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Modeling, dynamical analysis and numerical simulation of a new 3D cubic Lorenz-like system

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Little seems to be considered about the globally exponentially asymptotical stability of parabolic type equilibria and the existence of heteroclinic orbits in the Lorenz-like system with high-order nonlinear terms. To achieve this target, by adding the nonlinear terms yz and x^2y to the second equation of the system, this paper introduces the new 3D cubic Lorenz-like system: $\dot{x} = a(y - x)$, $\dot{y} = b_1y + b_2yz + b_3xz + b_4x^2y$, $\dot{z} = -cz + y^2$, which does not belong to the generalized Lorenz systems family. In addition to giving rise to generic and degenerate pitchfork bifurcation, Hopf bifurcation, hidden Lorenz-like attractors, singularly degenerate heteroclinic cycles with nearby chaotic attractors, etc., one still rigorously proves that not only the parabolic type equilibria $S_x = \{(x, x, \frac{x^2}{c}) | x \in \mathbb{R}, c \neq 0\}$ are globally exponentially asymptotically stable, but also there exists a pair of symmetrical heteroclinic orbits with respect to the z -axis, as most other Lorenz-like systems. This study may offer new insights into revealing some other novel dynamic characteristics of the Lorenz-like system family.

In 1963, the introduction of the Lorenz attractor^{1,2,3} motivated scholars to reveal the forming mechanism of it and other various strange attractors^{2,4–22}. Based on boundary problem and contraction map, Shilnikov et al.²³ developed an effective tool to study the existence of homoclinic and heteroclinic orbits. When detecting homoclinic and heteroclinic trajectories of Lorenz-like systems, by aid of Lyapunov function, Leonov²⁴ formulated another effective method, i.e., fishing principle, which also was applied to solve the Tricomi problem²⁵. Recently, Belykh et al.¹⁹ pioneered a new way and developed an elegant geometrical method of synthesizing a piecewise-smooth ODE system that can switch between several linear systems with known exact solutions that can display a resembling the celebrated Lorenz attractor whose structure and bifurcations can be described rigorously without any computer assistance. Moreover, Belykh et al.²⁰ performed a rigorous analysis of its homoclinic bifurcations that the emergence of sliding motions leads to novel bifurcation scenarios in which bifurcations of unstable homoclinic orbits of a saddle can yield stable limit cycles, which are in sharp contrast with their smooth analogs that can generate only unstable (saddle) dynamics. In addition, Gonchenko et al.^{21,22} studied geometrical and dynamical properties of the discrete Lorenz-like attractors and conjoined Lorenz twins in three-dimensional maps and flows. In contrast to self-excited Lorenz-like attractors, some hidden ones were coined in the Lorenz-like systems^{12–14}. Meanwhile, Zhang and Chen¹⁵, and Kuznetsov et al.²⁶ generalized the second part of the celebrated Hilbert's 16th problem²⁷ on the number and mutual disposition of attractors and repellers in the chaotic multidimensional dynamical systems, and, in particular, their dependence on the degree of polynomials in the model. From the point of view of boundedness and Lyapunov exponents, Liao et al.^{28,29} argued that the former attracts trajectories of the studied system with the way from outside to inside, and the latter pushes the trajectories with the way from inside to outside, which are two basic sufficient conditions that guarantee the studied continuous system to exhibit chaotic motions.

With the presence of powerful computational tools, scientists shifted to computer-assisted proof for the Lorenz attractor^{30–32}. Dated back to 1999, based on the Lorenz system and the method of chaotification, Chen and Ueta⁷ reported the finding of a new chaotic attractor in a new system, i.e., the Chen attractor. Following this thought, many researchers later proposed many other systems which exhibit various strange attractors, the

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Lü attractor⁸, Li attractor⁹, Rabinovich attractor¹⁰, Wang-Chen attractor¹¹, Sprott attractor¹⁷, and others^{2,18–22}. Among these chaotic systems, a large number of systems^{2,14,18,33–48} related to the Lorenz system, i.e., Lorenz-like/type systems, have been intensively studied by researchers, which in turn might account for revealing the nature of the Lorenz system itself. For example, the broken version of coexisting pseudo and true singularly degenerate heteroclinic cycles, or explosion version of normally hyperbolic stable foci could create most of the Lorenz-like attractors, i.e., two-, three-, four-wing/scroll self-excited or hidden chaotic/hyperchaotic attractors^{14,18,33–48}, shedding light on the forming mechanism of chaos.

In 2006, Li et al. formulated a method for proving the existence of heteroclinic orbits to the origin and two nontrivial equilibria of the Chen system, i.e., combining Lyapunov function, the definitions of both α -limit set and ω -limit set⁴⁹. Later on, other researchers^{14,37,38,42,43,47,48,50–60} applied it to other Lorenz-like systems one after another, which thus can be considered as a general dynamical property for the Lorenz system family. However, we find that this method is not applicable to the simple Lorenz-like system (1). Fortunately, performing a similar study as the method of chaotification and introducing the nonlinear terms yz and x^2y to the second equation of it, we introduce a new 3D cubic Lorenz-like system, i.e., the one (2), and present its following dynamical properties:

- (1) The parabolic type equilibria are globally exponentially asymptotically stable.
- (2) The existence of a pair of heteroclinic orbits to the origin and a pair of symmetrical equilibria.

Our study outcome not only uncovers the interesting dynamics of the cubic Lorenz-like system family, but also provides a reference on predicting the similar dynamical behaviors of other models, especially the higher dimensional ones.

Therefore, in the ongoing pursuit to determine which experimental conditions may require a more complicated model, the present work may offer characteristics of that 3D cubic Lorenz-like system which may be suitable for comparison with experimental data.

The rest of this paper is arranged as follows. Section "Preliminary" introduces some basic concepts. In Section "The new 3D cubic Lorenz-like system", one formulates a new 3D cubic Lorenz-like system and presents some basic dynamical properties of it, i.e., the Chen-like attractor and Lyapunov exponents, bifurcation analysis, singularly degenerate heteroclinic cycles or normally hyperbolic stable foci with nearby chaotic attractors. Section "Basic behaviors" studies the stability and bifurcation of equilibria by utilizing the center manifold theorem, Routh-Hurwitz criterion, the theory of pitchfork bifurcation, Hopf bifurcation and Lyapunov function. In Section "Existence of heteroclinic orbit", combining concepts of α -limit set, ω -limit set and the theory of Lyapunov function, one proves the existence of heteroclinic orbits. Conclusion remarks are drawn in Section "Conclusions".

Preliminary

Consider the differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\xi})$, where $\mathbf{x} \in \mathbb{R}^n$ and $\boldsymbol{\xi} \in \mathbb{R}^m$ are vectors representing phase variables and control parameters respectively. Assume that \mathbf{f} is of class C^∞ in $\mathbb{R}^n \times \mathbb{R}^m$. Suppose that system has an equilibrium point $\mathbf{x} = \mathbf{x}_0$ at $\boldsymbol{\xi} = \boldsymbol{\xi}_0$. If at least one eigenvalue of the Jacobian matrix associated with linearized vector field about x_0 is zero or has a zero real part, then x_0 is said to be non-hyperbolic or semi-hyperbolic.

In this paper, system (2) has a line of semi-hyperbolic equilibria $S_z = \{(0, 0, z) | z \in \mathbb{R}\}$, given by the z -axis. As the value of z varies, S_z are saddles, or foci or nodes normally hyperbolic to the z -axis.

In this paper, we define the set $S_x = \{(x, x, \frac{x^2}{c}) | x \in \mathbb{R}, c \neq 0\}$ to the parabolic type equilibria.

Referring to⁶¹, the generic pitchfork bifurcation is that the restriction of a system to the center manifold is locally topologically equivalent near the bifurcating equilibrium point to one of the following normal forms, $\dot{\xi} = m\xi \pm \xi^3$. As stated in^{18,47,48,51,52,62,63}, for system (2), the degenerate pitchfork bifurcation is defined to be the symmetric bifurcation occurring as the certain parameter crosses the zero value, i.e., $c = 0$, due to the line of equilibria existing for $c = 0$. The main difference between the generic and degenerate pitchfork bifurcation is that, for $c = 0$, the flow of the studied system restricted to the 1D center manifold coincides with the center manifold of the system at the origin, associated with the invariant z -axis, which is filled by equilibrium points if $c = 0$.

Let the set of points: S (either connected or disconnected) be equilibria of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\xi})$ and $D \subset \mathbb{R}^n$ to be a domain containing S . Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that $V(S) = 0$ and $V(x) > 0$ in $D \setminus S$, $\dot{V}(x) \leq 0$ in D . The derivative of $V(x)$ along the trajectories of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\xi})$, denoted by $\dot{V}(x)$, is given by $\dot{V}(x) = \sum_{i=1}^n \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i(x)$. Then, S is stable. If $\dot{V}(x) < 0$ in $D \setminus S$, then S is asymptotically stable. Moreover, if $D = \mathbb{R}^n$, then S is globally asymptotically stable. In addition, $\forall \varepsilon > 0$, $V_0 = V(t_0)$, if $V \leq V_0 e^{-2\varepsilon(t-t_0)} \rightarrow 0$, $t \rightarrow +\infty$, then S is globally exponentially asymptotically stable.

The new 3D cubic Lorenz-like system

Based on the Lorenz-like system¹:

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = -xz + cy, a, b, c \in \mathbb{R}, \\ \dot{z} = -bz + y^2, \end{cases} \quad (1)$$

one in this section proposes the following 3D autonomous chaotic system:

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = b_1y + b_2yz + b_3xz + b_4x^2y, \\ \dot{z} = -cz + y^2, \end{cases} \quad (2)$$

where $a, c, b_i \in \mathbb{R}, i = 1, 2, 3, 4$.

Remark 3.1 Referring to¹, the results on stability and Hopf bifurcation of $P_{\pm} = (\pm\sqrt{bc}, \pm\sqrt{bc}, c)$ of system (1) are erroneous. To this end, we firstly derive the right result by Routh-Hurwitz criterion and Projection Method. Secondly, that system undergoes Bautin bifurcation (generalized or degenerate Hopf bifurcation) at P_{\pm} when parameters a, b, c satisfy the golden proportion $a = \frac{1+\sqrt{5}}{2}c, b = \frac{-1+\sqrt{5}}{2}c$. Finally, a hidden Lorenz-like attractor coexisting with one saddle in the origin and two stable equilibria is coined based on bifurcation diagrams. The manuscript has been uploaded to the web site: “<https://github.com/ljbcThree/Kim>”, and the interested readers can download it.

Set $\mathbf{X} = (x, y, z)^T$, system (2) is rewritten as

$$\dot{\mathbf{X}} = \begin{pmatrix} -a & a & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & -c \end{pmatrix} \mathbf{X} + x \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b_3 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ b_2yz + b_4x^2y \\ y^2 \end{pmatrix}. \quad (3)$$

Apparently, system (2) is not topologically equivalent to the generalized Lorenz systems family², and it shows a Chen-like attractor with three Lyapunov exponents: $(\lambda_{LE_1}, \lambda_{LE_2}, \lambda_{LE_3}) = (0.271306, 0.000084, -1.203645)$ when $(a, b_1, b_2, b_3, b_4, c) = (3, 2.5, -7, -100, 8, 0.3)$ and $(x_0^1, y_0^1, z_0^1) = (0.1382, 0.1618, 0) \times 1e^{-7}$, as depicted in Fig. 1.

Furthermore, the chaotic dynamics are examined in the following two cases:

(1) $(a, b_2, b_3, b_4, c) = (3, -7, -50, 8, 0.3), b_1 \in [0, 4]$

In this case, based on Proposition 4.1 and 4.7 in Section “Basic behaviors”, the equilibria S_{\pm} exist and are stable for $b_1 \in (0, 1.2210)$. This coincides well with the bifurcation diagram in Fig. 2a. In particular, at $b_1 = 1$, trajectories of system (2) change from the stable S_+ to the stable S_- , which is a sign to chaos as the ones^{14,38}, especially the hidden one illustrated in Fig. 3. Particularly, when $1.180 \leq b_1 < 1.2210$, there exist chaotic attractors coexisting with stable S_{\pm} and the saddle S_0 . While $1.2210 < b_1 < 2.869$, system (2) experiences chaotic behaviors coexisting with unstable S_{\pm} and the saddle S_0 . But there are periodic three windows in the chaotic band for $1.576 < b_1 < 1.98$. When $2.869 < b_1 < 4$, there is a period-doubling bifurcation window, which is an important route to chaos and is also similar to its special case [1, Figure 4, p.1888].

(2) $(a, b_1, b_2, b_3, b_4) = (3, 2.5, -7, -100, 8), c \in [0, 2]$

At this time, from Proposition 4.1, 4.2 and 4.7 in Section “Basic behaviors”, the non-isolated or line of semi-hyperbolic equilibria S_z exist for $c = 0$, and the equilibria S_{\pm} also exist and are stable for $c \in (3.0234, 10.8865)$. For $c = 0$, singularly degenerate heteroclinic cycles and normally hyperbolic stable foci S_z with nearby chaotic attractors exist, as shown in Fig. 2b, which is in accordance with Figs. 4 and 5, despite a little bit on the parameters b_3 and b_4 . When $0 < c < 0.6$, system (2) undergoes chaotic behaviors coexisting with unstable S_{\pm} and the saddle S_0 . While $0.6 < c < 0.2$, there is a period-doubling bifurcation window, foreboding a coming chaos.

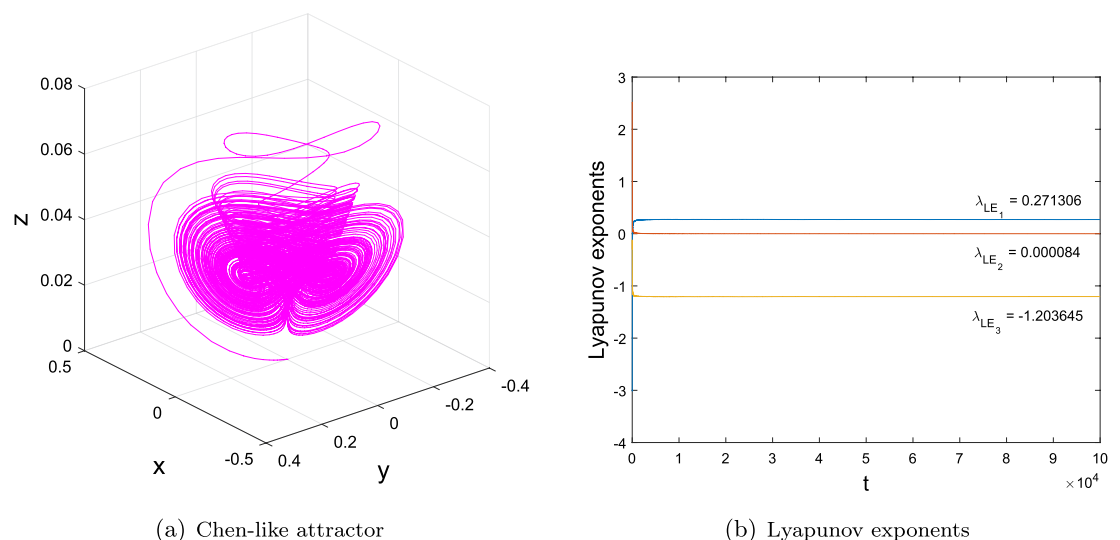


Figure 1. Phase portrait and Lyapunov exponents of system (2) with $(a, b_1, b_2, b_3, b_4, c) = (3, 2.5, -7, -100, 8, 0.3)$ and $(x_0^1, y_0^1, z_0^1) = (0.1382, 0.1618, 0) \times 1e^{-7}$.

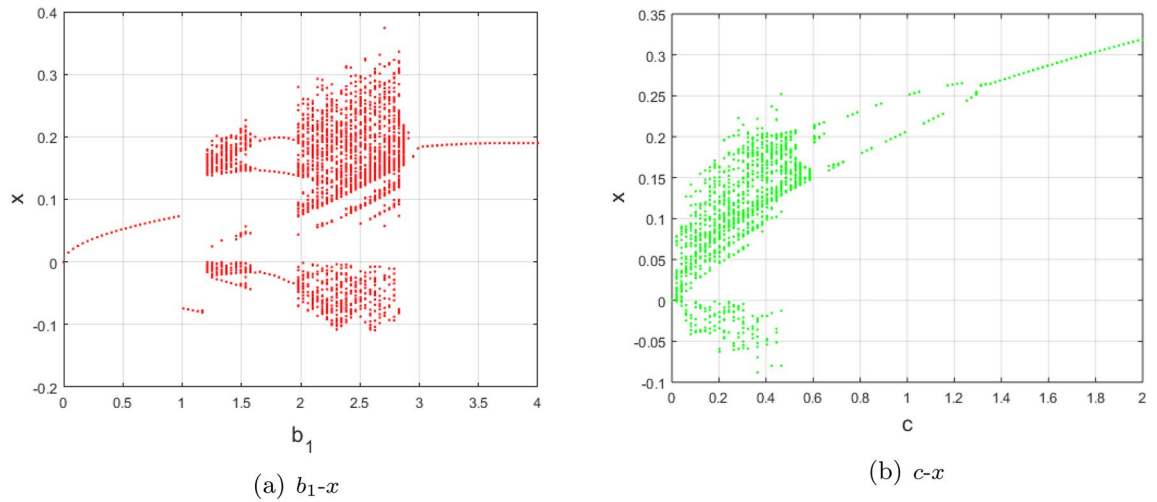


Figure 2. Bifurcation diagrams of system (2) with (a) $(a, b_2, b_3, b_4, c) = (3, -7, -50, 8, 0.3), b_1 \in [0, 4]$ and (b) $(a, b_1, b_2, b_3, b_4) = (3, 2.5, -7, -100, 8), c \in [0, 2]$, and initial value $(x_0^1, y_0^1, z_0^1) = (0.1382, 0.1618, 0) \times 1e^{-7}$.

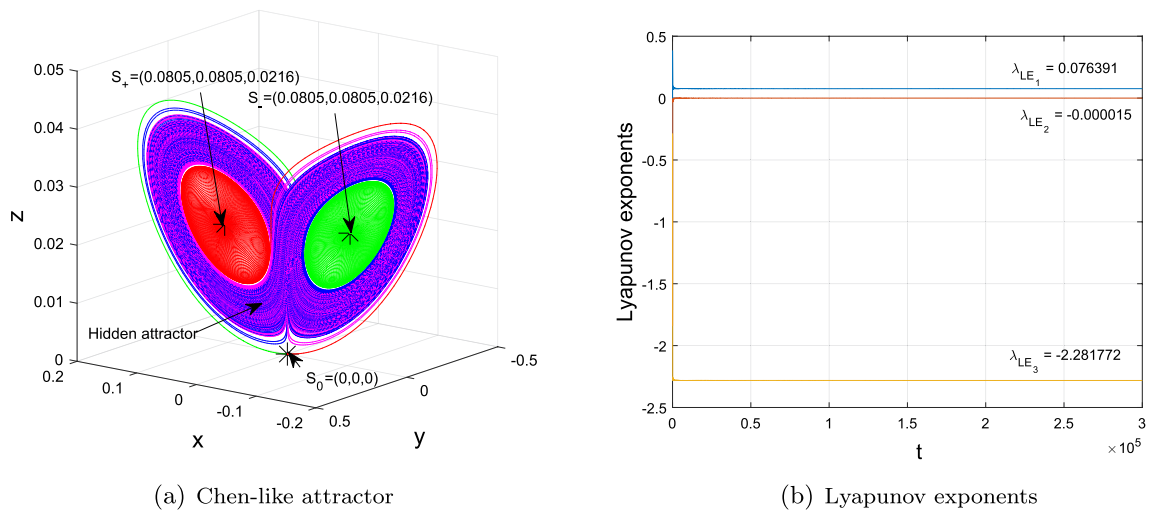


Figure 3. (a) The hidden attractor with $(a, b_1, b_2, b_3, b_4, c) = (3, 1.179, -7, -50, 8, 0.3)$ and initial conditions $(\pm 0.05, \pm 0.03, 0.03)$, (b) Lyapunov exponents. Outgoing separatrices of unstable zero equilibrium S_0 with initial conditions $(x_0^{1,2}, y_0^{1,2}, z_0^1) = (\pm 0.1382, \pm 0.1618, 0) \times 1e^{-7}$ tend to two symmetric stable equilibria $S_{\pm} = (\pm 0.0805, \pm 0.0805, 0.0216)$.

Remark 3.2 In contrast with other Lorenz-like systems^{12–14,51} and system (1) i.e., a special case of system (2), it is a difficult task to detect the hidden Lorenz-like attractors in system (2), which might contribute to the power of nonlinear terms and the number of parameters.

Further, based on the dynamics of S_z in Proposition 4.4 in Section "Existence of heteroclinic orbit" and through a detailed numerical study, we may state the following numerical result.

Numerical Result 3.1 If $c = 0$ and $a[b_1 + (b_2 + b_3)z_1] > 0$ for $z_1 \in \mathbb{R}$, then the 1D unstable manifolds $W^u(S_z^1)$ ($S_z^1 = (0, 0, z_1)$) of each normally hyperbolic saddle S_z^2 given in Proposition 4.4 tend to one of the normally hyperbolic stable nodes (resp. foci) $S_z^2 = (0, 0, z_2)$ as $t \rightarrow \infty$, where z_2 satisfies $b_1 + b_2 z_2 - a < 0$, $a[b_1 + (b_2 + b_3)z_2] < 0$ and $(\tau_1)^2 + 4\rho_1 = (a - b_1 - b_2 z_2)^2 + 4a[b_1 + (b_2 + b_3)z_2] \geq 0$ (resp. < 0), which together with the line of equilibria between S_z^1 and S_z^2 forms singularly degenerate heteroclinic cycles. With a small perturbation of $c > 0$, the broken version of singularly degenerate heteroclinic cycles, or explosions of normally hyperbolic stable nodes or foci creates chaotic attractors.

Take, for instance, $(a, b_1, b_2, b_3, b_4, c) = (3, 2.5, -7, -138, 9, 0)$ and $(x_0^{1,2}, y_0^{1,2}) = (\pm 1.382, \pm 1.618) \times 10^{-6}$, $z_0^{2,1,3,4,5,6,7,8,9} = -0.05, 0, 0.01, 0.01701, 0.01715, 0.01719, 0.0173, 0.02, 0.035$. At this time, the dynamics of S_z are included in Table 1 when the value of z varies.

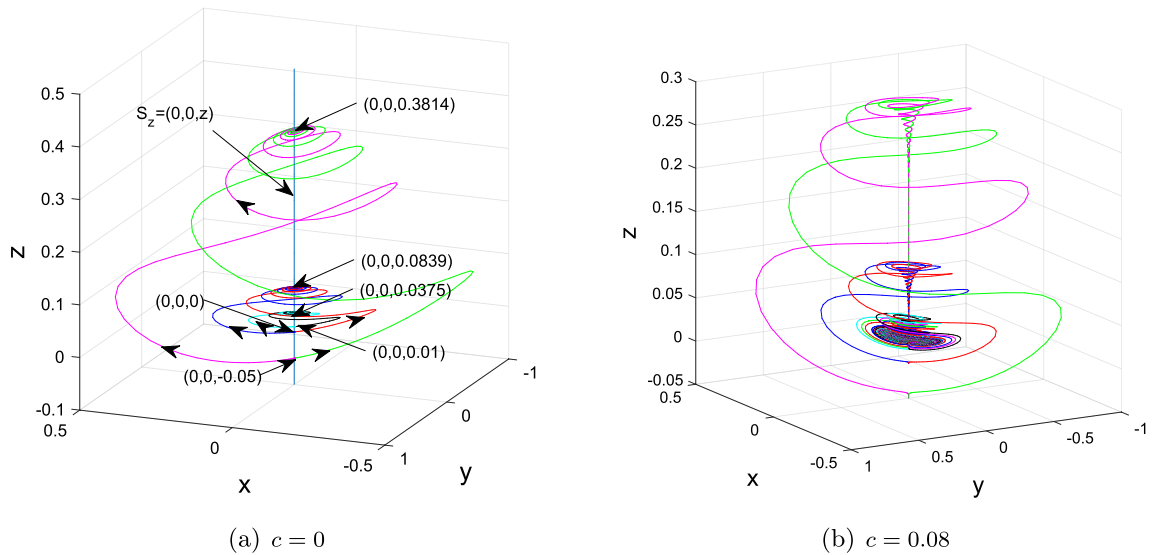


Figure 4. Chen-like attractor created through collapse of singularly degenerate heteroclinic cycles consisting of normally hyperbolic saddles $S_z^{1,2,3}$ and normally hyperbolic stable foci $(0, 0, 0.3814)$, $(0, 0, 0.0839)$ and $(0, 0, 0.0375)$ when $(a, b_1, b_2, b_3, b_4, c) = (3, 2.5, -7, -138, 9, 0)$ and $(x_0^{1,2}, y_0^{1,2}) = (\pm 1.382, \pm 1.618) \times 10^{-6}$, $z_0^{2,1,3} = -0.05, 0, 0.01$.

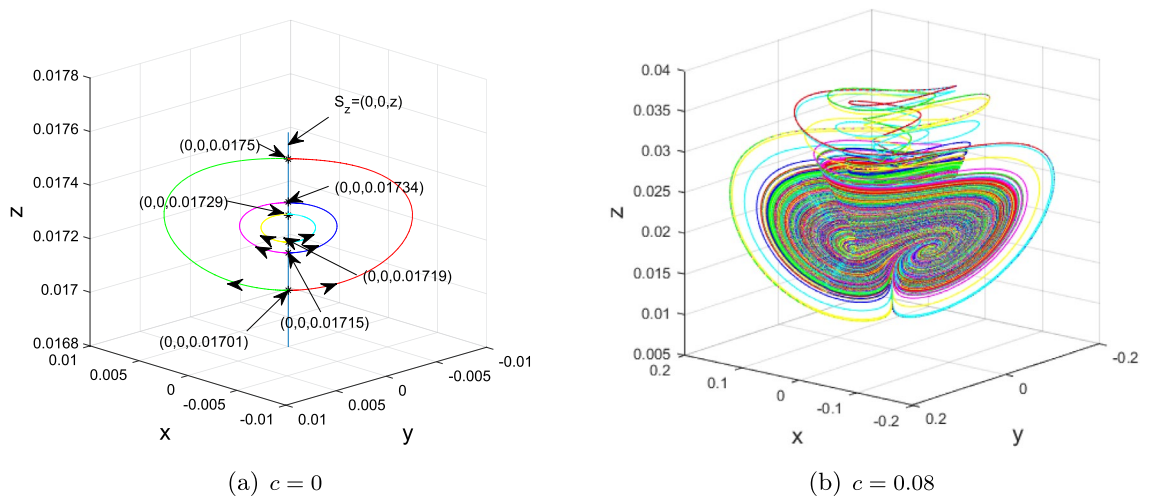


Figure 5. Chen-like attractor created through collapse of singularly degenerate heteroclinic cycles consisting of normally hyperbolic saddles $S_z^{4,5,6}$ and normally hyperbolic stable nodes $(0, 0, 0.0175)$, $(0, 0, 0.01734)$ and $(0, 0, 0.01729)$ when $(a, b_1, b_2, b_3, b_4, c) = (3, 2.5, -7, -138, 9, 0)$ and $(x_0^{1,2}, y_0^{1,2}) = (\pm 1.382, \pm 1.618) \times 10^{-6}$, $z_0^{4,5,6} = 0.01701, 0.01715, 0.01719$.

z	$[35.3499, \infty)$	$(0.0175, 35.3499)$	$(0.0172, 0.0175)$	0.0172	$(-\infty, 0.0172)$
S_z	Stable node	Stable focus	Stable node	a 1D W_{loc}^s and a 2D W_{loc}^c	Saddle

Table 1. The dynamics of S_z with $(a, b_1, b_2, b_3, b_4, c) = (3, 2.5, -7, -138, 9, 0)$ and $z \in \mathbb{R}$.

The 1D $W^u(S_z^{1,2,3})$ (resp. $W^u(S_z^{4,5,6})$) of normally hyperbolic saddles $S_z^{2,1,3} = (0, 0, -0.05)$, $(0, 0, 0)$, $(0, 0, 0.01)$ (resp. $S_z^{4,5,6} = (0, 0, 0.01701)$, $(0, 0, 0.01715)$, $(0, 0, 0.01719)$) tend upward the normally hyperbolic stable foci $(0, 0, 0.3814)$, $(0, 0, 0.0839)$ and $(0, 0, 0.0375)$ (resp. nodes $(0, 0, 0.0175)$, $(0, 0, 0.01734)$ and $(0, 0, 0.01729)$) in S_z as $t \rightarrow \infty$, forming singularly degenerate heteroclinic cycles, which further also collapse into Chen-like attractor depicted in Figs. 4, 5 and 7 when $c = 0.08$. Moreover, as shown in Fig. 6, explosions of normally hyperbolic stable foci $(0, 0, 0.0173)$, $(0, 0, 0.02)$ and $(0, 0, 0.035)$ also create Chen-like attractors. Figures 4, 5 and 6 only depict some of them. The existence of infinitely many S_z given in Proposition 4.3 suggests that there exists an infinite set of singularly degenerate heteroclinic cycles and normally hyperbolic stable nodes and foci.

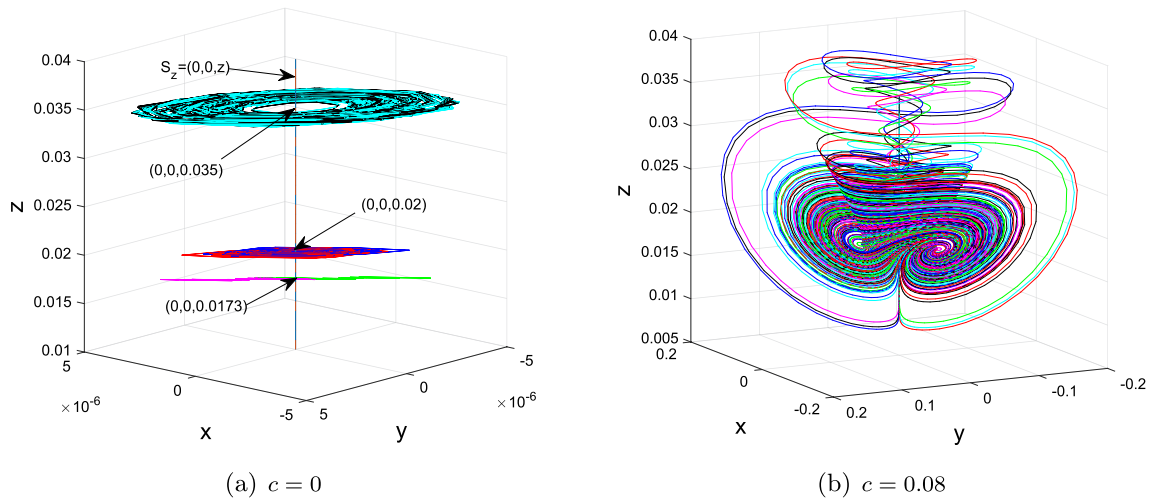


Figure 6. Chen-like attractor created through explosions of normally hyperbolic stable foci when $(a, b_1, b_2, b_3, b_4) = (3, 2.5, -7, -138, 9)$ $(x_0^{1,2}, y_0^{1,2}) = (\pm 1.382, \pm 1.618) \times 10^{-6}$, $z_0^{7,8,9} = 0.0173, 0.02, 0.035$.

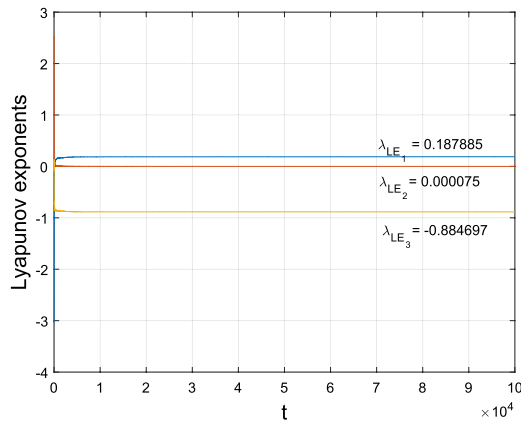


Figure 7. Lyapunov exponents of the bifurcated Chen-like attractor when $(a, b_1, b_2, b_3, b_4, c) = (3, 2.5, -7, -138, 9, 0.08)$.

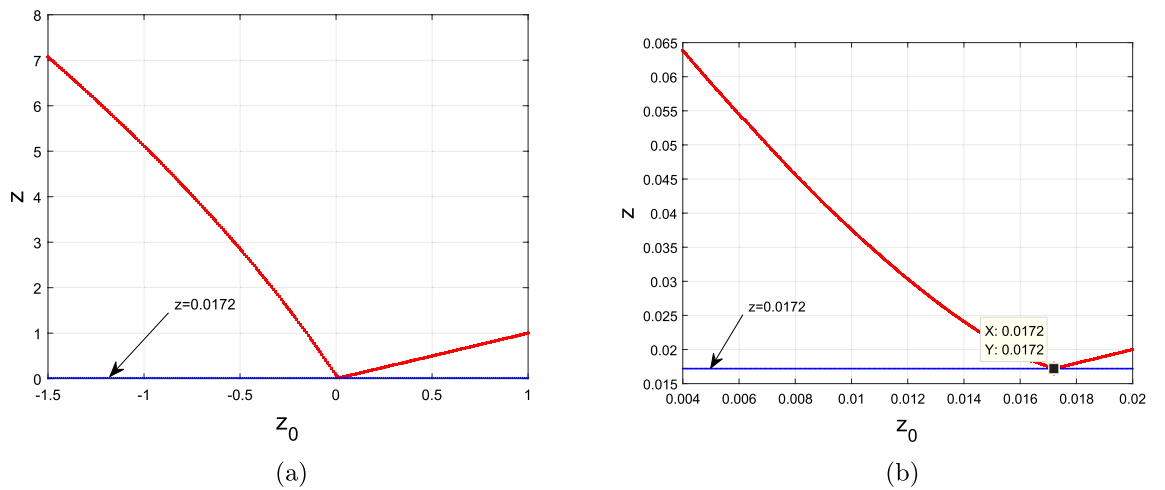


Figure 8. Bifurcation diagrams of the system (2) with $(a, b_1, b_2, b_3, b_4, c) = (3, 2.5, -7, -138, 9, 0)$ and $(x_0^1, y_0^1) = (1.382, 1.618) \times 10^{-6}$ and (a) $z_0 \in [-1.5, 1]$, (b) $z_0 \in [0.004, 0.02]$.

Remark 3.3 Set $(a, b_1, b_2, b_3, b_4, c) = (3, 2.5, -7, -138, 9, 0)$.

(1) If $z < 0.01701$, the trajectories of system (2) starting from the unstable manifold of $S_z = (0, 0, z)$ ultimately toward the normally hyperbolic stable foci S_z with $z > 0.0175$, forming singularly degenerate heteroclinic cycles, as depicted in Figs. 4a and 8.

(2) If $0.01701 \leq z < 0.0172$, the trajectories of system (2) starting from the unstable manifold of $S_z = (0, 0, z)$ ultimately toward the normally hyperbolic stable nodes S_z with $0.0172 < z \leq 0.0175$, forming singularly degenerate heteroclinic cycles, as depicted in Figs. 5a and 8.

(3) If $z > 0.0172$, all of $S_z = (0, 0, z)$ are normally hyperbolic stable nodes or node-foci, as shown in Figs. 6a and 8.

Basic behaviors

In this section, the stability and bifurcation of equilibria of system (2) are studied by aid of the center manifold theorem, Routh-Hurwitz criterion, the theory of pitchfork bifurcation and Hopf bifurcation, Lyapunov function and so on.

Firstly, from the algebraic structure of system (2), one easily presents the distribution of equilibrium points in the following proposition.

Proposition 4.1 (i) When $c = 0$, $S_z = \{(0, 0, z) | z \in \mathbb{R}\}$ is a line of semi-hyperbolic equilibria of system (2).

(ii) When $b_1 = 0, c \neq 0$ and $cb_4 + b_3 + b_2 = 0$, $S_x = \{(x, x, \frac{x^2}{c}) | x \in \mathbb{R}, c \neq 0\}$ is the parabolic type equilibria.

(iii) While $cb_1[cb_4 + b_3 + b_2] < 0$, system (2) has three equilibria: $S_0 = (0, 0, 0)$ and a pair of symmetrical equilibria

$$S_{\pm} = (\pm \sqrt{-\frac{cb_1}{cb_4 + b_3 + b_2}}, \pm \sqrt{-\frac{cb_1}{cb_4 + b_3 + b_2}}, -\frac{b_1}{cb_4 + b_3 + b_2}).$$

Secondly, for the convenience of determining the stability and bifurcation of equilibria, one has to calculate Jacobian matrix associated vector field of system (2):

$$J = \begin{pmatrix} -a & a & 0 \\ [3pt] b_3z + 2b_4xy & b_1 + b_2z + b_4x^2 & b_2y + b_3x \\ [3pt] 0 & 2y & -c \end{pmatrix}.$$

One can easily calculate the characteristic equations of points of S_z, S_0, S_x and S_{\pm} :

(1) The one of each of S_z is

$$\lambda[\lambda^2 - (b_1 + b_2z - a)\lambda - a(b_1 + (b_2 + b_3)z)] = 0$$

$$\text{with } \lambda_1 = 0, \lambda_{2,3} = \frac{(b_1 + b_2z - a) \pm \sqrt{(b_1 + b_2z - a)^2 + 4a(b_1 + (b_2 + b_3)z)}}{2}.$$

(2) The one of S_0 is

$$(\lambda + a)(\lambda - b_1)(\lambda + c) = 0$$

$$\text{with } \lambda_1 = -a, \lambda_2 = b_1 \text{ and } \lambda_3 = -c.$$

(3) For $b_1 = 0, c \neq 0$ and $cb_4 + b_3 + b_2 = 0$, the one of each of S_x is

$$\lambda[\lambda^2 + (a + c + \frac{b_3x^2}{c})\lambda + ac + x^2(b_3 + 2cb_4)] = 0$$

$$\text{with } \lambda_1 = 0, \lambda_{2,3} = \frac{-(a + c + \frac{b_3x^2}{c}) \pm \sqrt{(a + c + \frac{b_3x^2}{c})^2 - 4x^2(ac + b_3 + 2cb_4)}}{2}.$$

(4) The one of S_{\pm} is:

$$\lambda^3 + (a + c - \frac{b_1b_3}{cb_4 + b_3 + b_2})\lambda^2 + c(a + \frac{b_1(2b_2 + b_3 + 2ab_4)}{cb_4 + b_3 + b_2})\lambda + 2acb_1 = 0. \tag{4}$$

Proposition 4.2 (1) A generic pitchfork bifurcation happens at S_0 when b_1 crosses the null value and $cb_1(cb_4 + b_3 + b_2) < 0$. (2) If c passes through the null value and $cb_1(cb_4 + b_3 + b_2) < 0$, then system (2) undergoes a degenerate pitchfork bifurcation at S_z .

Proof (1) Assume $c(b_4 + b_3 + b_2) \neq 0, c \neq 0$ and $b_1 = 0$. The matrix associated with the vector field (2) linearized about S_0 has the eigenvalues: $\lambda_1 = -a, \lambda_2 = b_1 = 0$ and $\lambda_3 = -c$ with the corresponding eigenvectors

$$(\xi_1, \xi_2, \xi_3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Set $\bar{b}_1 = b_1 - 0$. System (2) becomes

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = \bar{b}_1 y + b_2 y z + b_3 x z + b_4 x^2 y, \\ \dot{z} = -c z + y^2. \end{cases} \tag{5}$$

Next, the following transformation

$$(x, y, z)^T = (\xi_2, \xi_1, \xi_3)(u, v, s)^T,$$

converts system (5) into

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix} \begin{pmatrix} u \\ v \\ s \end{pmatrix} + \begin{pmatrix} \bar{b}_1 u + b_2 u s + b_3(u + v)s + b_4(u + v)^2 u \\ -(\bar{b}_1 u + b_2 u s + b_3(u + v)s + b_4(u + v)^2 u) \\ -c s + u^2 \end{pmatrix}. \tag{6}$$

Based on the center manifold theorem, one can study the two-parameter family of first-order ordinary differential equations on the center manifold of S_0 :

$$W_{loc}^c(S_0) = \{(u, \bar{b}_1, v, s) \in \mathbb{R}^4 \mid v = V(u, \bar{b}_1), s = S(u, \bar{b}_1), \\ V(0, 0) = S(0, 0) = 0, DV(0, 0) = DS(0, 0) = 0\}$$

to determine the stability of S_0 near $\bar{b}_1 = 0$.

Therefore, one arrives at

$$\begin{aligned} V(u, \bar{b}_1) &= -\frac{b_2 + c b_4}{ac} u^3 + O(\| (u, \bar{b}_1) \|^3), \\ S(u, \bar{b}_1) &= \frac{1}{c} u^2 + O(\| (u, \bar{b}_1) \|^3), \end{aligned} \tag{7}$$

through substituting the expanded expressions of $V(u, \bar{b}_1)$ and $S(u, \bar{b}_1)$:

$$\begin{aligned} V(u, \bar{b}_1) &= \sum_{i+j=2}^{\infty} v_{ij} u^i \bar{b}_1^j, \\ S(u, \bar{b}_1) &= \sum_{i+j=2}^{\infty} s_{ij} u^i \bar{b}_1^j, \end{aligned}$$

into system (6).

Further, the restricted vector field of system (6) on its center manifold

$$\begin{cases} \dot{u} = \bar{b}_1 u + \frac{cb_4 + b_3 + b_2}{c} u^3 + O(\| (u, \bar{b}_1) \|^4) \triangleq U(u, \bar{b}_1), \\ \dot{\bar{b}}_1 = 0, \end{cases} \tag{8}$$

is obtained by substituting those expressions in (7) into system (6).

Since $U(0, 0) = 0, \frac{\partial U}{\partial u} \Big|_{u=0, \bar{b}_1=0} = 0$ and

$$\begin{cases} \frac{\partial U}{\partial \bar{b}_1} \Big|_{u=0, \bar{b}_1=0} = 0, \\ \frac{\partial^2 U}{\partial u^2} \Big|_{u=0, \bar{b}_1=0} = 0, \\ \frac{\partial^2 U}{\partial u \partial \bar{b}_1} \Big|_{u=0, \bar{b}_1=0} = 1, \\ \frac{\partial^3 U}{\partial u^3} \Big|_{u=0, \bar{b}_1=0} = \frac{cb_4 + b_3 + b_2}{c} \neq 0, \end{cases}$$

a generic pitchfork bifurcation happens at S_0 according to the pitchfork bifurcation theory^{23,64–66}.

(2) When the parameter c crosses the zero value, the family of this vector field crosses this degenerate situation transversally. More precisely speaking, for $cb_1[cb_4 + b_3 + b_2] < 0$, the line of equilibria S_z existing for $c = 0$ disappears and equilibria S_0 and S_{\pm} appear in system (2).

The proof is over. □

Proposition 4.3 Assume $a > 0, b_1 > 0$ and $c > 0$. The saddle S_0 has a 1D $W_{loc}^u(S_0)$ that is locally characterized by

$$W_{loc}^u(S_0) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid \begin{aligned} |x| &\ll 1, \\ y &= \frac{a+b_1}{a} x + O(x^2), \\ z &= \frac{(a+b_1)^2}{2b_1 a^2} x^2 + O(x^3), \end{aligned} \right\} \tag{9}$$

and a 2D $W_{loc}^s(S_0)$ containing the z -axis.

Proof The proof is similar to the ones in^{37,52,54,56}. One only sketches it. For $a > 0, b_1 > 0$ and $c > 0$, the eigenvalues of S_0 are $\lambda_1 = -a < 0, \lambda_2 = b_1 > 0$ and $\lambda_3 = -c < 0$. Thus S_0 has a 2D W_{loc}^s containing z -axis, and 1D $W_{loc}^u(S_0)$ whose appropriate expression is

$$W_{loc}^u(S_0) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{l} |x| \ll 1, \\ y = H(x), \\ z = K(x), \end{array} A_1 \begin{pmatrix} 1 \\ H'(0) \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ H'(0) \end{pmatrix}, \right. \\ \left. H(0) = K(0) = 0, \right\}$$

with $A_1 = \begin{pmatrix} -a & a \\ 0 & b_1 \end{pmatrix}$. Assuming that $y = H(x) = H_1x + H_2x^2 + O(x^3)$ and $z = K(x) = K_1x + K_2x^2 + O(x^3)$, and substituting them into system (2), one obtains the following first-order differential equation $K'(x)[a(H(x) - x)] = -cK(x) + (H(x))^2$, and

$$\begin{cases} b_1H_1 = H_1a(H_1 - 1), \\ b_1H_2 + b_2K_1H_1 + b_3K_1 = 2H_2a(H_1 - 1) + H_1H_2a, \\ cK_1 = K_1a(H_1 - 1), \\ H_1^2 = 2K_2a(H_1 - 1) + aK_1H_2. \end{cases} \tag{10}$$

In addition, the matrix equation $\begin{pmatrix} -a & a \\ 0 & b_1 \end{pmatrix} \begin{pmatrix} 1 \\ H'(0) \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ H'(0) \end{pmatrix}$ suggests $H_1 = H'(0) = \frac{a+b_1}{a}$. Hence, from Eq. (10), one has $K_1 = 0, H_2 = 0$ and $K_2 = \frac{(a+b_1)^2}{2b_1a^2}$. The proposition is thus proved. \square

Proposition 4.4 (1) When $c = 0, a, b_1, b_2, b_3, b_4, z \in \mathbb{R}$, the local dynamical behaviors of S_z are totally summarized in Table 2, where $\rho_1 = a[b_1 + (b_2 + b_3)z], \tau_1 = -(a - b_1 - b_2z)$, and $\sigma_1 = (\tau_1)^2 + 4\rho_1 = (a - b_1 - b_2z)^2 + 4a[b_1 + (b_2 + b_3)z]$. While $b_1 = 0, c \neq 0, cb_4 + b_3 + b_2 = 0$ and $z \in \mathbb{R}$, Table 3 lists the local dynamics of S_x , where $\rho_2 = ac + x^2(b_3 + 2cb_4), \tau_2 = -[a + c + \frac{b_3x^2}{c}]$, and $\sigma_2 = (\tau_2)^2 - 4\rho_2 = (a + c + \frac{b_3x^2}{c})^2 - 4[ac + x^2(b_3 + 2cb_4)]$.

(2) Moreover, for $c = 2a > 0, b_1 = b_3 = 0$ and $b_2 = -cb_4 < 0$, each point of S_x is globally exponentially asymptotically stable.

ρ_1	τ_1	σ_1	Property of S_z
< 0	< 0	< 0	Stable foci normally hyperbolic to S_z
	< 0	≥ 0	Stable nodes normally hyperbolic to S_z
	$= 0$	< 0	Fold-Hopf bifurcation may occur
	> 0	< 0	Unstable foci normally hyperbolic to S_z
	> 0	≥ 0	Unstable nodes normally hyperbolic to S_z
$= 0$	< 0		A 1D W_{loc}^s and a 2D W_{loc}^c
	$= 0$		A 3D W_{loc}^c
	> 0		A 2D W_{loc}^c and a 1D W_{loc}^u
> 0			Saddles normally hyperbolic to S_z

Table 2. The local dynamical behaviors of points of S_z .

ρ_2	τ_2	σ_2	Property of S_x
> 0	< 0	< 0	Stable foci normally hyperbolic to S_x
	< 0	≥ 0	Stable nodes normally hyperbolic to S_x
	$= 0$	< 0	Fold-Hopf bifurcation may occur
	> 0	< 0	Unstable foci normally hyperbolic to S_x
	> 0	≥ 0	Unstable nodes normally hyperbolic to S_x
$= 0$	< 0		A 1D W_{loc}^s and a 2D W_{loc}^c
	$= 0$		A 3D W_{loc}^c
	> 0		A 2D W_{loc}^c and a 1D W_{loc}^u
< 0			Saddles normally hyperbolic to S_x

Table 3. The dynamical behaviors of points of S_x .

Proof (1) Firstly, the local stability of points of S_z and S_x easily follows from the linear analysis and is omitted here.

(2) Secondly, we discuss the global stability of points of S_x , i.e., each point of S_x is globally exponentially asymptotically stable. For $c = 2a > 0, b_1 = b_3 = 0$ and $b_2 = -cb_4 < 0$, set the following Lyapunov function:

$$U = \frac{1}{2}[2a(y - x)^2 + b_4(-cz + x^2)^2]$$

with

$$\begin{aligned} \frac{dU}{dt} \Big|_{(2)} &= -2a^2(y - x)^2 - 2ab_4(-cz + x^2)^2 \\ &= -2a[a(y - x)^2 + b_4(-cz + x^2)^2] \\ &= -2aU - ab_4(-cz + x^2)^2 \\ &\leq -2aU, \end{aligned} \tag{11}$$

which yields

$$0 \leq U \leq U_0 e^{-2a(t-t_0)} \rightarrow 0, \quad t \rightarrow +\infty. \tag{12}$$

Namely, points of S_x are globally exponentially asymptotically stable. The proof is finished. \square

Remark 4.5 In contrast with other Lorenz-like systems^{14,37,38,42,43,47-60} and the special case of system (2), it follows from Proposition 4.4 that system (1) has no globally exponentially asymptotically stable parabolic type equilibria.

Proposition 4.6 (1) If $a < 0$ or $b_1 > 0$ or $c < 0$, then S_0 is unstable. If $a > 0, b_1 < 0$ and $c > 0$, then S_0 is stable.

(2) If $c = 0$, then the dynamics of S_0 are the same to the ones of S_z with $z = 0$ and listed in Table 2.

Proof The local stability of S_0 easily follows from the linear analysis and is omitted here. \square

Proposition 4.7 Denote $W = \{(a, b_1, b_2, b_3, b_4, c) \in \mathbb{R}^6 | a > 0, cb_1[cb_4 + b_3 + b_2] < 0\}$, $W_2 = W \setminus W_1$,

$$W_1 = \{(a, b_1, b_2, b_3, b_4, c) \in W : a + c - \frac{b_1 b_3}{cb_4 + b_3 + b_2} > 0, c(a + \frac{b_1(2b_2 + b_3 + 2ab_4)}{cb_4 + b_3 + b_2}) > 0, 2acb_1 > 0\}$$

$$W_1^1 = \{(a, b_1, b_2, b_3, b_4, c) \in W_1 : \Gamma < 0\},$$

$$W_1^2 = \{(a, b_1, b_2, b_3, b_4, c) \in W_1 : \Gamma = 0\}$$

$$W_1^3 = \{(a, b_1, b_2, b_3, b_4, c) \in W_1 : \Gamma > 0\},$$

where $\Gamma = c[a + c - \frac{b_1 b_3}{cb_4 + b_3 + b_2}](a + \frac{b_1(2b_2 + b_3 + 2ab_4)}{cb_4 + b_3 + b_2}) - 2ab_1$. S_{\pm} are unstable when $(a, b_1, b_2, b_3, b_4, c) \in W_1^1 \cup W_2$ whereas S_{\pm} are asymptotically stable when $(a, b_1, b_2, b_3, b_4, c) \in W_1^3$. When $(a, b_1, b_2, b_3, b_4, c) \in W_1^2$, system (2) undergoes Hopf bifurcation at S_{\pm} respectively.

Proof According to Routh-Hurwitz criterion and Eq. (4), S_{\pm} are unstable when $(a, b_1, b_2, b_3, b_4, c) \in W_1^1 \cup W_2$ whereas S_{\pm} are asymptotically stable when $(a, b_1, b_2, b_3, b_4, c) \in W_1^3$.

While $(a, b_1, b_2, b_3, b_4, c) \in W_1^2$, Eq. (4) has one negative real root $\lambda_1 = -(a + c - \frac{b_1^* b_3}{cb_4 + b_3 + b_2}) < 0$ and a pair of conjugate purely imaginary roots $\lambda_{2,3} = \pm \omega i$, where $\omega = \sqrt{c(a + \frac{b_1^*(2b_2 + b_3 + 2ab_4)}{cb_4 + b_3 + b_2})}$ and b_1^* satisfies $(a + c - \frac{b_1^* b_3}{cb_4 + b_3 + b_2})(a + \frac{b_1^*(2b_2 + b_3 + 2ab_4)}{cb_4 + b_3 + b_2}) - 2ab_1^* = 0$. Then calculating the derivatives on both sides of Eq. (4) with respect to the parameter b_1 and substituting λ_2 and b_1 with ωi and b_1^* into the derivative yield

$$\frac{dRe(\lambda_2)}{db_1} \Big|_{b_1=b_1^*} = \frac{2ac(cb_4 + b_3 + b_2) + b_3\omega^2 + c\lambda_1(2b_2 + b_3 + 2ab_4)}{2(\omega^2 + \lambda_1^2)(cb_4 + b_3 + b_2)} \neq 0.$$

Hence, the transversal condition holds. So, the Hopf bifurcations happen at S_{\pm} . Figure 9 illustrates that the numerical simulation agrees with the theoretical analysis. The proof is finished. \square

In the following Section "Conclusions", one studies the existence of heteroclinic orbits of system (2). For the convenience of discussion in the sequel, the following notations are introduced.

Denote by $\phi_t(q_0) = (x(t; x_0), y(t; y_0), z(t; z_0))$ a solution of system (2) with the initial condition $q_0 = (x_0, y_0, z_0)$. Let γ^+ (resp. γ^-) be the positive (resp. negative) branch of the unstable manifold $W^u(S_0)$ corresponding to $x_+ > 0$ (resp. $x_+ < 0$) for large negative t , i.e., $\gamma^{\pm} = \{\phi_t^{\pm}(q_0) | \phi_t^{\pm}(q_0) = (\pm x_+(t; x_0), \pm y_+(t; y_0), z_+(t; z_0)) \in W_{\pm}^u, t \in \mathbb{R}\}$.

Existence of heteroclinic orbit

Combining the Lyapunov function, concepts of both α -limit set and ω -limit set^{14,37,38,42,43,47-60}, one in this section rigorously proves the existence of a pair of heteroclinic orbits of system (2). Firstly, the unstable manifold of S_0 has been characterized in Proposition 4.3.

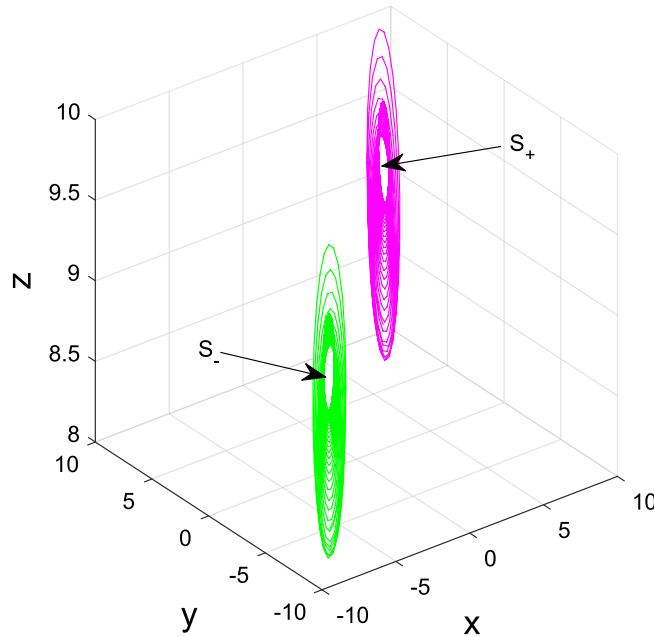


Figure 9. Hopf bifurcations at S_{\pm} for system (2) when $(a, b_1, b_2, b_3, b_4, c) = (3, 9.3563, -7, -1, 1, 7)$ and $(x_0^{3,4}, y_0^{3,4}, z_0^{10}) = (\pm 7.8, \pm 7.8, 8.2)$.

Secondly, set the Lyapunov function $V(\phi_t(q_0)) = V(x, y, z) = ac(c - 2a)b_4(y - x)^2 + a(c - 2a)b_4^2(-cz + x^2)^2 + \frac{1}{2}[-b_4(c - 2a)x^2 + cb_1]^2$. Proceeding as in ^{14,37,38,42,43,47,48,50–60}, one formulates the following result.

Proposition 5.1 When $c - 2a > 0, 0 < b_1 < a, b_4 > 0, b_2 = -cb_4$ and $b_3 = -(c - 2a)b_4$, one derives the following assertions.

- (i) If there exist t_1 and t_2 such that $t_1 < t_2$ and $V(\phi_{t_1}(q_0)) = V(\phi_{t_2}(q_0))$, then q_0 is one of equilibria of system (2).
- (ii) If $\phi_t(q_0) \rightarrow S_0$ as $t \rightarrow -\infty$, and $x(t; x_0) > 0$ for some $t \in \mathbb{R}$, then $V(S_0) > V(\phi_t(q_0))$ and $x(t; x_0) > 0$ for all $t \in \mathbb{R}$. Consequently, $q_0 \in \gamma^+$.

Proof i) For $c - 2a > 0, 0 < b_1 < a, b_4 > 0, b_2 = -cb_4$ and $b_3 = -(c - 2a)b_4$, one can compute the derivative of $V(\phi_t(q_0))$ along the solution $\phi_t(q_0)$:

$$\frac{dV(\phi_t(q_0))}{dt} \Big|_{(2)} = 2ac(c - 2a)b_4[(b_1 - a)(y - x)^2 - b_4(-cz + x^2)^2] \leq 0. \tag{13}$$

So, for all $t \in (t_1, t_2)$, the condition (i) implies

$$y(t; y_0) - x(t; x_0) \equiv -cz(t; z_0) + x^2(t; x_0) \equiv 0. \tag{14}$$

In virtue of system (2), $\dot{x}(t; x_0) = a(y - x) = 0$ suggests $x(t) = x_0$ and $\dot{y}(t; y_0) = 0, \forall t \in \mathbb{R}$. $-cz + x^2 = 0$ implies $-cz + y^2 = 0$ for all $t \in \mathbb{R}$, i.e. $\dot{z}(t; z_0) = 0$. In a word, q_0 is one of equilibria, i.e.

$$\dot{x}(t; x_0) \equiv \dot{y}(t; y_0) \equiv \dot{z}(t; z_0) \equiv 0. \tag{15}$$

ii) Firstly, one proves $V(S_0) > V(\phi_t(q_0)), \forall t \in \mathbb{R}$. Otherwise, suppose $0 < V(S_0) \leq V(\phi_{t_0}(q_0))$ for at least a $t_0 \in \mathbb{R}$. This also yields that q_0 is one of equilibria of system (2), which leads to $q_0 = S_0$ and $x(t; x_0) = 0, \forall t \in \mathbb{R}$ according to $\lim_{t \rightarrow -\infty} \phi_t(q_0) = S_0$. A contradiction occurs! Therefore, $V(S_0) > V(\phi_t(q_0))$, for all $t \in \mathbb{R}$.

Next, one proves $x(t; x_0) > 0, \forall t \in \mathbb{R}$. Assume by contrary that $x(t'; x_0) \leq 0$ for some $t' \in \mathbb{R}$. From the hypothesis of (ii), there exists a $\tau \in \mathbb{R}$ such that $x(\tau, x_0) = 0$. Since $V(S_0) > V(\phi_t(q_0)), \forall t \in \mathbb{R}$, one has $\phi_\tau(q_0) \in \{(x, y, z) : V(S_0) > V(x, y, z)\} \cap \{(0, y, z)\} = \{(x, y, z) : ac(c - 2a)b_4y^2 + a(c - 2a)b_4^2c^2z^2 + \frac{1}{2}c^2b_1^2 < \frac{1}{2}c^2b_1^2\} = \emptyset$, which is a contradiction. Hence, it follows that $x(t; x_0) > 0, \forall t \in \mathbb{R}$. The proof of the proposition is finished. \square

Based on Proposition 5.1, the existence of heteroclinic orbits to S_0 and S_{\pm} is derived in the following statement.

Proposition 5.2 Consider $c - 2a > 0, 0 < b_1 < a, b_4 > 0, b_2 = -cb_4$ and $b_3 = -(c - 2a)b_4$. One has the statements as follows.

- (a) Neither homoclinic orbits nor heteroclinic orbits to S_+ and S_- exist in system (2).
- (b) System (2) has a pair of symmetrical heteroclinic orbits to S_0 and S_{\pm} .

Proof a) Firstly, one proves that neither heteroclinic orbits nor homoclinic orbits to S_+ and S_- exist in system (2). Assume by contrary that $\phi_t(q_0)$ is a heteroclinic orbit or a homoclinic orbit to S_+ and S_- , i.e.

$$\lim_{t \rightarrow -\infty} \phi_t(q_0) = s_-, \lim_{t \rightarrow \infty} \phi_t(q_0) = s_+,$$

where s_- and s_+ satisfy either $\{s_-, s_+\} = \{S_-, S_+\}$ or $s_- = s_+ \in \{S_0, S_-, S_+\}$. From Eq. (13), one arrives at $V(s_-) \geq V(\phi_t(q_0)) \geq V(s_+)$ and thus $V(s_-) = V(s_+)$ in either case, which also yields $V(\phi_t(q_0)) = V(s_+)$. According to Proposition 5.1(i), q_0 is one of equilibria. Therefore, system (2) has neither heteroclinic orbits nor homoclinic orbits joining S_+ and S_- .

b) Next, one proves that there exists a single heteroclinic orbit joining S_0 and S_+ : $\gamma^+(t)$. As $t \rightarrow \infty$, it follows Proposition 4.2 that $\phi_t(q_0)$ approaches neither S_0 nor S_- . Hence, $\lim_{t \rightarrow \infty} \phi_t(q_0) = S_+$.

Finally, let us show that if system (2) has a second heteroclinic orbit to S_0 and S_+ , then it coincides with γ^+ .

Suppose $\phi_t^1(q_0)$ is a solution of system (2) that

$$\lim_{t \rightarrow -\infty} \phi_t^1(q_0) = s_-^1, \lim_{t \rightarrow \infty} \phi_t^1(q_0) = s_+^1,$$

where s_-^1 and s_+^1 satisfy $\{s_-^1, s_+^1\} = \{S_0, S_+\}$. Since V is decreasing, one has $V(s_-^1) \geq V(\phi_t^1(q_0)) \geq V(s_+^1)$ and $V(S_0) > V(S_+)$. Therefore, one obtains $s_-^1 = S_0$ and $s_+^1 = S_+$, i.e.,

$$\lim_{t \rightarrow -\infty} \phi_t^1(q_0) = S_0, \lim_{t \rightarrow \infty} \phi_t^1(q_0) = S_+.$$

It follows from Proposition 5.1(ii) that $\phi_t^1(q_0) = \gamma^+$.

Since system (2) is symmetrical with respect to the z -axis, γ^- is another unique heteroclinic orbit to S_0 and S_- . Figure 10 verifies the correctness of the theoretical result. Thus proof is completed. \square

Remark 5.3 It follows from Proposition 5.2 that the special case of system (2), i.e., the one (1) has not heteroclinic orbits to S_0 and S_{\pm} for $b_2 = b_4 = 0$. How to prove the existence of heteroclinic orbits of system (1) will be our future work, if they exist. Compared with other Lorenz-like systems^{14,37,38,42,43,47,48,50-60}, it is more difficult to construct the Lyapunov function for system (2) with more nonlinear terms.

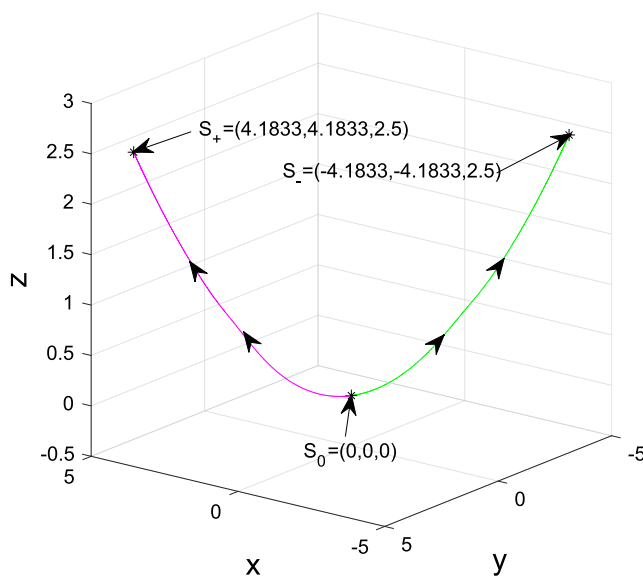


Figure 10. Heteroclinic orbits to S_0 and S_{\pm} of system (2) for $(a, b_1, b_2, b_3, b_4, c) = (3, 2.5, -7, -1, 1, 7)$ and initial values $(x_0'', y_0'', z_0'') = (\pm 1.382 \times 1e^{-6}, \pm \frac{5.5}{3} \times 1.382 \times 1e^{-6}, \frac{6.05}{9} \times 1.382 \times 1.382 \times 1e^{-12})$ on the unstable manifolds of S_0 , i.e., the $W_{loc}^u(S_0)$ characterized in Proposition 4.3.

Conclusions

This note reports a new 3D cubic Lorenz-like system, which contains the existing one as special cases and generates rich dynamics, such as generic and degenerate pitchfork bifurcation, Hopf bifurcation, infinitely many singularly degenerate heteroclinic cycles with nearby Chen-like attractors, etc. Using Lyapunov functions, we prove that the parabolic type equilibria are globally exponentially asymptotically stable, and there exists a pair of heteroclinic orbits to the origin and two symmetrical equilibria.

In future work, other important dynamics of that system, such as homoclinic orbit, invariant algebraic surface, positively invariant set, the forming mechanism of chaotic attractor and so on, require further analytical descriptions to complete its mathematical treatment. We also hope that the basic ideas and the self-contained approach presented in this paper can be applied to explore other similar chaotic/hyperchaotic systems, i.e.,

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = b_1y + b_2yz + b_3xz + b_4x^2y + p_1w, \\ \dot{z} = -cz + y^2, \\ \dot{w} = q_1w + q_2y, \end{cases} \quad (16)$$

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = b_2yz + b_3xz + b_4x^2y + p_1w, \\ \dot{z} = -cz + y^2, \\ \dot{w} = q_1w + q_3x, \end{cases} \quad (17)$$

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = b_1y + b_2yz + b_3xz + b_4x^2y + p_1w + p_2x, \\ \dot{z} = -cz + y^2 + xy, \\ \dot{w} = q_1w + q_2y + q_3x, \end{cases} \quad (18)$$

where $a \neq 0, c, b_i, p_1, p_2, q_1, q_2, q_3 \in \mathbb{R}, i = 1, 2, 3, 4$, etc. Preliminary studies show that there might exist chaotic/hyperchaotic attractors, generic and degenerate pitchfork bifurcation, Hopf bifurcation, singularly degenerate heteroclinic cycles, globally exponentially asymptotically stable parabolic type equilibria, a pair of heteroclinic orbits to the origin and two symmetrical equilibria in system (16–18). We also guess that the dynamics also exist in higher dimensional analogues.

Data availability

All data generated or analyzed during this study are included in this published article.

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Author contributions

H.W.: Conceptualization, software, writing—original draft, visualization, investigation. G.K.: Supervision, software, methodology, investigation, visualization. J.P.: Software, visualization, validation, writing—review & editing. Q.S.: Validation, writing—review & editing.

Competing interests

The authors declare no competing interests.

Additional information

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