x_n\|_2^\omega\right) &\leq \nu_1^*\|x_1\|_2^\omega + \nu_2^*\|x_2\|_2^\omega + \ldots + \nu_n^*\|x_n\|_2^\omega \\
&= \nu(I - S)\left(\|x_1\|_2^\omega, \|x_2\|_2^\omega, \ldots, \|x_n\|_2^\omega\right)^T \leq \nu\sqrt{\omega}(\theta_1, \theta_2, \ldots, \theta_n)^T \\
&= \sqrt{\omega}\sum_{j=1}^{n}\nu_i\theta_i.
\end{aligned}
\tag{24}
$$

Thus, we obtain

$$\left(\int_0^\omega |x_i^2(t)|dt\right)^{\frac{1}{2}} = \|x_i\|_2^\omega \leq \frac{\sqrt{\omega}\sum_{i=1}^{n}\nu_i\theta_i}{min\{\nu_1^*, \nu_2^*, \ldots, \nu_n^*\}} \triangleq \sqrt{\omega}N. \tag{25}$$

Obviously, we can see that there exists $t_i \in [0, \omega]$ such that

$$|x_i(t)| \leq \sqrt{\omega}N, \ i = 1, 2, \ldots, n. \tag{26}$$

Since for $t_i \in [0, \omega]$,

$$x_i(t) = x_i(t_i) + \int_{t_i}^{t} \dot{x}_i(s)ds,$$

thus we obtain

$$|x_i(t)| \leq N + \int_0^\omega \dot{x}_i(s)ds, \ i = 1, 2, .., n,$$

we can derive from (17) and (19) that

$$
\begin{aligned}
\int_0^\omega |\dot{x}_i(t)|dt &< a_i^l \int_0^\omega |x_i(t)|dt + \sum_{j=1}^{n} b_{ij}^u F_j\left(\int_0^\omega |x_j(t)|dt\right) \\
&\quad + \sum_{j=1}^{n} c_{ij}^u G_j \int_0^\omega |x_j(t - \tau_{ij}(t))|dt + \sum_{j=1}^{n} p_{ij}^u H_j\left(\int_0^\omega \int_{-\infty}^{t} k_{ij}(t - s)|x_j(s)|dsdt\right) \\
&\quad + \omega\left(\sum_{j=1}^{n}\left(b_{ij}^u|f_j(0)| + c_{ij}^u|g_j(0)| + \frac{M}{\alpha}p_{ij}^u|h_j(0)|\right) + J_i^I\right) \\
&= \int_0^\omega |\dot{x}_i(t)|dt < a_i^l \int_0^\omega |x_i(t)|dt + \sum_{j=1}^{n} b_{ij}^u F_j\left(\int_0^\omega |x_j(t)|dt\right) + \sum_{j=1}^{n} c_{ij}^u G_j \int_0^\omega |x_j(t - \tau_{ij}(t))|dt \\
&\quad + \sum_{j=1}^{n} p_{ij}^u H_j\left(\int_{-\infty}^{\omega} |x_j(s)| \int_s^\omega k_{ij}(t - s)dtds\right) + \omega\left(\sum_{j=1}^{n}\left(b_{ij}^u|f_j(0)| + c_{ij}^u|g_j(0)|\right) + \frac{M}{\alpha}p_{ij}^u|h_j(0)| + J_i^I\right) \\
&= \int_0^\omega |\dot{x}_i(t)|dt < a_i^l \int_0^\omega |x_i(t)|dt + \sum_{j=1}^{n} b_{ij}^u F_j\left(\int_0^\omega |x_j(t)|dt\right) + \sum_{j=1}^{n} c_{ij}^u G_j \int_0^\omega |x_j(t - \tau_{ij}(t))|dt \\
&\quad + \sum_{j=1}^{n} p_{ij}^u H_j\left(\int_0^\omega |x_i(s)|\left(\int_s^\omega k_{ij}(t - s)dt\right)ds\right) + \omega\left(\sum_{j=1}^{n}\left(b_{ij}^u|f_j(0)| + c_{ij}^u|g_j(0)|\right) + \frac{M}{\alpha}p_{ij}^u|h_j(0)| + J_i^I\right) \\
&\leq \int_0^\omega |\dot{x}_i(t)|dt < a_i^l \int_0^\omega |x_i(t)|dt + \sum_{j=1}^{n} b_{ij}^u F_j\left(\int_0^\omega |x_i(t)|dt\right) + \sum_{j=1}^{n} c_{ij}^u G_j \int_0^\omega |x_j(t - \tau_{ij}(t))|dt \\
&\quad + \sum_{j=1}^{n} \frac{M}{\alpha}p_{ij}^u H_j\left(\int_0^\omega |x_i(s)|ds\right) + \omega\left(\sum_{j=1}^{n}\left(b_{ij}^u|f_j(0)| + c_{ij}^u|g_j(0)|\right) + \frac{M}{\alpha}p_{ij}^u|h_j(0)| + J_i^I\right) \\
&\leq a_i^l\sqrt{\omega}\|x_i\|_2^\omega + \sum_{j=1}^{n}\left(b_{ij}^u F_j + \frac{c_{ij}^u G_j}{\sqrt{1 - \tau_{ij}^D}} + \frac{M}{\alpha}p_{ij}^u H_j\right)\sqrt{\omega}\|x_i\|_2^\omega + \omega\left(\sum_{j=1}^{n}\left(b_{ij}^u|f_j(0)| + c_{ij}^u|g_j(0)|\right) + \frac{M}{\alpha}p_{ij}^u|h_j(0)| + J_i^I\right) \\
&\leq \sqrt{\omega}N + \left(a_i^l + \sum_{j=1}^{n}\left(b_{ij}^u F_j + \frac{c_{ij}^u G_j}{\sqrt{1 - \tau_{ij}^D}} + \frac{M}{\alpha}p_{ij}^u H_j\right)\right) + \omega\left(\sum_{j=1}^{n}\left(b_{ij}^u|f_j(0)| + c_{ij}^u|g_j(0)|\right) + \frac{M}{\alpha}p_{ij}^u|h_j(0)| + J_i^I\right) \triangleq R_i,
\end{aligned}
$$

for each $i = 1, 2, .., n$. Then, it follows

$$|x_i(t)| \leq N + R_i \triangleq H_i, \ i = 1, 2, .., n. \tag{27}$$

One may readily verify that H_i, $i = 1, 2, .., n$ is independent of λ. Again taking (H4) into account, we can get from Lemma 1, that there exists a vector $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)^T > (0, 0, \ldots, 0)^T$, such that $(I - S)\zeta > (0, 0, \ldots, 0)^T$. Hence, we can choose a sufficiently large constant σ such that

$\zeta^* = (\zeta_1^*, \zeta_2^*, \ldots, \zeta_n^*) = (\sigma\zeta_1, \sigma\zeta_2, \ldots, \sigma\zeta_n)^T > \sigma\zeta$, and that

$$\zeta_i^* = \sigma\zeta_i > H_i (i = 1, 2, \ldots, n), \ and (I - S)\zeta^* > \theta. \tag{28}$$

In order to finish our proof we will proceed in three steps:

Step1 let us take

$$\Delta = \{x(t) \in C_\omega : -\zeta^* < x(t) < \zeta^*, \forall t \in \mathbb{R}\}. \tag{29}$$

Hence Δ is an open bounded set of C_ω and for any $\lambda \in (0, 1)$, $x \notin \partial\Delta$. Thus, the condition (1) in Mawhin coincidence theorem is satisfied.

Step 2 we shall use contradiction to demonstrate the condition (2) in Lemma 2. Let us consider that there exists a solution $u = (u_1, u, \ldots, u_n)^T \in \partial\Delta \cap \mathbb{R}^n$ of the inclusion

$$0 \in \frac{1}{\omega} \int_0^\omega F(t, u) dt = g_0(u).$$

Then u is a constant vector on $\mathbb{R}^n$ such that $|u_i| = \zeta_i$ for $i \in (1, 2, .., n)$.

Therefore, we have for $1 < i \leq n$

$$0 \in g_0(u) = -u_i \frac{1}{\omega} \int_0^\omega a_i(t) u_i(t)] dt + \sum_{j=1}^n f_j(u) \frac{1}{\omega} \int_0^\omega K[b_{ij}(u_j(t))] dt$$

$$+ \sum_{j=1}^n g_j(u) \frac{1}{\omega} \int_0^\omega K[c_{ij}(u_j(t))] dt + \sum_{j=1}^n \frac{1}{\omega} h_j(u) \int_0^\omega K[b_{ij}(u_j(t))] dt \int_{-\infty}^t k_{ij}(t-s) ds + \frac{1}{\omega} \int_0^\omega I_i(t) dt, \tag{30}$$

or

$$0 = -u_i \frac{1}{\omega} \int_0^\omega a_i(t) dt + \sum_{j=1}^n f_j(u) \frac{1}{\omega} \int_0^\omega \gamma_{ij}(t) dt + \sum_{j=1}^n g_j(u) \frac{1}{\omega} \int_0^\omega \eta_{ij}(t)) dt$$

$$+ \sum_{j=1}^n \frac{M}{\alpha} h_j(u) \frac{1}{\omega} \int_0^\omega v_{ij}(t) dt + \frac{1}{\omega} \int_0^\omega I_i(t) dt, \ i = 1, 2, .., n, \tag{31}$$

where $\gamma_{ij}(t) \in K[b_{ij}(u_i)]$, $\eta_{ij}(t) \in K[c_{ij}(u_i)]$ and $v_{ij}(t) \in K[p_{ij}(u_i)]$. Then, there exists some $t^* \in [0, \omega)$ such that

$$-u_i a_i(t^*) + \sum_{j=1}^n (\gamma_{ij}(t^*) f_j(u) + \eta_{ij}(t^*) g_j(u) + v_{ij}(t^*) \frac{M}{\alpha} h_j(u)) + I_i(t^*) = 0, \ i = 1, 2, .., n. \tag{32}$$

It follows

$$\zeta_i^* = |u_i| = \frac{1}{a_i(t^*)} [\sum_{j=1}^n \left(\gamma_{ij}(t^*) f_j(u) + \eta_{ij}(t^*) g_j(u) + v_{ij}(t^*) \frac{M}{\alpha} h_j(u) \right) + I_i(t^*)]$$

$$\leq \frac{1}{a_i^l} [\sum_{j=1}^n (b_{ij}^u (F_j(|u_j|) + |f_j(0)|) + c_{ij}^u (G_j(|u_j|) + |g_j(0)|) + \frac{M}{\alpha} p_{ij}^u (H_j(|u_j|) + |h_j(0)|)) + I_i^u,$$

$$\leq \frac{1}{a_i^l} [\sum_{j=1}^n (b_{ij}^u (F_j(|u_j|) + |f_j(0)|) + \frac{c_{ij}^u (G_j(|u_j|) + |g_j(0)|)}{\sqrt{1 - \tau_{ij}^D}} + \frac{M}{\alpha} p_{ij}^u (H_j(|u_j|) + |h_j(0)|)) + I_i^u] \tag{33}$$

$$= \sum_{j=1}^n s_{ij} |u_j| + \theta_i = \sum_{j=1}^n s_{ij} \zeta_j^* + \theta_i.$$

Therefore $(I - S)\zeta^* \leq \theta$, which contradicts the fact $(I - S)\zeta^* > \theta$ and the condition 2 of Lemma 2 holds.

Step 3 In order to prove condition 3 let us define homotopic set-valued map

$\phi : \Delta \cap \mathbb{R}^n \times [0, 1] \to A_\omega$

$(u, h) \longmapsto h \, diag(-\bar{a}_1, -\bar{a}_2, \ldots, -\bar{a}_n) u + (1 - h) g_0(u),$

where
$\bar{a}_i = \frac{1}{\omega} \int_0^{\omega} a_i(t)dt, i = 1, 2, \ldots, n.$

if $u = (u_1, u_2, \ldots, u_n)^T \in \partial\Delta \cap \mathbb{R}^n$ then u is a constant vector on $\mathbb{R}^n$ such that $|u_i| = \zeta_i^*$ for some $i \in \{1, 2, \ldots, n\}$.

It follows that

$$(\phi(u, h))_i = -u_i \frac{1}{\omega} \int_0^{\omega} a_i(t)u_i(t)dt + (1-h)[\sum_{j=1}^{n} f_j(u)\frac{1}{\omega}\int_0^{\omega} K[b_{ij}(u_j(t))]dt + \sum_{j=1}^{n} g_j(u)\frac{1}{\omega}\int_0^{\omega} K[c_{ij}(u_j(t))]dt$$

$$+ \sum_{j=1}^{n} \frac{M}{\alpha} h_j(u)\frac{1}{\omega}\int_0^{\omega} K[p_{ij}(u_j(t))]dt + \frac{1}{\omega}\int_0^{\omega} I_i(t)dt].$$

(34)

Which implies that

$$0 \notin (\phi(u, h))_i, i = 1, 2, \ldots, n.$$

(35)

If this is not true, then $0 \in (\phi(u, h))_i, i = 1, 2, \ldots, n,$, i.e.,

$$0 \in -u_i \frac{1}{\omega} \int_0^{\omega} a_i(t)u_i(t)dt + (1-h)[\sum_{j=1}^{n} f_j(u)\frac{1}{\omega}\int_0^{\omega} K[b_{ij}(u_i(t))]dt + \sum_{j=1}^{n} g_j(u)\frac{1}{\omega}\int_0^{\omega} K[c_{ij}(u_i(t))]dt$$

$$+ \sum_{j=1}^{n} \frac{M}{\alpha} h_j(u)\frac{1}{\omega}\int_0^{\omega} K[p_{ij}(u_i(t))]dt + \frac{1}{\omega}\int_0^{\omega} I_i(t)dt].$$

(36)

Similarly, there exist $\gamma_{ij}(t) \in K[b_{ij}(u_i)], \eta_{ij}(t) \in K[c_{ij}(u_i)]$ and $\nu_{ij}(t) \in K[p_{ij}(u_i)], i = 1, 2, \ldots, n$ such that

$$0 = (-u_i)\frac{1}{\omega}\int_0^{\omega} a_i(t)dt + (1-h)[\sum_{j=1}^{n} f_j(u)\frac{1}{\omega}\int_0^{\omega}\gamma_{ij}(t)dt + \sum_{j=1}^{n} g_j(u)\frac{1}{\omega}\int_0^{\omega}\eta_{ij}(t))dt + \sum_{j=1}^{n}\frac{M}{\alpha}h_j(u)\frac{1}{\omega}\int_0^{\omega}\nu_{ij}(t))dt$$

$$+ \frac{1}{\omega}\int_0^{\omega} I_i(t)dt], i = 1, 2, .., n,$$

(37)

consequently, there exists some $t^{**} \in [0, \omega]$ such that

$$0 = -u_i a_i(t^{**}) + (1-h)[\sum_{j=1}^{n}\gamma_{ij}(t^{**})f_j(u) + \eta_{ij}(t^{**})g_j(u)$$

$$+ \frac{M}{\alpha}\nu_{ij}(t^{**})h_j(u) + I_i(t^{**})] i = 1, 2, \ldots, n.$$

(38)

We derive from (38) that

$$\zeta_i^* = |u_i| = \frac{1-h}{a_i(t^{**})}[\sum_{j=1}^{n}\gamma_{ij}(t^{**})f_j(u) + \eta_{ij}(t^{**})g_j(u) + \frac{M}{\alpha}\nu_{ij}(t^{**})h_j(u) + I_i(t^{**})]$$

$$\leq \frac{1}{a_i^l}[\sum_{j=1}^{n}(b_{ij}^u(F_j(|u_j|) + |f_j(0)|) + c_{ij}^u(G_j(|u_j|) + |g_j(0)|) + \frac{M}{\alpha}p_{ij}^u(H_j(|u_j|) + |h_j(0)|)) + I_i^u]$$

$$\leq \frac{1}{a_i^l}[\sum_{j=1}^{n}(b_{ij}^u(F_j(|u_j|) + |f_j(0)|) + \frac{c_{ij}^u(G_j(|u_j|) + |g_j(0)|)}{\sqrt{1-\tau_{ij}^D}} + \frac{M}{\alpha}p_{ij}^u(H_j(|u_j|) + |h_j(0)|)) + I_i^u]$$

$$= \sum_{j=1}^{n} q_{ij}|u_j| + \theta_i = \sum_{j=1}^{n} q_{ij}\zeta_j^* + \theta_i,$$

which yields that $(I - S)\zeta^* \leq \theta$, which contradicts $(I - S)\zeta^* > \theta$. Thus, (30) holds. Which implies that $(0, 0, \ldots, 0)^T \notin \phi(u, h)$ for any $u = (u_1, u_2, \ldots, u_n)^T \partial\Delta \cap \mathbb{R}^n, h \in [0, 1]$. Thus, using the solution properties of the topological degree and the homotopy invariance, we have

$$deg\{g_0, \Delta \cap \mathbb{R}^n, 0\} = deg\{\phi(u, 0), \Delta \cap \mathbb{R}^n, 0\} = deg\{\phi(u, 1), \Delta \cap \mathbb{R}^n, 0\} = deg\{(-\bar{a}_1 u_1, -\bar{a}_2 u_2, \ldots, -\bar{a}_n u_n)^T, \Delta \cap \mathbb{R}^n, 0\}$$

$$= sign \begin{vmatrix} -\bar{a}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\bar{a}_1 \end{vmatrix} = (-1)^n \neq 0.$$

(39)

This means that $\triangle$ satisfies all the conditions in Lemma 2, then the system (1) possesses at least one $\omega-$periodic solution.

The proof is finished. □

Uniqueness and global exponential stability

Now, we will prove the uniqueness and global exponential stability of the ω-periodic solution for the system (1). Mainly, when the system (1) is considered autonomous, we will find the sufficient conditions on the existence, uniqueness and global exponential stability of fixed point of the system.

Definition 6 *(Stability)* We denote $x^*(t, \varphi)$ a periodic solution of the system (1). The periodic solution $x^*(t, \psi)$ is said to be globally exponentially stable if for any solution $x(t, \varphi)$ of the system (1), there are constants $M \geq 1$ and $\mu > 0$ such that for any $\varphi \in C_\tau$

$\|x(t, \varphi) - x^*(t, \psi)\| \leq M\|\varphi - \psi\|_C e^{\mu t}, \ t \geq 0.$

Let us firstly introduce the following lemma.

Lemma 3 [10] *If $f_j(\pm T_j) = 0$, $g_j(\pm T_j) = 0$ and $h_j(\pm T_j) = 0 (j = 1, \ldots, n)$ then for every $x_j, y_j \in \mathbb{R}$ we have*

$$K[b_{ij}(x_j(t))]f_j(x_j) - K[b_{ij}(y_j(t))]f_j(y_j) \leq b_{ij}^u F_j |x_j - y_j|, \tag{40}$$

and

$$K[c_{ij}(x_j(t))]G_j(x_j) - K[c_{ij}(y_j(t))]g_j(y_j) \leq c_{ij}^u G_j |x_j - y_j|, \tag{41}$$

and

$$K[p_{ij}(x_j(t))]H_j(x_j) - K[p_{ij}(y_j(t))]h_j(y_j) \leq p_{ij}^u H_j |x_j - y_j|, \tag{42}$$

for $i, j = 1, 2, \ldots, n$.

Lemma 4 [43] *Let $x(t) : [0, +\infty[\to \mathbb{R}^n$ an absolutely continuous on any compact interval of $[0, +\infty[$ and $V(x) : \mathbb{R}^n \to \mathbb{R}$ is Clarke's regular, then $x(t)$ and $V(x) : \mathbb{R}^n \to \mathbb{R}$ are differential for all $t \in [0, +\infty[$. We get*

$$\frac{d}{dt} v(t) = \gamma(t)^T \dot{x}(t), \forall \gamma(t) \in \partial V(x(t)),$$

where $\partial V(x(t))$ is Clarke's generalized gradient.

Next, we consider the assumption below.

(H5) *$I - S$ is an M−matrix, I is the identity matrix of size n, $S = (s_{ij})_{n \times n}$ and*

$s_{ij} = \frac{1}{a_i^l} \left(b_{ij}^u F_j + c_{ij}^u G_j + \frac{M}{\alpha} p_{ij}^u H_j \right), i, j = i, j = 1, 2, \ldots, n.$

Theorem 2 *Suppose that $f_j(\pm T_j) = 0$, $g_j(\pm T_j) = 0$ and $h_j(\pm T_j) = 0$ for $j = 1, \ldots, n$, and the assumption (H5) holds. If there exists periodic solution $x^*(t, \psi)$ for system (1), then $x^*(t, \psi)$ is a unique periodic solution of system (1) and is globally exponentially stable, and for any other solution $x(t, \varphi)$ of system (1), there exist constants $M, \mu > 0$ such that*

$$\left\| x_i(t, \varphi) - x_i^*(t, \psi) \right\| \leq R\|\varphi - \psi)\|_c e^{-\mu t},$$

for any $t > 0$.

Proof Consider $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ any solution of system (1) and $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ is an $\omega-$periodic solution of system (1). We get:

$$\frac{dx_i(t)}{dt} \in -a_i x_i(t) + \sum_{j=1}^{n} K[b_{ij}(x_j(t))]f_j(x_j(t)) + \sum_{j=1}^{n} K[c_{ij}(x_j(t - \tau_j(t)))]g_j(x_j(t - \tau_j(t)))$$

$$+ \sum_{j=1}^{n} K[p_{ij}(x_j(t))] \int_{-\infty}^{t} h_j(x_j(s)) ds + J_i],$$

$$\frac{dx_i^*(t)}{dt} \in -a_i x_i^*(t) + \sum_{j=1}^{n} K[b_{ij}(x_j^*(t))]f_j(x_j^*(t)) + \sum_{j=1}^{n} K[c_{ij}(x_j^*(t - \tau_j(t)))]g_j(x_j(t - \tau_j(t)))$$

$$+ \sum_{j=1}^{n} K[p_{ij}(x_j^*(t))] \int_{-\infty}^{t} k_{ij}(t - s) h_j(x_j(s)) ds + J_i].$$

Assume that $y_i(t) = x_i(t) - x_i^*(t)$, then

$$\frac{dy_i(t)}{dt} \in -a_i y_i(t) + \sum_{j=1}^{n} B_{ij}(y_j(t), x_j^*(t)) + \sum_{j=1}^{n} C_{ij}(y_j(t - \tau_{ij}(t)), x_j^*(t - \tau_{ij}(t))) + \sum_{j=1}^{n} \frac{M}{\alpha} P_{ij}(y_j(t), x_j^*(t)),$$

(43)

where $B_{ij}(u, v)$, $C_{ij}(u, v)$ and $P_{ij}(u, v)$ are given as following

$$\begin{cases} B_{ij}(u, v) &= K[b_{ij}(u + v)]f_j(u, v) - K[b_{ij}(v)]f_j(x_j(t)) \\ C_{ij}(u, v) &= K[c_{ij}(u + v)]g_j(u, v) - K[c_{ij}(v)]g_j(x_j(t)) \\ P_{ij}(u, v) &= K[p_{ij}(u + v)]h_j(u, v) - K[p_{ij}(v)]h_j(x_j(t)) \end{cases}.$$

Similarly, there exist $\gamma_{ij}(y_j(t)) \in B_{ij}(y_j(t), x_j^*(t))$, $\eta_{ij}(y_j(t - \tau_{ij}(t))) \in C_{ij}(y_j(t - \tau_{ij}(t)), x_j^*(t - \tau_{ij}(t)))$ and $v_{ij}(y_j(t)) \in P_{ij}(y_j(t), x_j^*(t))$ verify,

$$\frac{dy_i(t)}{dt} \in -a_i y_i(t) + \sum_{j=1}^{n} \gamma_{ij}(y_j(t), y_j(t)) + \sum_{j=1}^{n} \eta_{ij}(y_j(t - \tau_{ij}(t)), y_j(t - \tau_{ij}(t))) + \sum_{j=1}^{n} \frac{M}{\alpha} v_{ij}(y_j(t), y_j(t)),$$

(44)

for every $t \in [0, T), i = 1, 2, \ldots, n$

Taking (44) and Lemma 3 into account, we obtain

$$\begin{cases} \gamma_{ij}(y_j(t)) &\leq b_{ij}^u F_j |y_j(t)| \\ \eta_{ij}(y_j(t - \tau_{ij}(t))) &\leq c_{ij}^u G_j |y_j(t - \tau_{ij}(t))| \\ v_{ij}(y_j(t)) &\leq p_{ij}^u H_j |y_j(t)| \end{cases}$$

Obviously, basing on (H5), the matrix $diag(a_1^l, a_1^l, \ldots, a_n^l) - (b_{ij}^u F_j + c_{ij}^u G_j + \frac{M}{\alpha} p_{ij}^u H_j)_{n \times n}$ is also a nonsingular Mmatrix. In addition, there exists a positive $\beta_i (i = 1, 2, \ldots, n)$ such that

$$\beta_i a_i^l - (b_{ij}^u F_j + c_{ij}^u G_j + \frac{M}{\alpha} p_{ij}^u H_j) > 0, \ i = 1, 2, \ldots, n,$$

As a result, there exists a sufficiently small positive number μ such that

$$\beta_i \left(a_i^l - \mu \right) - \sum_{j=1}^{n} \beta_j (b_{ij}^u F_j + c_{ij}^u G_j e^{\mu \tau} + \frac{M}{\alpha} p_{ij}^u H_j) > 0, \ i = 1, 2, \ldots, n.$$

(45)

We consider the Lyapunov function:

$$V(t) = max\left\{ \frac{e^{\mu t} |y_i(t)|}{\beta_j}, i = 1, 2, \ldots, n. \right\}.$$

$V(t)$ is differential for all $t \geq 0$ because any solution $x(t)$ of system (1) including the ω-periodic solution $x^*(t)$ are absolutely continuous.

The function $|y_i(t)|$ is locally Lipschitz continuous in y_i on $\mathbb{R}$. Hence, the Clarke's generalized gradient of function $|y_i(t)|$ at $y_i(t)$ is

$$\partial(|y_i(t)|) = \bar{co}[sign(y_i(t))] = \begin{cases} -1 & if \ y_i(t) < 0, \\ [-1, 1] & if \ y_i(t) = 0, \\ -1 & if \ y_i(t) > 0. \end{cases}$$

For a given $t \geq 0$, there exists a $k \in \{1, \ldots, n\}$ such that $V(t) = \frac{e^{\mu t} |y_k(t)|}{\beta_k}$, and let $v_k(t) = sign(y_k(t))$, if $y_k(t) \neq 0$, while $v_k(t)$ can be arbitrarily chosen in $[-1, 1]$, if $y_k(t) = 0$. From Lemma 4 and system (44), it follows for all $t \geq 0$:

$$\dot{V}(t) = \mu V(t) + V_k(t) \frac{e^{\mu t} \dot{y}_k(t)|}{\beta_k} \leq -(a_k(t) - \mu)V(t) + \frac{e^{\mu t}}{\beta_k} \sum_{j=1}^{n} |\gamma_{ij}(y_j(t))|$$

$$+ \frac{e^{\mu t}}{\beta_k} \sum_{j=1}^{n} |\eta_{ij}(y_j(t - \tau_{ij}(t)))| + \frac{e^{\mu t}}{\beta_k} \sum_{j=1}^{n} \frac{M}{\alpha} |v_{ij}(y_j(t))|$$

$$\leq -\left(a_k^l - \mu \right)V(t) + \frac{1}{\beta_k} \left[\sum_{j=1}^{n} b_{ij}^u F_j \beta_j V(t) + \sum_{j=1}^{n} c_{ij}^u G_j \beta_j e^{\mu \tau_{ij}(t)} V(t - \tau_{ij}(t)) + \sum_{j=1}^{n} \frac{M}{\alpha} p_{ij}^u H_j \beta_j V(t) \right]$$

$$\leq -\frac{1}{\beta_k} \left[-\left(a_k^l - \mu \right)\beta_k - \sum_{j=1}^{n} \left(b_{ij}^u F_j + c_{ij}^u G_j e^{\mu \tau} + \frac{M}{\alpha} p_{ij}^u H_j \right) \beta_j \right] V(t) \leq 0,$$

(46)

when $V(t + s) \leq V(t)$ for any $s \in [-\tau, 0]$. Let $\bar{V}(t) = \sup_{-\tau \leq s \leq 0} V(t + s)$, then we get

$$\frac{\bar{V}(t)}{dt} \leq 0, \forall t \geq -\tau$$

(47)

Therefore

$$|y_j(t)| \le \beta_i V(t) e^{-\mu t} \le \beta_i V(0) e^{-\mu t}, \tag{48}$$

for all $i = 1, \ldots, n$. Thus, for any $t > 0$,

$$\|y(t)\| \le y_k(t)(t) e^{-\mu t} \sum_{i=1}^{n} \beta_i / \beta_k.$$

Moreover,

$$\|x(t) - x^*(t)\| \le R \|\varphi - \psi\| c^{e^{-\mu t}},$$

where $R = \sum_{i=1}^{n} \beta_i / \beta_{min}$, $x(t) = x(t, \varphi)$ and $x^*(t) = x^*(t, \psi)$.

Hence, the ω-periodic solution $x^*(t)$ of system (1) is globally exponentially stable. Then, the periodic solution $x^*(t)$ of system (1) is unique. The proof is complete. $\qquad\square$

Theorem 3 *Consider that $f_j(\pm T_j) = 0$, $g_j(\pm T_j) = 0$ and $h_j(\pm T_j) = 0 (j = 1, \ldots, n)$, and the assumption (H4) is satisfied. Then system (1) has a unique periodic solution $x^*(t, \psi)$, and it is globally exponentially stable.*

Next, we demonstrate the existence and global exponential stability of the equilibrium point for autonomous neural network model (1).

Let $a_i^l = a_i$, $b_{ij}^u = max\{|\hat{b}_{ij}|, |\check{b}_{ij}|\}, c_{ij}^u = max\{|\hat{c}_{ij}|, |\check{c}_{ij}|\} p_{ij}^u = max\{|\hat{p}_{ij}|, |\check{p}_{ij}|\}$ in the assumption (H4) and (H5) for system (1).

Firstly, for autonomous system (1), using Theorems 1 and 3 we can get the following result.

Corollary 1 *Consider that $f_j(\pm T_j) = 0$ and $g_j(\pm T_j) = 0$, $h_j(\pm T_j) = 0$ and $\tau_{ij}(t) \equiv \tau_{ij}$, where $\tau_{ij}(i, j = 1, 2, \ldots, n)$ are all nonnegative constants. if (H5) is satisfied, then there exists a unique equilibrium point x^* for system (1), which is globally exponentially stable.*

Proof Clearly, system (1) is an ω-periodic system, then, basing on Theorems 1 and 2, for any constant $\omega > 0$ system (1) possesses a unique ω-periodic solution $x^*(t)$ and it is globally exponentially stable.

Let $x^*(t)$ be unique for all $\omega > 0$, then we have $x^*(t + \omega) = x^*(t)$ for any constants $\omega > 0$ and $t \ge 0$. Hence $x^*(t) \equiv x^*$ for all $t \ge 0$.

Thus $x^* = x^*(0)$ is an equilibrium point of system (1) and x^* is unique and globally exponentially stable.

Theorem 4 *Consider that $f_j(\pm T_j) = 0$ and $g_j(\pm T_j) = 0 (j = 1, 2, \ldots, n)$. Since (H5) holds, there exists an uniqueness equilibrium point x^* for system (1), which is globally exponentially stable.*

From the assumption (H5), there exist positive constants $\beta_i(i = 1, 2, \ldots, n)$ such that

$$\beta_i a_i^l - \sum_{j=1}^{n} \beta_j \left(b_{ij}^u F_j + c_{ij}^u G_j + \frac{M}{\alpha} p_{ij}^u H_j \right) > 0, i = 1, \ldots, n.$$

After that, let a set-valued map $\Gamma(u) = (\Gamma_1(u), \Gamma_1(u), \ldots, \Gamma_n(u))^T$, and

$$
\begin{aligned}
\Gamma_i(u) = \beta_i \big[\sum_{j=1}^{n} K[b_{ij}(\frac{u_j}{\beta_j a_i})] f_j \left(\frac{u_j}{\beta_j a_i} \right) + \sum_{j=1}^{n} K[c_{ij}(\frac{u_j}{\beta_j a_i})] g_j \left(x_j \left(\frac{u_j}{\beta_j a_i} \right) \right) \\
+ \sum_{j=1}^{n} K[p_{ij}(\frac{u_j}{\beta_j a_i})] \int_{-\infty}^{t} k_{ij}(t-s) h_j \left(x_j \left(\frac{u_j}{\beta_j a_i} \right) \right) ds + J_i \big]
\end{aligned}
\tag{49}
$$

for $i = 1, 2, \ldots, n$, where $u = (u_1, \ldots, u_n)^T$.

Using Lemma 3, for any two vectors $u = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$ and $v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$, we have

$$|\Gamma_i(u) - \Gamma_i(v)| = \beta_i[\sum_{j=1}^{n} K[b_{ij}(\frac{u_j}{\beta_j a_i})]f_j\left(\frac{u_j}{\beta_j a_i}\right) - K[b_{ij}(\frac{u_j}{\beta_j a_i})]f_j\left(\frac{v_j}{\beta_j a_i}\right) + \sum_{j=1}^{n} K[c_{ij}(\frac{u_j}{\beta_j a_i})]g_j\left(x_j\left(\frac{u_j}{\beta_j a_i}\right)\right)$$

$$- \sum_{j=1}^{n} K[c_{ij}(\frac{u_j}{\beta_j a_i})]g_j\left(x_j\left(\frac{v_j}{\beta_j a_i}\right)\right) + \sum_{j=1}^{n}\frac{M}{\alpha}K[p_{ij}(\frac{u_j}{\beta_j a_i}]h_j\left(\frac{u_j}{\beta_j a_i}\right)ds - \sum_{j=1}^{n}\frac{M}{\alpha}K[p_{ij}(\frac{u_j}{\beta_j a_i})]h_j\left(\frac{v_j}{\beta_j a_i}\right)],$$

$$(50)$$

for $i = 1, 2, \ldots, n$, *then,*

$$\|\Gamma_i(u) - \Gamma_i(v)\| \le \sigma \|u - v\|, \tag{51}$$

where

$$\sigma = \max_{1 \le i \le n}\left\{ \frac{1}{\beta_i a_i^l}\sum_{j=1}^{n}\beta_j\left(b_{ij}^u F_j + c_{ij}^u G_j + \frac{M}{\alpha}p_{ij}^u H_j\right)\right\}, \tag{52}$$

and $0 < \sigma < 1$. *Thus, the map* $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ *is a contraction mapping on* $\mathbb{R}^n$. *It follows that, there is a unique fixed point* $u^* \in \mathbb{R}^n$ *such that* $u^* \in \Gamma(u^*)$, *i.e.,*

$$u_i^* \in \beta_i[\sum_{j=1}^{n} K[b_{ij}(\frac{u_j^*}{\beta_j a_i})]f_j\left(\frac{u_j^*}{\beta_j a_i}\right) + \sum_{j=1}^{n} K[c_{ij}(\frac{u_j^*}{\beta_j a_i})]g_j\left(\frac{u_j^*}{\beta_j a_i}\right) + \sum_{j=1}^{n}\frac{M}{\alpha}K[p_{ij}(\frac{u_j^*}{\beta_j a_i})]h_j\left(\frac{u_j^*}{\beta_j a_i}\right) + J_i]$$

for $i = 1, \ldots, n$. *Let* $x_j^* = \frac{u_j^*}{\beta_j a_i}$ *for* $i = 1, \ldots, n$, *then*

$$u_i^* \in -a_i x_i^* + \sum_{j=1}^{n} K[b_{ij}(x_j^*)]f_j\left(x_j^*\right) + \sum_{j=1}^{n} K[c_{ij}(x_j^*)]g_j\left(x_j^*\right) + \sum_{j=1}^{n}\frac{M}{\alpha}K[p_{ij}(x_j^*)]h_j\left(x_j^*\right) + J_i.$$

where $i = 1, \ldots, n$, *and u is unique, we obtain that system* (1) *has a unique equilibrium* x^*.

Thus, following the proof of Theorem 1, *we prove easily that equilibrium* x^* *of system* (1) *is globally exponentially stable.*

Finite-time periodic synchronization

In this section, we will examine the finite-time synchronization problem of delayed memristive neural networks.

For this purpose, we consider the delayed memristive neural network model (1) as the drive system, and a controlled response system is modeled by the following functional differential equation:

$$\dot{y}_i(t) = -a_i(t)y_i(t) + \sum_{j=1}^{n}[b_{ij}(t)f_j(y_j(t)) + c_{ij}(t)g_j(y_j(t - \tau_{ij}(t)))$$

$$+ p_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)h_j(y_j(s))ds + J_i(t)] + v_i(t) \tag{53}$$

where $y_i(t)$ is the controller to be designed.

Definition 7 The memristive neural network (1) is said to be completely synchronized onto (53) in finite time if by designing a suitable controller $v_i(t)$ to system (53), there exists a constant $t_1 > 0$ (t_1 depends on the initial value), satisfying

$$\lim_{t \to t_1}\|y_i(t) - x_i(t)\| = 0;$$

$$\|y_i(t) - x_i(t)\| \equiv 0, \text{ for } i = 1, 2, ..n, \ t > t_1$$

We take $e_i(t) = x_i(t) - y_i(t)$ the error term. Then, one can obtain the following result.

Theorem 5 *We consider that then system* (1) *exists at least one w-periodic solution. If there exists a positive definite matrix S satisfying*

$$Z_1 = \begin{pmatrix} -A + \frac{1}{2}S + \underline{B}F + \frac{M}{\alpha}\underline{P}H & \frac{1}{2}\underline{C}G \\ * & -\frac{1}{2}S \end{pmatrix} < 0.$$

$$Z_2 = \begin{pmatrix} -A + \frac{1}{2}S + \overline{B}F + \frac{M}{\alpha}\underline{P}H & \frac{1}{2}\overline{C}G \\ * & -\frac{1}{2}S \end{pmatrix} < 0,$$

where $F = diag(F_1, F_2, \ldots F_n)$, $G = diag(G_1, G_2, \ldots G_n)$, $H = diag(H_1, H_2, \ldots H_n)$, $\underline{B} = diag(\underline{b}_1, \underline{b}_2, \ldots \underline{b}_n)$, $\overline{B} = diag(\overline{b}_1, \overline{b}_2, \ldots \overline{b}_n)$, $\underline{C} = diag(\underline{c}_1, \underline{c}_2, \ldots \underline{c}_n)$, $\overline{C} = diag(\overline{c}_1, \overline{c}_2, \ldots \overline{c}_n)$, *then system* (53) *can synchronize onto system* (1) *in a finite time* $t_1 = \frac{\sqrt{2}}{k} V^{\frac{1}{2}}(0)$ *and to adapt to changes in the process that occur with time, we define the adaptive controller*

$$v_i(t) = -o_i(t)e_i(t), \tag{54}$$

and adaptive updated law, where

$$\dot{o}_i(t) = \varepsilon_i \left(e_i^2(t) - l_i \frac{e_i(t)}{o_i(t)} sign(e_i(t)) - k \frac{e_i(t)}{o_i(t)} sign(e_i(t)) - \frac{k}{\sqrt{\varepsilon_i}} sign(o_i(t)) \right) - k\sqrt{\lambda_{max}(P)} \left(\int_{t-\tau}^{0} e_i^2(s)ds \right)^{\frac{1}{2}},$$

and

$$V(0) = \tfrac{1}{2} e^T(0)e(0) + \tfrac{1}{2} \int_{-\tau}^{0} e^T(s)Se(s)ds + \tfrac{1}{2} \sum_{i=1}^{n} \tfrac{1}{\varepsilon_i} o_i^2(0).$$

$\varepsilon_i > 0$ *is a constant,* $k > 0$ *is a tunable constant,*

$\iota_i > 0, i = 1, 2, \ldots, n$, *are the control parameters to be determined and satisfies:*

$$\iota_i \geq |A|T_i + \sum_{j=1}^{n} F_j |\overline{b}_{ij} - \underline{b}_{ij}| T_i + \sum_{j=1}^{n} \sum |\overline{b}_{ij} - \underline{b}_{ij}| G_j \sum_{j=1}^{n} \frac{M}{\alpha} |\overline{P}_{ij} - \underline{p}_{ij}| H_j.$$

Proof Set $\Lambda = diag\{o_1(t), o_2(t), \ldots o_n(t)\}$. Consider the following Lyapunov functional:

$$V(t) = \tfrac{1}{2} e^T(t)e(t) + \tfrac{1}{2} \int_{t-\tau}^{0} e^T(s)Se(s)ds + \tfrac{1}{2} \sum_{i=1}^{n} \tfrac{1}{\varepsilon_i} o_i^2(t). \tag{55}$$

The master model (1) and the slave model (53) are state-dependent switching systems, hence, we can divide the error system into the following four cases at time t.

Case 1 If $|x_i(t)| > T_i$, $|y_i(t)| \geq T_i$, at time t, then the master system (1) and the slave system (53) decrease respectively, to the following models:

$$\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} \underline{b}_{ij}(t)f_j(x_j(t)) + \underline{c}_{ij}(t)g_j(x_j(t - \tau_{ij}(t))) + \underline{p}_{ij}(t) \int_{-\infty}^{t} k_{ij}(t-s)h_j(x_j(s))ds + J_i(t), \tag{56}$$

and

$$\dot{y}_i(t) = -a_i(t)y_i(t) + \sum_{j=1}^{n} \underline{b}_{ij}(t)f_j(y_j(t)) + \underline{c}_{ij}(t)g_j(y_j(t - \tau_{ij}(t))) + \underline{p}_{ij}(t) \int_{-\infty}^{t} k_{ij}(t-s)h_j(y_j(s))ds + u_i(t) + J_i(t). \tag{57}$$

Correspondingly, the error system can be written as

$$\dot{e}_i(t) = -a_i(t)e_i(t) + \sum_{j=1}^{n} \underline{b}_{ij}(t)f_j(e_j(t)) + \underline{c}_{ij}(t)g_j(e_j(t - \tau_{ij}(t))) + \underline{p}_{ij}(t) \int_{-\infty}^{t} k_{ij}(t-s)h_j(e_j(s))ds + u_i(t). \tag{58}$$

Let us denote $f_j(e_j(t)) = f_j(x_j(t)) - f_j(y_j(t))$; $g_j(e_j(t - \tau)) = g_j(x_j(t - \tau)) - g_j(y_j(t - \tau))$ and $h_j(e_j(t)) = h_j(x_j(t)) - h_j(y_j(t))$. Under assumption (H2), evaluating the derivation of $V(t)$ along the trajectory of error system gives

$$\dot{V}(t) = e^T(t)\left(-Ae(t) + \underline{B}f(e(t)) + \underline{C}ge(t-\tau) + \frac{M}{\alpha}\underline{P}h(e(t)) + u(t)\right) + \frac{1}{2}e^T(t)Se(t)$$

$$- \frac{1}{2}e^T(t-\tau)Se(t-\tau)$$

$$+ \frac{1}{2}\sum_{i=1}^{n} o_i(t)(e_i^2(t) - l_i\frac{e_i(t)}{o_i(t)}sign(e_i(t)) - k\frac{e_i(t)}{o_i(t)}sign(e_i(t))$$

$$- \frac{k}{\sqrt{\varepsilon_i}}sign(o_i(t)) - k\sqrt{\lambda_{max}(S)}\left(\int_{t-\tau}^{t} e_i^2(s)ds\right)^{\frac{1}{2}}\right)$$

$$\leq -e^T(t)Ae(t) + e^T\underline{B}Fe(t) + e^T\underline{C}Ge(t-\tau) + e^T\frac{M}{\alpha}\underline{P}He(t) - e^T(t)\Lambda e(t) \qquad (59)$$

$$+ \frac{1}{2}e^T(t)Se(t) - \frac{1}{2}e^T(t-\tau)Se(t-\tau)$$

$$+ e^T(t)\Lambda e(t) - \sum_{i=1}^{n}\iota_i|e_i(t)| - k\sum_{i=1}^{n}|e_i(t)| - k\sum_{i=1}^{n}\frac{1}{\sqrt{\varepsilon_i}}|o_i(t)| - k\left(\int_{t-\tau}^{t} e^T(s)Se(s)ds\right)^{\frac{1}{2}}$$

$$\leq \left(e^T(t), e^T(t-\tau)\right)Z_1\left(e^T(t), eT(t-\tau)\right)^T - \sum_{i=1}^{n}\iota_i|e_i(t) - k\sum_{i=1}^{n}|e_i(t)||$$

$$- k\sum_{i=1}^{n}\frac{1}{\sqrt{\varepsilon_i}}|o_i(t)| - k\left(\int_{t-\tau}^{t} e^T(s)Se(s)ds\right)^{\frac{1}{2}}.$$

Using previous results, we obtain

$$\dot{V}(t) \leq -k\left(\sum_{i=1}^{n}|e_i(t)|^2\right)^{\frac{1}{2}} - k\left(\int_{t-\tau}^{t} e^T(s)Se(s)ds\right)^{\frac{1}{2}} - k\sum_{i=1}^{n}\frac{1}{\sqrt{\varepsilon_i}}|o_i(t)|.$$

By Lemma 1, one has

$$\dot{V}(t) \leq -\sqrt{2}k\left[\frac{1}{2}e^T(t)e(t) + \frac{1}{2}\int_{t-\tau}^{t} e^T(s)Se(s)ds + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\varepsilon_i}o_i^2(t)\right]^{\frac{1}{2}} = -\sqrt{2}kV^{\frac{1}{2}}(t).$$

Case 2 Let $|x_i(t)| > T_i, |y_i(t)| > T_i$ at time t, then the master system (1) and the slave system (53) decrease to the following systems:

$$\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n}\overline{b}_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n}\overline{c}_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n}\overline{p}_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)h_j(x_j(s))ds + J_i(t),$$

$$(60)$$

and

$$\dot{y}_i(t) = -a_i(t)y_i(t) + \sum_{j=1}^{n}\overline{b}_{ij}(t)f_j(y_j(t)) + \overline{c}_{ij}(t)g_j(y_j(t-\tau_{ij}(t))) + \overline{p}_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)h_j(y_j(s))ds + J_i(t) + u_i(t).$$

$$(61)$$

Hence, we obtain the following error system

$$\dot{e}_i(t) = -a_i(t)e_i(t) + \sum_{j=1}^{n}\overline{b}_{ij}(t)f_j(e_j(t)) + \overline{c}_{ij}(t)g_j(e_j(t-\tau_{ij}(t))) + \overline{p}_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)h_j(e_j(s))ds + w_i(t).$$

$$(62)$$

Similarly, we write

$$\dot{V}(t) \leq \left(e^T(t), e^T(t-\tau)\right)Z_2\left(e^T(t), e(t-\tau)\right)^T - \sum_{i=1}^{n}l_i|e_i(t) - k\sum_{i=1}^{n}|e_i(t)||$$

$$- k\sum_{i=1}^{n}\frac{1}{\sqrt{\varepsilon_i}}|o_i(t)| - k\left(\int_{t-\tau}^{t} e^T(s)Se(s)ds\right)^{\frac{1}{2}}.$$

$$(63)$$

According to Lemmas 1, it follows

$$\dot{V}(t) \leq -\sqrt{2}k\left[\frac{1}{2}e^T(t)e(t) + \frac{1}{2}\int_{t-\tau}^t e^T(s)Se(s)ds + \frac{1}{2}\sum_{i=1}^n \frac{1}{\varepsilon_i}o_i^2(t)\right]^{\frac{1}{2}}$$

$$= -\sqrt{2}kV^{\frac{1}{2}}(t).$$

Case 3 If $|x_i(t)| > Ti, |y_i(t)| \leq T_i$ at time t, then the master system (1) and the slave system (53) reduce to (60) and (61). Correspondingly, the error system can be written as

$$\dot{e}_i(t) = -a_i(t)e_i(t) + \sum_{j=1}^n \overline{b}_{ij}(t)f_j(e_j(t)) + \sum_{j=1}^n \overline{c}_{ij}(t)g_j(e_j(t - \tau_{ij}(t)))$$

$$+ \sum_{j=1}^n \overline{p}_{ij}(t)\int_{-\infty}^t k_{ij}(t-s)h_j(e_j(s))ds + \left(\overline{a}_i(t) - \underline{a}_i(t)\right)y_i(t)$$

$$+ \sum_{j=1}^n \left(\underline{b}_{ij}(t) - \overline{b}_{ij}(t)\right)f_j(y_j(t)) + \sum_{j=1}^n \left(\underline{c}_{ij}(t) - \overline{c}_{ij}(t)\right)g_j(y_j(t - \tau_{ij}(t))) \quad (64)$$

$$+ \sum_{j=1}^n \left(\underline{p}_{ij}(t) - \overline{p}_{ij}(t)\right)\int_{-\infty}^t k_{ij}(t-s)h_j(y_j(s))ds + u_i(t).$$

evaluating the derivation of V(t) along the trajectory of (68), we have

$$\dot{V}_i(t) = \sum_{i=1}^n e_i(t)\left[-a_i(t)e_i(t) + \sum_{j=1}^n \overline{b}_{ij}(t)f_j(e_j(t)) + \sum_{j=1}^n \overline{c}_{ij}(t)g_j(e_j(t - \tau_{ij}(t)))\right.$$

$$+ \sum_{j=1}^n \overline{p}_{ij}(t)\int_{-\infty}^t k_{ij}(t-s)h_j(e_j(s))ds + |\underline{a}_i - \overline{a}_i||y_i(t)|$$

$$+ \sum_{j=1}^n |\underline{b}_{ij} - \overline{b}_{ij}||f_j(y_j(t))| + \sum_{j=1}^n |\underline{c}_{ij} - \overline{c}_{ij}||g_j(y_j(t - \tau_{ij}(t)))| + \sum_{j=1}^n |\underline{p}_{ij} - \overline{p}_{ij}|\int_{-\infty}^t k_{ij}(t-s)|h_j(y_j(s))|ds$$

$$+ u_{i(t)}] + \frac{1}{2}e^T(t)Se(t) - \frac{1}{2}e^T(t-\tau)Se(t-\tau) + \sum_{i=1}^n o_i^2(t) - l_i\frac{e_i(t)}{o_i(t)}sign(e_i(t)) - k\frac{e_i(t)}{o_i(t)}sign(e_i(t))$$

$$- \frac{k}{\sqrt{\varepsilon_i}}sign(o_i(t)) - k\sqrt{\lambda_{max}(S)}\left(\int_{t-\tau}^0 e_i^2(s)ds\right)^{\frac{1}{2}}\right]$$

$$\leq -e^T(t)Ae(t) + e^T\overline{B}Fe(t) + e^T\overline{C}Ge(t-\tau) + \frac{M}{\alpha}\overline{P}He(t) - e^T(t)\Lambda e(t) + \frac{1}{2}e^T(t)Se(t)$$

$$- \frac{1}{2}e^T(t-\tau)Se(t-\tau) + e^T(t)\Lambda e(t)$$

$$- \sum_{i=1}^n |e_i(t)| - k\sum_{i=1}^n |e_i(t)| - k\sum_{i=1}^n \frac{1}{\sqrt{\varepsilon_i}}|o_i(t)| - k\left(\int_{t-\tau}^t e^T(s)Se(s)ds\right)^{\frac{1}{2}} + \sum_{j=1}^n [|a_i|T_i$$

$$+ \sum_{j=1}^n F_j|\underline{b}_{ij} - \overline{b}_{ij}|T_i + \sum_{j=1}^n |\underline{c}_{ij} - \overline{c}_{ij}|G_j - l_i]|e_i(t)|$$

$$+ \sum_{j=1}^n \frac{M}{\alpha}H_j|\underline{p}_{ij} - \overline{p}_{ij}|$$

$$\leq \left(e^T(t), e^T(t-\tau)\right)Z_2\left(e^T(t), e(t-\tau)\right)^T - k\sum_{i=1}^n |e_i(t)| - k\sum_{i=1}^n \frac{1}{\sqrt{\varepsilon_i}}|o_i(t)|$$

$$- k\left(\int_{t-\tau}^t e^T(s)Se(s)ds\right)^{\frac{1}{2}} + \sum_{j=1}^n [|a_i|T_i + \sum_{j=1}^n F_j|\overline{b}_i - \underline{b}_{ij}|T_i$$

$$+ \sum_{j=1}^n |\overline{c}_{ij} - \underline{c}_{ij}|G_j + \sum_{j=1}^n \frac{M}{\alpha}H_j|\underline{p}_{ij} - \overline{p}_{ij}|T_i - l_i]|e_i(t)|.$$

In consideration of the definition l_i and Z_2, one has $\dot{V}_i(t) \leq -\sqrt{2}kV^{\frac{1}{2}}(t)$.

Case 4. Let $|x_i(t)| \leq T_i, |y_i(t)| > T_i$ at time t, then the master system (1) and the slave system (53) reduce to (60) and (62). Then, we obtain the following error system:

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$$\dot{e}_i(t) = -a_i(t)e_i(t) + \sum_{j=1}^{n}\overline{b}_{ij}(t)f_j(e_j(t)) + \sum_{j=1}^{n}\overline{c}_{ij}(t)g_j(e_j(t-\tau_{ij}(t)))$$

$$+ \sum_{j=1}^{n}\overline{p}_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)h_j(e_j(s))ds + a_i(t)x_i(t)$$

$$+ \sum_{j=1}^{n}\left(\overline{b}_i(t) - \underline{b}_{ij}(t)\right)f_j(x_j(t)) + \sum_{j=1}^{n}\left(\overline{c}_{ij}(t) - \underline{c}_{ij}(t)\right)g_j(x_j(t-\tau_{ij}(t))) \quad (65)$$

$$+ \sum_{j=1}^{n}\left(\overline{p}_{ij}(t) - \underline{p}_{ij}(t)\right)\int_{-\infty}^{t} k_{ij}(t-s)h_j\big(x_j(s)\big)ds + u_i(t).$$

Consider $|x_i(t)| \leq T_i$, we obtain

$$\dot{V}_i(t) \leq \left(e^T(t), e^T(t-\tau)\right)Z_2\left(e^T(t), e(t-\tau)\right)^T - k\sum_{i=1}^{n}|e_i(t)| - k\sum_{i=1}^{n}\frac{1}{\sqrt{\varepsilon_i}}|o_i(t)|$$

$$- k\left(\int_{t-\tau}^{t} e^T(s)Se(s)ds\right)^{\frac{1}{2}} + \sum_{j=1}^{n}[|a_i|T_i$$

$$+ \sum_{j=1}^{n}F_j|\underline{b}_i - \overline{b}_{ij}|T_i + \sum_{j=1}^{n}|\underline{c}_{ij} - \overline{c}_{ij}|G_j + \sum_{j=1}^{n}|\underline{c}_{ij} - \overline{c}_{ij}|\frac{M}{\alpha}H_j - l_i]|e_i(t)| \leq -\sqrt{2}kV^{\frac{1}{2}}(t)|$$

Or $V(t) = 0$ for $t \geq t_1$ with $t_1 = \frac{\sqrt{2}}{k}V^{\frac{1}{2}}(0)$, hence $e_i(t) = 0$ for $t \geq t_1, i = 1, 2, \ldots, n$. According to definition 5, the salve system (53) is finite-timely synchronized onto the master system (1) under the designed controller (54). This completes the proof. □

Numerical example
In this section, numerical example is given to show the effectiveness of our results. We consider the two-dimensional mermristor-based recurrent neural networks described by the following system:

$$\dot{x}_i(t) = -a_ix_i(t) + \sum_{j=1}^{2}(b_{ij}f_j(x_j(t)) + c_{ij}g_j(x_j(t-\tau_{ij}(t)))) + p_{ij}\int_{-\infty}^{t} k_{ij}(t-s)h_j(x_j(s))ds + J_i(t),$$

where $i = 1, 2, 2, a_1 = [3\ 4], \tau_{ij}(t) = \frac{1}{5}cos(t)$ and for all $x \in \mathbb{R}$

$$f_j(x) = g_j(x) = h_j(x) = \phi_j(x) = \frac{|x+1| - |x-1|}{2}$$

$$b_{11}(x_1(t)) = \begin{cases} -0.1, & |x_1(t)| < 1 \\ 1, & |x_1(t)| > 1 \end{cases}, b_{12}(x_1(t)) = \begin{cases} cos(t), & |x_1(t)| < 1 \\ -0.5, & |x_1(t)| > 1 \end{cases}$$

$$b_{21}(x_2(t)) = \begin{cases} -0.5 * sin(t), & |x_2(t)| < 1 \\ 1, & |x_2(t)| > 1 \end{cases}, b_{22}(x_2(t)) = \begin{cases} 0.1 * cos(t), & |x_2(t)| < 1 \\ -1, & |x_2(t)| > 1 \end{cases}$$

$$c_{11}(x_1(t)) = \begin{cases} 0.5, & |x_1(t)| < 1 \\ 1, & |x_1(t)| > 1 \end{cases}, c_{12}(x_1(t)) = \begin{cases} 2sin(t), & |x_1(t)| < 1 \\ -0.3 * cos(-t), & |x_1(t)| > 1 \end{cases}$$

$$c_{21}(x_2(t)) = \begin{cases} 0.2 * sin(t), & |x_2(t)| < 1 \\ sin(t), & |x_2(t)| > 1 \end{cases}, c_{22}(x_2(t)) = \begin{cases} 0.1 * cos(t), & |x_2(t)| < 1 \\ -1, & |x_2(t)| > 1 \end{cases}$$

$$p_{11}(x_1(t)) = \begin{cases} 0.5, & |x_1(t)| < 1 \\ -0.5, & |x_1(t)| > 1 \end{cases}, p_{12}(x_1(t))] = \begin{cases} 2sin(t), & |x_1(t)| < 1 \\ 1.5, & |x_1(t)| > 1 \end{cases}$$

$$p_{21}(x_2(t)) = \begin{cases} -0.2 * sin(t), & |x_2(t)| < 1 \\ 1, & |x_2(t)| > 1 \end{cases}, p_{22}(x_2(t))] = \begin{cases} 0.5 * cos(t), & |x_2(t)| < 1 \\ 1, & |x_2(t)| > 1 \end{cases}$$

$$M = 0.4, \alpha = 5.$$
$$J = [0.1 * sin(t);\ 0.2 * cos(t)];$$

We easily calculate
$I - S = \begin{pmatrix} 0.49 & 0.84 \\ 0.65 & 0.07 \end{pmatrix}$. Thus, the conditions required in Theorem 1 are satisfied. When I (t) is a periodic function, in the view of Theorem 1, this neural network has at least one periodic solution. It is clear that $I - S$ is
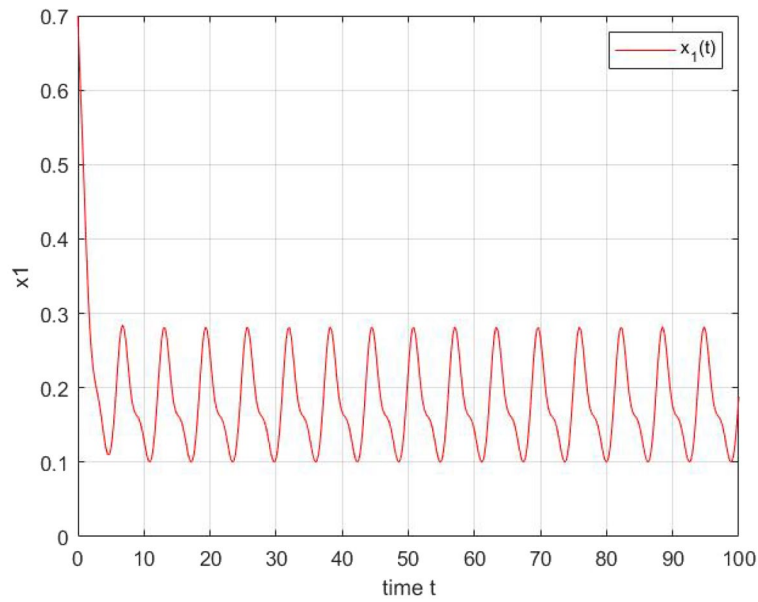
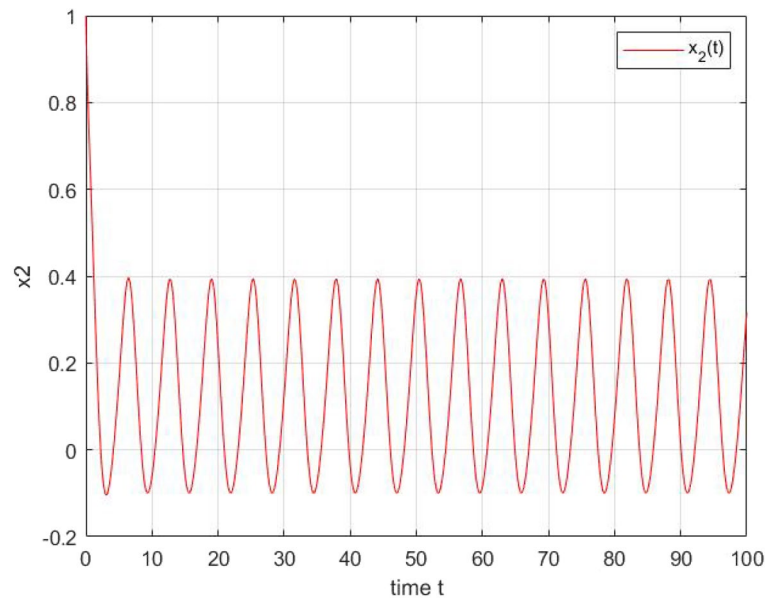**Figure 2.** The state trajectories of $x_1(t)$.

Figure 3. The state trajectories of $x_2(t)$.

an M-matrix. Then theorem 4 holds and the system has a unique equilibrium point x^*, which is globally exponentially stable.

After simulation of these two systems using matlab Toobox,we obtain the graphical illustration Figs. 2 and 3 shows the periodic dynamic behaviors of the output of the two neurons which are in accordance with theoretical results.

To prove the effectiveness of our result on finite-time synchronization we consider the master system the above simulated example and the following system is the slave.

Let consider the following response RNN:

$$\dot{y}_i(t) = -a_i x_i(t) + \sum_{j=1}^{2} b_{ij} f_j(y_j(t)) + \sum_{j=1}^{2} c_{ij} g_j(y_j(t-\tau_j)) + \sum_{j=1}^{2} p_{ij}(t) \int_{-\infty}^{t} k_{ij}(t-s) h_j(y_j(s)) ds + u_i(t).$$

We choose n=2 neurons and $u_i(t) = exp(-0.5 \times t)$ and the initial states $x = [0.5; 0.2]$; $y = [0.7; 0.3]$;

$$e_i(t) = x_i(t) - y_i(t), i = 1, 2.$$
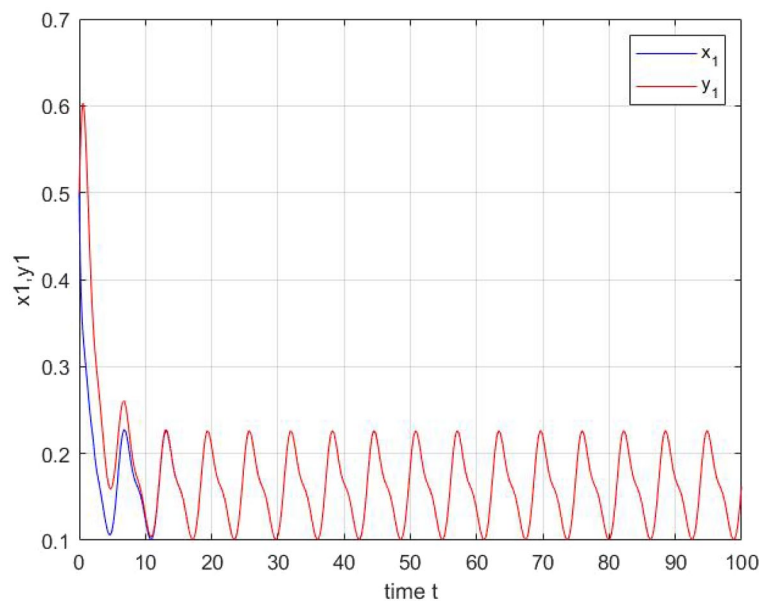
Figure 4. Time-domain behavior of the state variables x1(t) and y1(t).
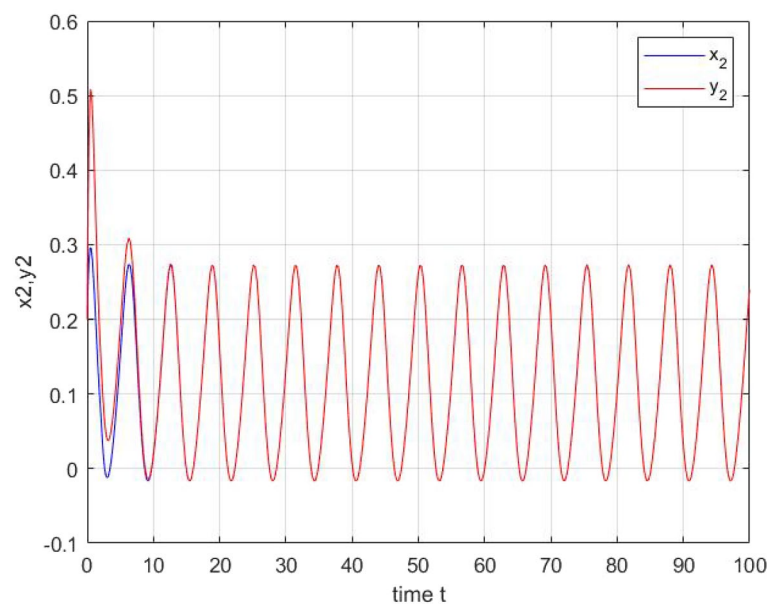
Figure 5. Time-domain behavior of the state variables x2(t) and y2(t).

We obtain in the following the simulation results: the two neurons tend to have the same trajectories in Figs. 4 and 5. Figs. 6 and 7 describes the time responses of finite-time synchronization errors and the trajectory turns to zero quickly as time goes and $t_1 = 4.4$ and $t_2 = 2.9$.

Conclusions

In this paper, we study a memristive recurrent neural networs by giving assumptions for the existence and uniqueness of periodic solution. In addition, we detemine sufficient conditions that ensure the global exponential stability of this solution. Further more, we garantee the finite-synchronization problem of delayed memristive by determining several assymptions.

Meanwhile, the theoretical proposed model can be tested in practical issues like brain computing interface, image processing, pattern recognition and intelligent control. In our ongoing future works, the proposed neural
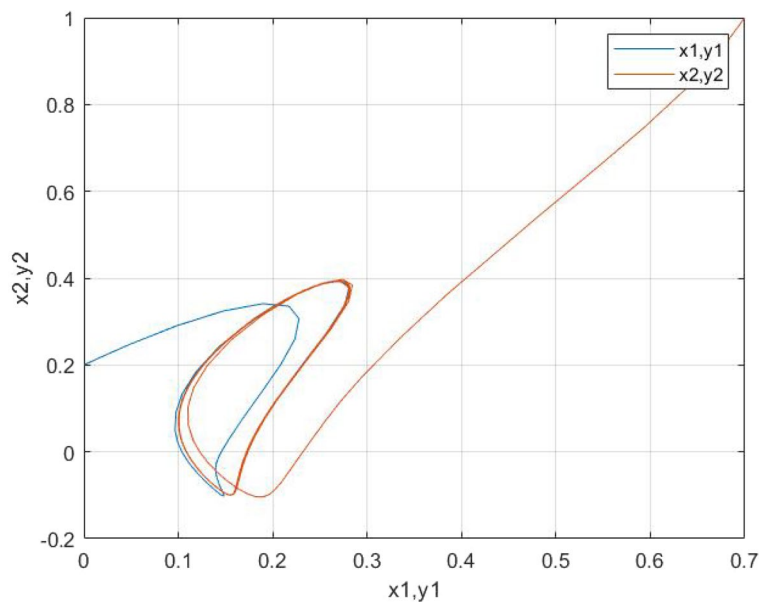
Figure 6. Phase plane behavior of the master system and the slave system.
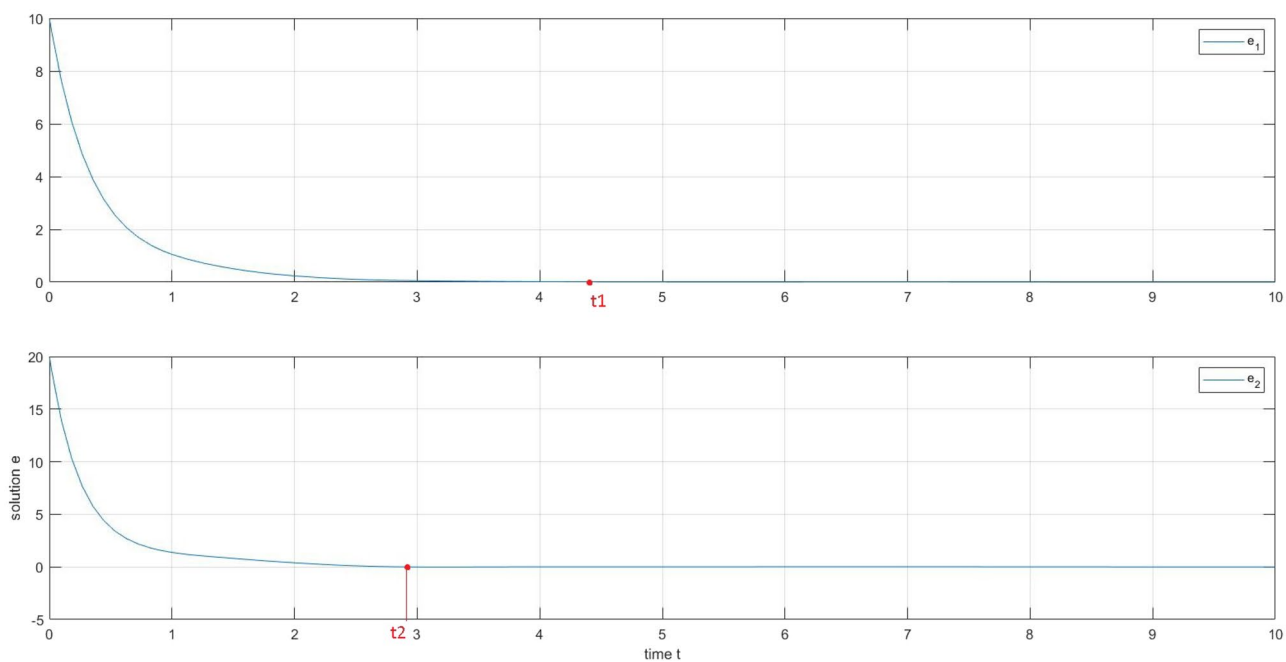
Figure 7. Finite-Time synchronization error.

network model will be adjusted to analyze the electroencephalography (EEG) data for implementing continuous vigilance estimation using EEG signals acquired by wearable dry electrodes in both simulated and real driving environments. Also, MNN synchronization and EEG signals can be combined to study the brain dynamics at rest following a perturbation.

Data availability
The data that support the findings of this study are available from author Hajer Brahmi but restrictions apply to the availability of these data, which were used under license for the current study, and so are not publicly available. Data are however available from the authors upon reasonable request and with permission of the author Hajer Brahmi.

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Author contributions

Competing interests

The authors declare no competing interests.

Additional information

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