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Strategies for single-shot discrimination of process matrices

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The topic of causality has recently gained traction quantum information research. This work examines the problem of single-shot discrimination between process matrices which are an universal method defining a causal structure. We provide an exact expression for the optimal probability of correct distinction. In addition, we present an alternative way to achieve this expression by using the convex cone structure theory. We also express the discrimination task as semidefinite programming. Due to that, we have created the SDP calculating the distance between process matrices and we quantify it in terms of the trace norm. As a valuable by-product, the program finds an optimal realization of the discrimination task. We also find two classes of process matrices which can be distinguished perfectly. Our main result, however, is a consideration of the discrimination task for process matrices corresponding to quantum combs. We study which strategy, adaptive or non-signalling, should be used during the discrimination task. We proved that no matter which strategy you choose, the probability of distinguishing two process matrices being a quantum comb is the same.

The topic of causality has remained a staple in quantum physics and quantum information theory for recent years. The idea of a causal influence in quantum physics is best illustrated by considering two characters, Alice and Bob, preparing experiments in two separate laboratories. Each of them receives a physical system and performs an operation on it. After that, they send their respective system out of the laboratory. In a causally ordered framework, there are three possibilities: Bob cannot signal to Alice, which means the choice of Bob's action cannot influence the statistics Alice records (denoted by $A \prec B$), Alice cannot signal to Bob ($B \prec A$), or neither party can influence the other ($A \parallel B$). A causally neutral formulation of quantum theory is described in terms of quantum combs¹.

One may wonder if Alice's and Bob's action can influence each other. It might seem impossible, except in a world with closed time-like curves (CTCs)². But the existence of CTCs implies some logical paradoxes, such as the grandfather paradox³. Possible solutions have been proposed in which quantum mechanics and CTCs can exist and such paradoxes are avoided, but modifying quantum theory into a nonlinear one⁴. A natural question arises: is it possible to keep the framework of linear quantum theory and still go beyond definite causal structures?

One such framework was proposed by Oreshkov, Costa and Brukner⁵. They introduced a new resource called a process matrix—a generalization of the notion of quantum state. This new approach has provided a consistent representation of correlations in causally and non-causally related experiments. Most interestingly, they have described a situation that two actions are neither causally ordered and one cannot say which action influences the second one. Thanks to that, the term of causally non-separable (CNS) structures started to correspond to superpositions of situations in which, roughly speaking, Alice can signal to Bob, and Bob can signal to Alice, jointly. A general overview of causal connection theory is described in Ref.⁶.

The indefinite causal structures could make a new aspect of quantum information processing. This more general model of computation can outperform causal quantum computers in specific tasks, such as learning or discriminating between two quantum channels^{7–9}. The problem of discriminating quantum operations is of the utmost importance in modern quantum information science. Imagine we have an unknown operation hidden in a black box. We only have information that it is one of two operations. The goal is to determine an optimal strategy for this process that achieves the highest possible probability of discrimination. For the case of a single-shot discrimination scenario, researchers have used different approaches, with the possibility of using entanglement in order to perform an optimal protocol. In Ref.¹⁰, Authors have shown that in the task of discrimination of unitary channels, the entanglement is not necessary, whereas for quantum measurements^{11–13}, we need to use entanglement. Considering multiple-shot discrimination scenarios, researchers have utilized parallel or adaptive approaches. In the parallel case, we establish that the discrimination between operations does not require pre-processing and post-processing. One example of such an approach is distinguishing unitary channels¹⁰, or von Neumann measurements¹⁴. The case when the black box can be used multiple times in an adaptive way was

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investigated by the authors of Refs.^{15,16}, who have proven that the use of adaptive strategy and a general notion of quantum combs can improve discrimination. It is worth noting that the problem of discrimination between process matrices is a direct analogy to the discrimination task of quantum states or channels. In general, the discrimination task of quantum objects helps study the geometry of sets of these objects by inducing the trace norm (for quantum states) or the diamond norm (for quantum channels and measurements). These norms constitute the basis for quantum capacity theory¹⁷, quantum error correction¹⁸ or quantum cryptography¹⁹.

In this work, we study the problem of discriminating process matrices in a single-shot scenario. We obtain that the probability of correct distinction process matrices is strictly related to the Holevo–Helstrom theorem for quantum channels. Additionally, we write this result as a semidefinite program (SDP) which is numerically efficient. The SDP program allows us to find an optimal discrimination strategy. We compare the effectiveness of the obtained strategy with the previously mentioned strategies. The problem gets more complex in the case when we consider the non-causally ordered framework. In this case, we consider the discrimination task between two process matrices having different causal orders.

This paper is organized as follows. In “[Mathematical preliminaries](#)” section we introduce necessary mathematical framework. “[Process matrices](#)” section is dedicated to the concept of process matrices. “[Discrimination task](#)” section presents the discrimination task between pairs of process matrices and calculate the exact probability of distinguishing them. Some examples of discrimination between different classes of process matrices are presented in “[Discrimination between different classes of process matrices](#)” section. In “[Free process matrices](#)” section, we consider the discrimination task between free process matrices, whereas in “[Process matrices representing quantum combs](#)” section we consider the discrimination task between process matrices being quantum combs. In “[Process matrices of the form \$W^{A \prec B}\$ and \$W^{B \prec A}\$](#) ” section, we show a particular class of process matrices having opposite causal structures which can be distinguished perfectly. Finally, “[SDP program for calculating the optimal probability of process matrices discrimination](#)” and “[Distance between process matrices](#)” sections are devoted to semidefinite programming, thanks to which, among other things, we obtain an optimal discrimination strategy. In “[Convex cone structure theory](#)” section, we analyze an alternative way to achieve this expression using the convex cone structure theory. Concluding remarks are presented in the final “[Conclusion and discussion](#)” section. In the Supplementary Materials, we provide technical details about the convex cone structure.

Mathematical preliminaries

Let us introduce the following notation. Consider two complex Euclidean spaces and denote them by \mathcal{X} , \mathcal{Y} . By $L(\mathcal{X}, \mathcal{Y})$ we denote the collection of all linear mappings of the form $A : \mathcal{X} \rightarrow \mathcal{Y}$. As a shorthand put $L(\mathcal{X}) := L(\mathcal{X}, \mathcal{X})$. By $\text{Herm}(\mathcal{X})$ we denote the set of Hermitian operators while the subset of $\text{Herm}(\mathcal{X})$ consisting of positive semidefinite operators will be denoted by $\text{Pos}(\mathcal{X})$. The set of quantum states, that is positive semidefinite operators ρ such that $\text{tr} \rho = 1$, will be denoted by $\Omega(\mathcal{X})$. An operator $U \in L(\mathcal{X})$ is unitary if it satisfies the equation $UU^\dagger = U^\dagger U = \mathbb{1}_{\mathcal{X}}$. The notation $U(\mathcal{X})$ will be used to denote the set of all unitary operators. We will also need a linear mapping of the form $\Phi : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ transforming $L(\mathcal{X})$ into $L(\mathcal{Y})$. The set of all linear mappings is denoted $M(\mathcal{X}, \mathcal{Y})$. There exists a bijection between set $M(\mathcal{X}, \mathcal{Y})$ and the set of operators $L(\mathcal{Y} \otimes \mathcal{X})$ known as the Choi²⁰ and Jamiołkowski²¹ isomorphism. For a given linear mapping $\Phi_M : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ corresponding Choi matrix $M \in L(\mathcal{Y} \otimes \mathcal{X})$ can be explicitly written as

$$M := \sum_{i,j=0}^{\dim(\mathcal{X})-1} \Phi_M(|i\rangle\langle j|) \otimes |i\rangle\langle j|. \quad (1)$$

We will denote linear mappings by Φ_M, Φ_N, Φ_R etc., whereas the corresponding Choi matrices as plain symbols: M, N, R etc. Let us consider a composition of mappings $\Phi_R = \Phi_N \circ \Phi_M$ where $\Phi_N : L(\mathcal{Z}) \rightarrow L(\mathcal{Y})$ and $\Phi_M : L(\mathcal{X}) \rightarrow L(\mathcal{Z})$ with Choi matrices $N \in L(\mathcal{Z} \otimes \mathcal{Y})$ and $M \in L(\mathcal{X} \otimes \mathcal{Z})$, respectively. Then, the Choi matrix of Φ_R is given by²²

$$R = \text{tr}_{\mathcal{Z}} \left[\left(\mathbb{1}_{\mathcal{Y}} \otimes M^{T_{\mathcal{Z}}} \right) (N \otimes \mathbb{1}_{\mathcal{X}}) \right], \quad (2)$$

where $M^{T_{\mathcal{Z}}}$ denotes the partial transposition of M on the subspace \mathcal{Z} . The above result can be expressed by introducing the notation of the link product of the operators N and M as

$$N * M := \text{tr}_{\mathcal{Z}} \left[\left(\mathbb{1}_{\mathcal{Y}} \otimes M^{T_{\mathcal{Z}}} \right) (N \otimes \mathbb{1}_{\mathcal{X}}) \right]. \quad (3)$$

Finally, we introduce a special subset of all mappings Φ , called quantum channels, which are completely positive and trace preserving (CPTP). In other words, the first condition reads

$$(\Phi \otimes \mathcal{I}_{\mathcal{Z}})(X) \in \text{Pos}(\mathcal{Y} \otimes \mathcal{Z}), \quad (4)$$

for all $X \in \text{Pos}(\mathcal{X} \otimes \mathcal{Z})$ and $\mathcal{I}_{\mathcal{Z}}$ is an identity channel acts on $L(\mathcal{Z})$ for any \mathcal{Z} , while the second condition reads

$$\text{tr}(\Phi(X)) = \text{tr}(X) \quad (5)$$

for all $X \in L(\mathcal{X})$.

In this work we will consider a special class of quantum channels called non-signaling channels (or causal channels)^{23,24}. We say that $\Phi_N : L(\mathcal{X}_I \otimes \mathcal{Y}_I) \rightarrow L(\mathcal{X}_O \otimes \mathcal{Y}_O)$ is a non-signaling channel if its Choi operator satisfies the following conditions

$$\begin{aligned} \text{tr}_{\mathcal{X}_O}(N) &= \frac{\mathbb{1}_{\mathcal{X}_I}}{\dim(\mathcal{X}_I)} \otimes \text{tr}_{\mathcal{X}_O \mathcal{X}_I}(N), \\ \text{tr}_{\mathcal{Y}_O}(N) &= \frac{\mathbb{1}_{\mathcal{Y}_I}}{\dim(\mathcal{Y}_I)} \otimes \text{tr}_{\mathcal{Y}_O \mathcal{Y}_I}(N). \end{aligned} \tag{6}$$

It can be shown²⁵ that each non-signaling channel is an affine combination of product channels. More precisely, any non-signaling channel $\Phi_N : L(\mathcal{X}_I \otimes \mathcal{Y}_I) \rightarrow L(\mathcal{X}_O \otimes \mathcal{Y}_O)$ can be written as

$$\Phi_N = \sum_i \lambda_i \Phi_{S_i} \otimes \Phi_{T_i}, \tag{7}$$

where $\Phi_{S_i} : L(\mathcal{X}_I) \rightarrow L(\mathcal{X}_O)$ and $\Phi_{T_i} : L(\mathcal{Y}_I) \rightarrow L(\mathcal{Y}_O)$ are quantum channels, $\lambda_i \in \mathbb{R}$ such that $\sum_i \lambda_i = 1$. For the rest of this paper, by $\mathbf{NS}(\mathcal{X}_I \otimes \mathcal{X}_O \otimes \mathcal{Y}_I \otimes \mathcal{Y}_O)$ we will denote the set of Choi matrices of non-signaling channels.

The most general quantum operations are represented by quantum instruments^{26,27}, that is, collections of completely positive (CP) maps $\{\Phi_{M_i}\}_i$ associated to all measurement outcomes, characterized by the property that $\sum_i \Phi_{M_i}$ is a quantum channel.

We will also consider the concept of quantum network and tester^{1,28}. We say that $\Phi_{R^{(N)}}$ is a deterministic quantum network (or quantum comb) if it is a concatenation of N quantum channels and $R^{(N)} \in L\left(\bigotimes_{i=0}^{2N-1} \mathcal{X}_i\right)$ fulfills the following conditions

$$\begin{aligned} R^{(N)} &\geq 0, \\ \text{tr}_{\mathcal{X}_{2k-1}}\left(R^{(k)}\right) &= \mathbb{1}_{\mathcal{X}_{2k-2}} \otimes R^{(k-1)}, \end{aligned} \tag{8}$$

where $R^{(k-1)} \in L\left(\bigotimes_{i=0}^{2k-3} \mathcal{X}_i\right)$ is the Choi matrix of the reduced quantum comb with concatenation of $k-1$ quantum channels, $k = 2, \dots, N$. We remind that a probabilistic quantum network $\Phi_{S^{(N)}}$ is equivalent to a concatenation of N completely positive trace non increasing linear maps. Then, the Choi operator $S^{(N)}$ of $\Phi_{S^{(N)}}$ satisfies $0 \leq S^{(N)} \leq R^{(N)}$, where $R^{(N)}$ is Choi matrix of a quantum comb. Finally, we recall the definition of a quantum tester. A quantum tester is a collection of probabilistic quantum networks $\{R_i^{(N)}\}_i$ whose sum is a quantum comb, that is $\sum_i R_i^{(N)} = R^{(N)}$, and additionally $\dim(\mathcal{X}_0) = \dim(\mathcal{X}_{2N-1}) = 1$.

We will also use the Moore–Penrose pseudo-inverse by abusing notation $X^{-1} \in L(\mathcal{Y}, \mathcal{X})$ for an operator $X \in L(\mathcal{X}, \mathcal{Y})$. Moreover, we introduce the vectorization operation of X defined by $|X\rangle\rangle = \sum_{i=0}^{\dim(\mathcal{X})-1} (X|i\rangle) \otimes |i\rangle$.

Process matrices

This section introduces the formal definition of the process matrix with its characterization and intuition. Next, we present some classes of process matrices considered in this paper.

Let us define the operator $\mathcal{X}Y$ as

$$\mathcal{X}Y = \frac{\mathbb{1}_{\mathcal{X}}}{\dim(\mathcal{X})} \otimes \text{tr}_{\mathcal{X}} Y, \tag{9}$$

for every $Y \in L(\mathcal{X} \otimes \mathcal{Z})$, where \mathcal{Z} is an arbitrary complex Euclidean space. We will also need the following projection operator

$$L_V(W) =_{\mathcal{A}_O} W +_{\mathcal{B}_O} W -_{\mathcal{A}_O \mathcal{B}_O} W -_{\mathcal{B}_I \mathcal{B}_O} W +_{\mathcal{A}_O \mathcal{B}_I \mathcal{B}_O} W -_{\mathcal{A}_I \mathcal{A}_O} W +_{\mathcal{A}_O \mathcal{A}_I \mathcal{B}_O} W, \tag{10}$$

where $W \in \text{Herm}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$.

Definition 1 We say that $W \in \text{Herm}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$ is a process matrix if it fulfills the following conditions

$$W \geq 0, \quad W = L_V(W), \quad \text{tr}(W) = \dim(\mathcal{A}_O) \cdot \dim(\mathcal{B}_O), \tag{11}$$

where the projection operator L_V is defined by Eq. (10).

The set of all process matrices will be denoted by \mathbf{W}^{PROC} . In the upcoming considerations, it will be more convenient to work with the equivalent characterization of process matrices which can be found in Ref.²⁹.

Definition 2 We say that $W \in \mathbf{W}^{\text{PROC}}$ is a process matrix if it fulfills the following conditions

$$\begin{aligned} W &\geq 0, \\ \mathcal{A}_I \mathcal{A}_O W &=_{\mathcal{A}_O \mathcal{A}_I \mathcal{B}_O} W, \\ \mathcal{B}_I \mathcal{B}_O W &=_{\mathcal{A}_O \mathcal{B}_I \mathcal{B}_O} W, \quad W =_{\mathcal{B}_O} W +_{\mathcal{A}_O} W -_{\mathcal{A}_O \mathcal{B}_O} W, \\ \text{tr}(W) &= \dim(\mathcal{A}_O) \cdot \dim(\mathcal{B}_O). \end{aligned} \tag{12}$$

The concept of process matrix can be best illustrated by considering two characters, Alice and Bob, performing experiments in two separate laboratories. Each party acts in a local laboratory, which can be identified by an input space \mathcal{A}_I and an output space \mathcal{A}_O for Alice, and analogously \mathcal{B}_I and \mathcal{B}_O for Bob. In general, a label i , denoting Alice’s measurement outcome, is associated with the CP map $\Phi_{M_i^A}$ obtained from the instrument $\{\Phi_{M_i^A}\}_i$. Analogously, the Bob’s measurement outcome j is associated with the map $\Phi_{M_j^B}$ from the instrument $\{\Phi_{M_j^B}\}_j$. Finally, the joint probability for a pair of outcomes i and j can be expressed as

$$p_{ij} = \text{tr} \left[W \left(M_i^A \otimes M_j^B \right) \right], \tag{13}$$

where $W \in \mathbf{W}^{\text{PROC}}$ is a process matrix that describes the causal structure outside of the laboratories. The valid process matrix is defined by the requirement that probabilities are well defined, that is, they must be non-negative and sum up to one. These requirements give us the conditions present in Definitions 1 and 2.

In the general case, the Alice’s and Bob’s strategies can be more complex than the product strategy $M_i^A \otimes M_j^B$ which defines the probability p_{ij} given by Eq. (13). If their action is somehow correlated, we can write the associated instrument in the following form $\{\Phi_{N_{ij}^{AB}}\}$. It was observed in Ref.²⁹ that this instrument describes a valid strategy, that is

$$\text{tr} \left(W \sum_{ij} N_{ij}^{AB} \right) = 1, \tag{14}$$

for all process matrix $W \in \mathbf{W}^{\text{PROC}}$ if and only if

$$\sum_{ij} N_{ij}^{AB} \in \mathbf{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O). \tag{15}$$

In this paper, we will consider different classes of process matrices. Initially, we define the subset of process matrices known as free objects in the resource theory of causal connection³⁰. Such process matrices will be defined as follows.

Definition 3 We say that $W^{A||B} \in \mathbf{W}^{\text{PROC}}$ is a free process matrix if it satisfies the following condition

$$W^{A||B} = \rho_{\mathcal{A}_I \mathcal{B}_I} \otimes \mathbb{1}_{\mathcal{A}_O \mathcal{B}_O}, \tag{16}$$

where $\rho_{\mathcal{A}_I \mathcal{B}_I} \in \Omega(\mathcal{A}_I \otimes \mathcal{B}_I)$ is an arbitrary quantum state and $\mathbb{1}_{\mathcal{A}_O \mathcal{B}_O} \in L(\mathcal{A}_O \otimes \mathcal{B}_O)$. The set of all process matrices of this form will be denoted by $\mathbf{W}^{A||B}$.

We often consider process matrices corresponding to quantum combs²². For example, a quantum comb $A \prec B$ (see in Fig. 1) shows that Alice’s and Bob’s operations are performed in causal order. This means that Bob cannot signal to Alice and the choice of Bob’s instrument cannot influence the statistics Alice records. Such process matrices are formally defined in the following way.

Definition 4 We say that $W^{A \prec B} \in \mathbf{W}^{\text{PROC}}$ is a process matrix representing a quantum comb $A \prec B$ if it satisfies the following conditions

$$\begin{aligned} W^{A \prec B} &= W'_{\mathcal{A}_I \mathcal{A}_O \mathcal{B}_I} \otimes \mathbb{1}_{\mathcal{B}_O}, \\ \text{tr}_{\mathcal{B}_I} W'_{\mathcal{A}_I \mathcal{A}_O \mathcal{B}_I} &= W''_{\mathcal{A}_I} \otimes \mathbb{1}_{\mathcal{A}_O}. \end{aligned} \tag{17}$$

The set of all process matrices of this form will be denoted by $\mathbf{W}^{A \prec B}$.

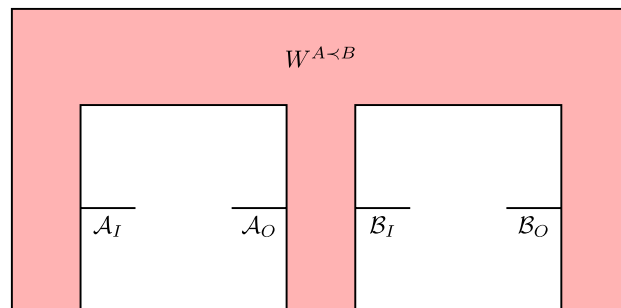


Figure 1. A schematic representation of a process matrix $W^{A \prec B}$ representing a quantum comb $A \prec B$.

One can easily observe that the set $\mathbf{W}^{A||B}$ is an intersection of the sets $\mathbf{W}^{A \prec B}$ and $\mathbf{W}^{B \prec A}$. Finally, the definition of the set $\mathbf{W}^{A \prec B}$, together with $\mathbf{W}^{B \prec A}$ allow us to provide their convex hull which is called as causally separable process matrices.

Definition 5 We say that $W^{SEP} \in \mathbf{W}^{PROC}$ is a causally separable process matrix if it is of the form

$$W^{SEP} = pW^{A \prec B} + (1-p)W^{B \prec A}, \quad (18)$$

where $W^{A \prec B} \in \mathbf{W}^{A \prec B}$, $W^{B \prec A} \in \mathbf{W}^{B \prec A}$ for some parameter $p \in [0, 1]$. The set of all causally separable process matrices will be denoted by \mathbf{W}^{SEP} .

There are however process matrices that do not correspond to a causally separable process and such process matrices are known as causally non-separable (CNS). The examples of such matrices were provided in Ref.^{5,31}. The set of all causally non-separable process matrices will be denoted by \mathbf{W}^{CNS} . In Fig. 2 we present a schematic plot of the sets of process matrices.

Discrimination task

This section presents the concept of discrimination between pairs of process matrices. It is worth emphasizing that the definition of a process matrix is a generalization of the concept of quantum states, channels, superchannels³² and even generalized supermaps^{33,34}. The task of discrimination between process matrices poses a natural extension of discrimination of quantum states³⁵, channels³⁶ or measurements¹³. The process matrices discrimination task can be described by the following scenario.

Let us consider two process matrices $W_0, W_1 \in \mathbf{W}^{PROC}$. The classical description of process matrices W_0, W_1 is assumed to be known to the participating parties. We know that one of the process matrices, W_0 or W_1 , describes the actual correlation between Alice's and Bob's laboratories, but we do not know which one. Our aim is to determine, with the highest possible probability, which process matrix describes this correlation. For this purpose, we construct a discrimination strategy S . In the general approach, such a strategy S is described by an instrument $S = \{S_0, S_1\}$. Due to the requirement given by Eq. (14), the instrument S must fulfill the condition $S_0 + S_1 \in \mathbf{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$. The result of composing a process matrix W with the discrimination strategy S results in a classical label which can take values zero or one. If the label zero occurs, we decide that the correlation is given by W_0 . Otherwise, we decide to choose W_1 . In this setting the maximum success probability $p_{succ}(W_0, W_1)$ of correct discrimination between two process matrices W_0 and W_1 can be expressed by

$$p_{succ}(W_0, W_1) = \frac{1}{2} \max_{S=\{S_0, S_1\}} [\text{tr}(W_0 S_0) + \text{tr}(W_1 S_1)]. \quad (19)$$

The following theorem provides the optimal probability of process matrices discrimination as a direct analogue of the Holevo–Helstrom theorem for quantum states and channels.

Theorem 1 Let $W_0, W_1 \in \mathbf{W}^{PROC}$ be two process matrices. For every choice of discrimination strategy $S = \{S_0, S_1\}$, it holds that

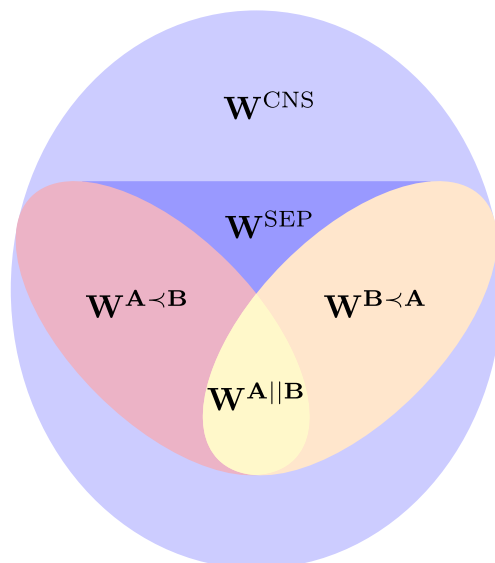


Figure 2. A schematic representation of the sets of process matrices \mathbf{W}^{PROC} .

$$\begin{aligned} & \frac{1}{2} \operatorname{tr}(S_0 W_0) + \frac{1}{2} \operatorname{tr}(S_1 W_1) \leq \\ & \frac{1}{2} + \frac{1}{4} \max \left\{ \|\sqrt{N}(W_0 - W_1)\sqrt{N}\|_1 : N \in \mathbf{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O) \right\}, \end{aligned} \tag{20}$$

where $\mathbf{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$ is the set of Choi matrices of non-signaling channels. Moreover, there exists a discrimination strategy S , which saturates the inequality Eq. (20).

Proof Let us define the sets

$$\mathbf{A} := \{(S_0, S_1) : S_0 + S_1 \in \mathbf{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O), S_0, S_1 \geq 0\}, \tag{21}$$

and

$$\begin{aligned} \mathbf{B} := \{(\sqrt{N}Q_0\sqrt{N}, \sqrt{N}Q_1\sqrt{N}) : N \in \mathbf{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O), \\ Q_0, Q_1 \geq 0, \\ Q_0 + Q_1 = \mathbb{1}_{\mathcal{A}_I \mathcal{A}_O \mathcal{B}_I \mathcal{B}_O}\}. \end{aligned} \tag{22}$$

We prove the equality between sets \mathbf{A} and \mathbf{B} . To show $\mathbf{B} \subseteq \mathbf{A}$, it suffices to observe that $\sqrt{N}Q_0\sqrt{N} + \sqrt{N}Q_1\sqrt{N} \in \mathbf{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$. To prove $\mathbf{A} \subseteq \mathbf{B}$ let us take $N := S_0 + S_1$. It implies that

$$\Pi_{\operatorname{im}(N)} = \sqrt{N}^{-1} N \sqrt{N}^{-1} = \sqrt{N}^{-1} S_0 \sqrt{N}^{-1} + \sqrt{N}^{-1} S_1 \sqrt{N}^{-1}. \tag{23}$$

Let us fix $\widetilde{Q}_0 := \sqrt{N}^{-1} S_0 \sqrt{N}^{-1}$ and $\widetilde{Q}_1 := \sqrt{N}^{-1} S_1 \sqrt{N}^{-1}$. Then, we have $\mathbb{1}_{\mathcal{A}_I \mathcal{A}_O \mathcal{B}_I \mathcal{B}_O} = \mathbb{1}_{\mathcal{A}_I \mathcal{A}_O \mathcal{B}_I \mathcal{B}_O} - \Pi_{\operatorname{im}(N)} + \Pi_{\operatorname{im}(N)} = \mathbb{1}_{\mathcal{A}_I \mathcal{A}_O \mathcal{B}_I \mathcal{B}_O} - \Pi_{\operatorname{im}(N)} + \widetilde{Q}_0 + \widetilde{Q}_1$. Finally, it suffices to take

$$Q_0 := \mathbb{1}_{\mathcal{A}_I \mathcal{A}_O \mathcal{B}_I \mathcal{B}_O} - \Pi_{\operatorname{im}(N)} + \widetilde{Q}_0, \tag{24}$$

and $Q_1 := \widetilde{Q}_1$. It implies that $\mathbf{A} = \mathbf{B}$. In conclusion, we obtain

$$\begin{aligned} & \frac{1}{2} \operatorname{tr}(S_0 W_0) + \frac{1}{2} \operatorname{tr}(S_1 W_1) \\ &= \frac{1}{2} \operatorname{tr}(\sqrt{N}Q_0\sqrt{N}W_0) + \frac{1}{2} \operatorname{tr}(\sqrt{N}Q_1\sqrt{N}W_1) \\ &= \frac{1}{2} \operatorname{tr}(Q_0\sqrt{N}W_0\sqrt{N}) + \frac{1}{2} \operatorname{tr}(Q_1\sqrt{N}W_1\sqrt{N}) \\ &\leq \frac{1}{2} + \frac{1}{4} \max \left\{ \|\sqrt{N}(W_0 - W_1)\sqrt{N}\|_1 : N \in \mathbf{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O) \right\}. \end{aligned} \tag{25}$$

Moreover, from Holevo–Helstrom theorem³⁶ there exists a projective binary measurement $Q = \{Q_0, Q_1\}$ such that the last inequality is saturated, which completes the proof. \square

Corollary 1 The maximum probability $p_{\text{succ}}(W_0, W_1)$ of correct discrimination between two process matrices W_0 and W_1 is given by

$$\begin{aligned} p_{\text{succ}}(W_0, W_1) = \\ \frac{1}{2} + \frac{1}{4} \max \left\{ \|\sqrt{N}(W_0 - W_1)\sqrt{N}\|_1 : N \in \mathbf{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O) \right\}. \end{aligned} \tag{26}$$

As a valuable by-product of Theorem 1, we receive a realization of process matrices discrimination scheme. The schematic representation of this setup is presented in Fig. 3. To distinguish the process matrices W_0 and W_1 , Alice and Bob prepare the strategy $S = \{S_0, S_1\}$ such that $S_0 + S_1 \in \mathbf{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$. To implement it, let us introduce complex Euclidean spaces $\mathcal{X}_1, \dots, \mathcal{X}_4$ such that $\dim(\otimes_{i=1}^4 \mathcal{X}_i) = \dim(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$. Alice and Bob prepare the quantum channel $\Phi_K : L(\mathcal{A}_I \otimes \mathcal{B}_I) \rightarrow L(\mathcal{A}_O \otimes \mathcal{B}_O \otimes \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3 \otimes \mathcal{X}_4)$ with the Choi matrix K given by

$$K = \left(\mathbb{1}_{\mathcal{X}_{1,2,3,4}} \otimes \sqrt{N} \right) (|\mathbb{1}\rangle) \langle \langle \mathbb{1} | \left(\mathbb{1}_{\mathcal{X}_{1,2,3,4}} \otimes \sqrt{N} \right), \tag{27}$$

where $|\mathbb{1}\rangle \in L(\mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3 \otimes \mathcal{X}_4 \otimes \mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$ and $N \in \mathbf{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$ maximizes the trace norm $\|\sqrt{N}(W_0 - W_1)\sqrt{N}\|_1$. It is worth noting that the quantum channel Φ_K is correctly defined due to the fact that $\operatorname{tr}_{\mathcal{X}_{1,2,3,4}} K \in \mathbf{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$. Afterwards, they perform the binary measurement $Q = \{Q_0, Q_1\}$, where the effect $Q_0 \in L(\mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3 \otimes \mathcal{X}_4)$ is defined by Eq. (24). Next, they decide which process matrix was used during the calculation assuming W_0 if the measurement label is 0. Otherwise, they assume W_1 .

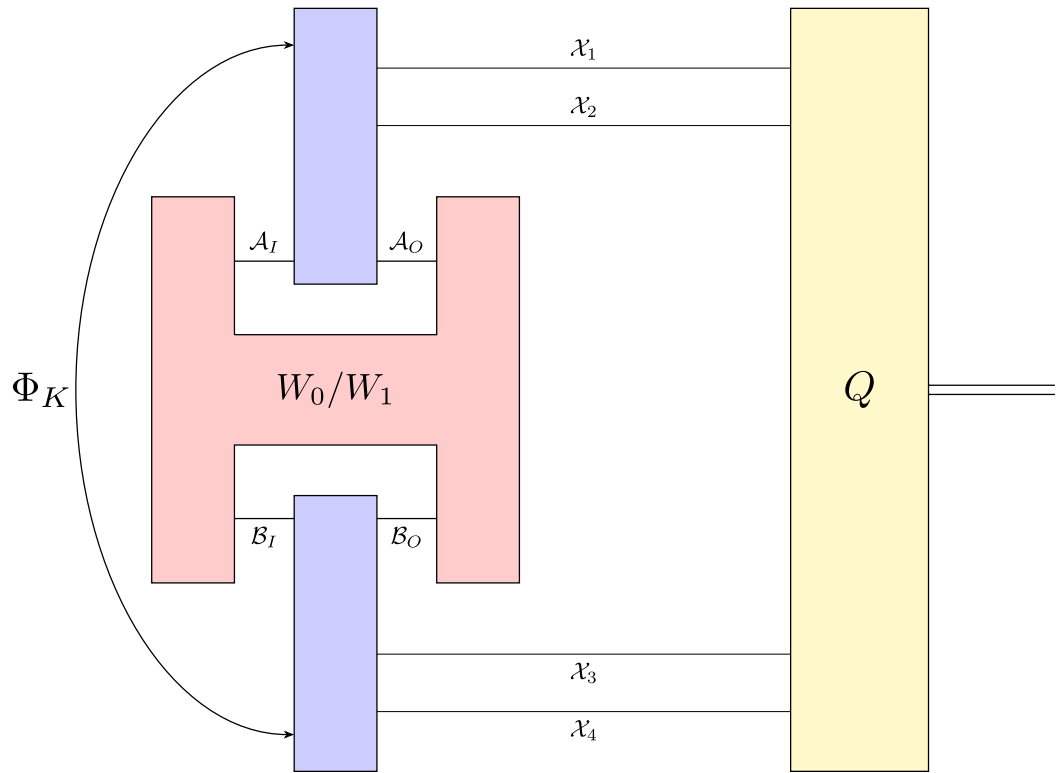


Figure 3. A schematic representation of the setup for distinguishing between process matrices W_0 and W_1 . The discrimination strategy is constructed by using the quantum channel $\Phi_K : L(\mathcal{A}_I \otimes \mathcal{B}_I) \rightarrow L(\mathcal{A}_O \otimes \mathcal{B}_O \otimes \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3 \otimes \mathcal{X}_4)$ and the binary measurement $Q = \{Q_0, Q_1\}$ defined in the proof of Theorem 1.

Discrimination between different classes of process matrices

This section presents some examples of discrimination between different classes of process matrices. We begin our consideration with the problem of discrimination between two free process matrices $\mathbf{W}^{A||B}$. Next, we will consider various cases of process matrices discrimination representing a quantum comb. First, we calculate exact probability of correct discrimination between two process matrices come from the same class $\mathbf{W}^{A<B}$. Next, we study the discrimination task assuming that one of the process matrices is of the form $\mathbf{W}^{A<B}$ and the other one is of the form $\mathbf{W}^{B<A}$. Finally, we construct a particular class of process matrices which can be perfectly distinguished.

Free process matrices. The following consideration confirms an intuition that the task of discrimination between free process matrices reduces to the problem of discrimination between quantum states.

From definition of $p_{\text{succ}}(W_0, W_1)$ we have

$$p_{\text{succ}}(W_0, W_1) = \frac{1}{2} \max_{S=\{S_0, S_1\}} [\text{tr}(W_0 S_0) + \text{tr}(W_1 S_1)]. \tag{28}$$

Let W_0 and W_1 be two process matrices of the form $W_0 = \rho \otimes \mathbb{1}_{\mathcal{A}_O \mathcal{B}_O}$ and $W_1 = \sigma \otimes \mathbb{1}_{\mathcal{A}_O \mathcal{B}_O}$, where $\rho, \sigma \in \Omega(\mathcal{A}_I \otimes \mathcal{B}_I)$. Then, p_{succ} is exactly equal to

$$\begin{aligned} & \max_{S=\{S_0, S_1\}} \left[\frac{1}{2} \text{tr}(W_0 S_0) + \frac{1}{2} \text{tr}(W_1 S_1) \right] \\ &= \max_{S=\{S_0, S_1\}} \left[\frac{1}{2} \text{tr}((\rho \otimes \mathbb{1}) S_0) + \frac{1}{2} \text{tr}((\sigma \otimes \mathbb{1}) S_1) \right] \\ &= \max_{S=\{S_0, S_1\}} \left[\frac{1}{2} \text{tr}(\rho \text{tr}_{\mathcal{A}_O \mathcal{B}_O} S_0) + \frac{1}{2} \text{tr}(\sigma \text{tr}_{\mathcal{A}_O \mathcal{B}_O} S_1) \right]. \end{aligned} \tag{29}$$

Let us observe $\text{tr}_{\mathcal{A}_O \mathcal{B}_O} S_0 + \text{tr}_{\mathcal{A}_O \mathcal{B}_O} S_1 = \mathbb{1}_{\mathcal{A}_I \mathcal{B}_I}$. So, $\{\text{tr}_{\mathcal{A}_O \mathcal{B}_O} S_0, \text{tr}_{\mathcal{A}_O \mathcal{B}_O} S_1\}$ is a binary measurement and therefore, from Holevo–Helstrom theorem for quantum states, we have

$$\max_{S=\{S_0, S_1\}} \left[\frac{1}{2} \text{tr}(\rho \text{tr}_{\mathcal{A}_O \mathcal{B}_O} S_0) + \frac{1}{2} \text{tr}(\sigma \text{tr}_{\mathcal{A}_O \mathcal{B}_O} S_1) \right] \leq \frac{1}{2} + \frac{1}{4} \|\rho - \sigma\|_1. \tag{30}$$

Now, assume that $E = \{E_0, E_1\}$ is the Holevo–Helstrom measurement (by taking E_0 and E_1 as positive and negative part of $\rho - \sigma$, respectively). Hence, we obtain

$$\max_{E=\{E_0, E_1\}} \left[\frac{1}{2} \text{tr}(\rho E_0) + \frac{1}{2} \text{tr}(\sigma E_1) \right] = \frac{1}{2} + \frac{1}{4} \|\rho - \sigma\|_1. \tag{31}$$

Observe, it suffices to take $S_0 := E_0 \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|$ and $S_1 := E_1 \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|$. Note that $S_0 + S_1 = \mathbb{1}_{\mathcal{A}_I \mathcal{B}_I} \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|$ is non-signaling channel. Therefore, we have

$$p_{\text{succ}}(W_0, W_1) = \frac{1}{2} + \frac{1}{4} \|\rho - \sigma\|_1, \tag{32}$$

which completes the consideration.

Due to the above consideration, we obtain the following corollary.

Corollary 2 Let $\rho, \sigma \in \Omega(\mathcal{A}_I \otimes \mathcal{B}_I)$ be quantum states and let $W_0, W_1 \in \mathbf{W}^{\mathcal{A}||\mathcal{B}}$ be two free process matrices of the form $W_0 = \rho \otimes \mathbb{1}$ and $W_1 = \sigma \otimes \mathbb{1}$. Then,

$$p_{\text{succ}}(W_0, W_1) = \frac{1}{2} + \frac{1}{4} \|\rho - \sigma\|_1. \tag{33}$$

Process matrices representing quantum combs. Here, we will compare the probability of correct discrimination between two process matrices being quantum combs of the form $W_0^{A \prec B}, W_1^{A \prec B} \in \mathbf{W}^{A \prec B}$ by using non-signalling strategy $S = \{S_0, S_1\}$ described by Eq. (54) or an adaptive strategy.

Before that, we will discuss the issue of adaptive strategy. The most general strategy of quantum operations discrimination is known as an adaptive strategy^{22,37}. An adaptive strategy is realized by a quantum tester¹. A schematic representation of this setup is presented in Fig. 4.

Let us consider a quantum tester $\{L_0, L_1\}$ which is a two-element collection of probabilistic quantum networks L_0 and L_1 whose sum is a quantum comb, and additionally $L_i * W \in \mathbb{R}, i = 0, 1$ for any process matrix $W \in \mathbf{W}^{A \prec B}$. The probability of correct discrimination between $W_0^{A \prec B}$ and $W_1^{A \prec B}$ by using an adaptive strategy is defined by equation

$$p_{\text{adapt}}(W_0^{A \prec B}, W_1^{A \prec B}) := \frac{1}{2} \max_{\{L_0, L_1\}} L_0 * W_0 + L_1 * W_1. \tag{34}$$

It turns out that we do not need adaptation in order to obtain the optimal probability of distinction. This is stated formally in the following theorem.

Theorem 2 Let $W_0^{A \prec B}, W_1^{A \prec B} \in \mathbf{W}^{A \prec B}$ be two process matrices representing quantum combs $A \prec B$. Then,

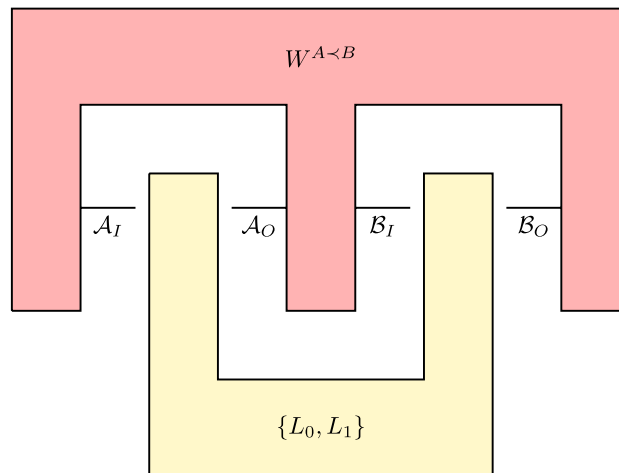


Figure 4. A schematic representation of an adaptive strategy discriminating two process matrices $W_0, W_1 \in \mathbf{W}^{A \prec B}$ by using a quantum tester $\{L_0, L_1\}$.

$$p_{succ}(W_0^{A \prec B}, W_1^{A \prec B}) = p_{adapt}(W_0^{A \prec B}, W_1^{A \prec B}). \tag{35}$$

Proof For simplicity, we will omit superscripts ($A \prec B$ and $B \prec A$). The inequality $p_{succ}(W_0, W_1) \leq p_{adapt}(W_0, W_1)$ is trivial by observing that we calculate maximum value over a larger set.

To show $p_{succ}(W_0, W_1) \geq p_{adapt}(W_0, W_1)$, let us consider the quantum tester $\{L_0, L_1\}$ which maximizes Eq. (34), that means

$$p_{adapt}(W_0, W_1) = \frac{1}{2}(L_0 * W_0 + L_1 * W_1). \tag{36}$$

Hence, from definition of $W^{A \prec B}$ we have

$$W^{A \prec B} = W' \otimes \mathbb{1}_{\mathcal{B}_O}, \tag{37}$$

and then we obtain

$$\frac{1}{2}(L_0 * W_0 + L_1 * W_1) = \frac{1}{2} \text{tr} (W'_0 \text{tr}_{\mathcal{B}_O} L_0 + W'_1 \text{tr}_{\mathcal{B}_O} L_1). \tag{38}$$

Observe that $\text{tr}_{\mathcal{B}_O}(L_0 + L_1) = \mathbb{1}_{\mathcal{B}_I} \otimes J$, where J is a Choi matrix of a channel $\Phi_J : L(\mathcal{A}_I) \rightarrow L(\mathcal{A}_O)$. Let us define a strategy $S = \{S_0, S_1\}$ such that

$$\begin{aligned} S_0 &= \text{tr}_{\mathcal{B}_O} L_0 \otimes \frac{\mathbb{1}_{\mathcal{B}_O}}{\dim(\mathcal{B}_O)}, \\ S_1 &= \text{tr}_{\mathcal{B}_O} L_1 \otimes \frac{\mathbb{1}_{\mathcal{B}_O}}{\dim(\mathcal{B}_O)}, \\ S &= S_0 + S_1. \end{aligned} \tag{39}$$

It easy to observe that $S \in \mathbf{NS}(\mathcal{A}_O \otimes \mathcal{A}_I \otimes \mathcal{B}_O \otimes \mathcal{B}_I)$. Then, we have

$$\frac{1}{2}(S_0 * W_0 + S_1 * W_1) = \frac{1}{2}(\text{tr} (W'_0 \text{tr}_{\mathcal{B}_O} L_0 + W'_1 \text{tr}_{\mathcal{B}_O} L_1)). \tag{40}$$

It implies that

$$p_{succ}(W_0, W_1) \geq p_{adapt}(W_0, W_1), \tag{41}$$

which completes the proof. □

Process matrices of the form $W^{A \prec B}$ and $W^{B \prec A}$. Now, we present some results for discrimination task assuming the one of the process matrices if of the form $W^{A \prec B}$ and the other one is of the form $W^{B \prec A}$. We will construct a particular class of such process matrices for which the perfect discrimination is possible.

Let us define a process matrix of the form

$$W^{A \prec B} = \rho \otimes |U\rangle\rangle\langle\langle U| \otimes \mathbb{1}, \tag{42}$$

where $\rho \in \Omega(\mathcal{A}_I)$, $|U\rangle\rangle\langle\langle U|$ is the Choi matrix of a unitary channel $\text{Ad}_{U^\top} : L(\mathcal{A}_O) \rightarrow L(\mathcal{B}_I)$ of the form $\text{Ad}_{U^\top}(X) = U^\top X U$ and $\mathbb{1} \in L(\mathcal{B}_O)$. A schematic representation of this process matrix we can see in Fig. 5.

Proposition 1 Let $W^{A \prec B}$ be a process matrix given by Eq. (42). Let us define a process matrix $W^{B \prec A}$ of the form

$$W^{B \prec A} = P_\pi W^{A \prec B} P_\pi, \tag{43}$$

where P_π is the swap operator replacing the systems $\mathcal{A}_I \rightarrow \mathcal{B}_I$ and $\mathcal{A}_O \rightarrow \mathcal{B}_O$. Then, the process matrix $W^{A \prec B}$ is perfectly distinguishable from $W^{B \prec A}$.

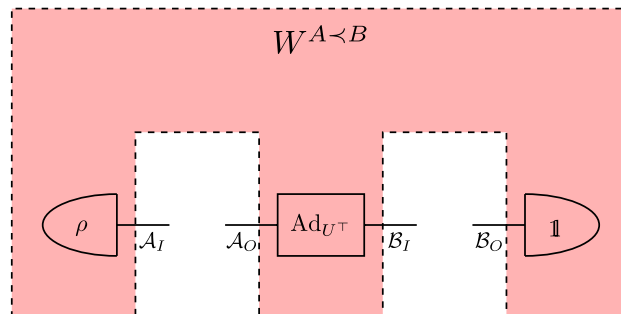


Figure 5. A schematic representation of process matrix $W^{A \prec B}$ given by Eq. (42).

Proof Let us consider the process matrix given by Eq. (42) described by Fig. 5. W.l.o.g. let d be a dimension of each of the systems. Let $\rho = \sum_{i=0}^{d-1} \lambda_i |x_i\rangle\langle x_i|$, where $\lambda_i \geq 0$ such that $\sum_i \lambda_i = 1$. Based on the spectral decomposition of ρ we create the unitary matrix V by taking i -th eigenvector of ρ , and the measurement Δ_V (in basis of ρ) given by

$$\Delta_V(X) = \sum_{i=0}^{d-1} \langle x_i | X | x_i \rangle |i\rangle\langle i| \otimes |x_i\rangle\langle x_i|. \tag{44}$$

Let us also define the permutation matrix $P_\sigma = \sum_{i=0}^{d-1} |x_{i+1 \bmod d}\rangle\langle x_i|$ corresponding to the permutation $\sigma = (0, 1, \dots, d - 1)$.

Alice and Bob prepare their discrimination strategy. Alice performs the local channel (see Fig. 6) given by

$$\Phi_A(\rho) = ((\mathcal{I} \otimes \text{Ad}_{\bar{U}}) \circ \Delta_V)(\rho). \tag{45}$$

Meanwhile, Bob performs his local channel (see Fig. 7) given by

$$\Phi_B(\rho) = ((\mathcal{I} \otimes \text{Ad}_{\bar{U}}) \circ \Delta_V \circ \text{Ad}_{P_\sigma})(\rho). \tag{46}$$

□

Let us consider the case $A \prec B$. The output after Alice’s action is described by

$$\Phi_A(\rho) = \sum_i \lambda_i |i\rangle\langle i| \otimes \bar{U} |x_i\rangle\langle x_i| U^\top. \tag{47}$$

Next, we apply the quantum channel Ad_{U^\top} (see Fig. 5), and hence we have

$$(\mathcal{I} \otimes \text{Ad}_{U^\top}) \circ \Phi_A(\rho) = \sum_i \lambda_i |i\rangle\langle i| \otimes |x_i\rangle\langle x_i|. \tag{48}$$

In the next step, Bob applies his channel as follows

$$(\mathcal{I} \otimes \Phi_B) \circ (\mathcal{I} \otimes \text{Ad}_{U^\top}) \circ \Phi_A(\rho) = \sum_i \lambda_i |i\rangle\langle i| \otimes |i + 1\rangle\langle i + 1| \otimes |x_{i+1}\rangle\langle x_{i+1}|. \tag{49}$$

Finally, we apply partial trace operation on the subspace \mathcal{B}_O (see Fig. 5), that means

$$\text{tr}_{\mathcal{B}_O} \left(\sum_i \lambda_i |i\rangle\langle i| \otimes |i + 1\rangle\langle i + 1| \otimes |x_{i+1}\rangle\langle x_{i+1}| \right) = \sum_i \lambda_i |i\rangle\langle i| \otimes |i + 1\rangle\langle i + 1|. \tag{50}$$

So, the quantum state obtained after the discrimination scenario in the case $A \prec B$ is given by

$$\sigma^{A \prec B} = \sum_i \lambda_i |i\rangle\langle i| \otimes |i + 1\rangle\langle i + 1|. \tag{51}$$

It implies that if Alice measures her system, she obtains the label i with probability λ_i whereas Bob obtains the label $(i + 1) \bmod d$ with the same probability. On the other hand, considering the case $B \prec A$, then the state obtained after the discrimination scenario is given by

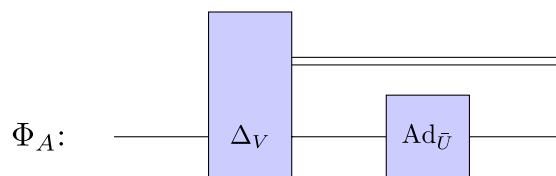


Figure 6. A schematic representation of Alice’s discrimination strategy described by Eq. (45).

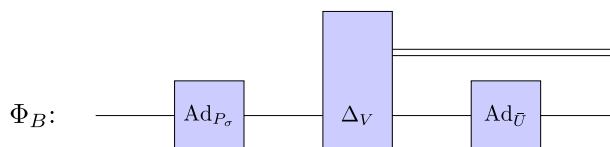


Figure 7. A schematic representation of Bob’s discrimination strategy described by Eq. (46).

$$\sigma^{B \leftarrow A} = \sum_i \lambda_i |i+1\rangle\langle i+1| \otimes |i+1\rangle\langle i+1|. \tag{52}$$

So, Bob and Alice obtain the same label $(i+1) \pmod d$ with probability λ_i . Then, the quantum channel Φ_K (realizing the discrimination strategy S) is created as a tensor product of Alice's and Bob's local channels, that means $\Phi_K = \Phi_A \otimes \Phi_B$. Due to that they perform the binary measurement $Q = \{Q_0, Q_1\}$, where the effect Q_1 is given by $Q_1 = \sum_{i=0}^{d-1} |i\rangle\langle i| \otimes |i\rangle\langle i|$. Hence, we have

$$p_{\text{succ}}(W^{A \leftarrow B}, W^{B \leftarrow A}) = \frac{1}{2} \text{tr}(\sigma^{A \leftarrow B} Q_0) + \frac{1}{2} \text{tr}(\sigma^{B \leftarrow A} Q_1) = 1. \tag{53}$$

In summary, the process matrices $W^{A \leftarrow B}$ and $W^{B \leftarrow A}$ are perfectly distinguishable by Alice and Bob which completes the proof. \square

SDP program for calculating the optimal probability of process matrices discrimination

In the standard approach, we would need to compute the probability of correct discrimination between two process matrices W_0 and W_1 . For this purpose, we use the semidefinite programming (SDP). This section presents the SDP program for calculating the optimal probability of discrimination between W_0 and W_1 .

Recall that the maximum value of such a probability can be noticed by

$$p_{\text{succ}}(W_0, W_1) = \frac{1}{2} \max_{S=\{S_0, S_1\}} [\text{tr}(W_0 S_0) + \text{tr}(W_1 S_1)], \tag{54}$$

with requirement that the optimal strategy $S = \{S_0, S_1\}$ is a quantum instrument such that $S_0 + S_1 \in \text{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$. Hence, we arrive at the primal and dual problems presented in the Program 1.

To optimize this problem we used the Julia programming language along with quantum package `QuantumInformation.jl`³⁸ and SDP optimization via SCS solver^{39,40} with absolute convergence tolerance 10^{-5} . The code is available on GitHub⁴¹.

It may happen that the values of primal and dual programs are equal. This situation is called strong duality. Slater's theorem provides the set of conditions which guarantee strong duality³⁶. It can be shown that Program 1 fulfills conditions of Slater's theorem (it suffices to take $Y_0, Y_1 = 0$ and $\alpha > \frac{1}{2} \max\{\Lambda^{\max}(W_0), \Lambda^{\max}(W_1)\}$, where $\Lambda^{\max}(X)$ is the maximum eigenvalue of X). Therefore, we can consider the primal and the dual problem equivalently.

Program 1 Semidefinite program for maximizing the probability of correct discrimination between two process matrices W_0 and W_1

SDP program for calculating the optimal probability of discrimination between W_0 and W_1	
Primal problem	
maximize:	$\frac{1}{2} \text{Tr}(W_0 S_0) + \frac{1}{2} \text{Tr}(W_1 S_1)$
subject to:	$\text{tr}_{\mathcal{A}_O}(S_0 + S_1) = \frac{\mathbb{1}_{\mathcal{A}_I}}{\dim(\mathcal{A}_I)} \otimes \text{tr}_{\mathcal{A}_O \mathcal{A}_I}(S_0 + S_1),$
	$\text{tr}_{\mathcal{B}_O}(S_0 + S_1) = \frac{\mathbb{1}_{\mathcal{B}_I}}{\dim(\mathcal{B}_I)} \otimes \text{tr}_{\mathcal{B}_O \mathcal{B}_I}(S_0 + S_1),$
	$\text{tr}(S_0 + S_1) = \dim(\mathcal{A}_I) \dim(\mathcal{B}_I),$
	$S_0 \in \text{Pos}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O),$
	$S_1 \in \text{Pos}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O).$
Dual problem	
minimize:	$\alpha \cdot \dim(\mathcal{A}_I) \dim(\mathcal{B}_I)$
subject to:	$\mathbb{1}_{\mathcal{A}_O} \otimes Y_0 - \frac{\mathbb{1}_{\mathcal{A}_I \mathcal{A}_O}}{\dim(\mathcal{A}_I)} \otimes \text{tr}_{\mathcal{A}_I}(Y_0) + \mathbb{1}_{\mathcal{B}_O} \otimes Y_1 +$
	$-\frac{\mathbb{1}_{\mathcal{B}_I \mathcal{B}_O}}{\dim(\mathcal{B}_I)} \otimes \text{tr}_{\mathcal{B}_I} Y_1 + \alpha \cdot \mathbb{1}_{\mathcal{A}_I \mathcal{A}_O \mathcal{B}_I \mathcal{B}_O} \geq \frac{1}{2} W_0,$
	$\mathbb{1}_{\mathcal{A}_O} \otimes Y_0 - \frac{\mathbb{1}_{\mathcal{A}_I \mathcal{A}_O}}{\dim(\mathcal{A}_I)} \otimes \text{tr}_{\mathcal{A}_I}(Y_0) + \mathbb{1}_{\mathcal{B}_O} \otimes Y_1 +$
	$-\frac{\mathbb{1}_{\mathcal{B}_I \mathcal{B}_O}}{\dim(\mathcal{B}_I)} \otimes \text{tr}_{\mathcal{B}_I} Y_1 + \alpha \cdot \mathbb{1}_{\mathcal{A}_I \mathcal{A}_O \mathcal{B}_I \mathcal{B}_O} \geq \frac{1}{2} W_1,$
	$Y_0 \in \text{Herm}(\mathcal{A}_I \otimes \mathcal{B}_I \otimes \mathcal{B}_O),$
	$Y_1 \in \text{Herm}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I),$
	$\alpha \in \mathbb{R}.$

We can observe a connection between Program 1 and the semidefinite program calculating the maximal probability of successful discrimination of two qubit-qubit quantum channels using two copies under different strategies⁷. Thanks to their idea, we obtain the following SDP program.

Primal problem	
given:	$W_0, W_1 \in \mathbf{W}^{\text{PROC}}$
maximize:	$\frac{1}{2} \text{Tr}(W_0 S_0) + \frac{1}{2} \text{Tr}(W_1 S_1)$
subject to:	$\{S_0, S_1\}$ is a tester such that $S_0 + S_1 \in \text{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$.
Dual problem	
given:	$W_0, W_1 \in \mathbf{W}^{\text{PROC}}$
minimize:	α
subject to:	$\frac{1}{2} W_i \leq \alpha \cdot X$, where $X \in \widetilde{\text{NS}}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$, and $\widetilde{\text{NS}}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$ lies in the dual affine space of the set $\text{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$ and $i \in \{0, 1\}$.

It can be shown, eg. in⁷ that $\widetilde{\text{NS}}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$ creates the set of all process matrices \mathbf{W}^{PROC} . Hence, the constrains $\frac{1}{2} W_i \leq \alpha X$, where $X \in \mathbf{W}^{\text{PROC}}$, can be explicitly written as in Program 1.

The SDP Program 1 is defined on the tensor product of four Hilbert spaces with local dimensions equal d . The number of variables of the primal and dual problem scales as $\Omega(d^8)$ and $\Omega(d^6)$, respectively, therefore, the standard SDP solvers can take a long time to find the solution for large d . One approach to deal with this problem is to use a simplified SDP programming approach like it was done for computing the diamond norm⁴². Another method is to exploit an iterative method based on the fixed point and the Complementary Slackness criterion, like in Ref.⁴³ for the fidelity function. We leave this problem for future research.

Distance between process matrices

In this section we present the semidefinite programs for calculating the distance in trace norm between a given process matrix $W \in \mathbf{W}^{\text{PROC}}$ and different subsets of process matrices, such that $\mathbf{W}^{\text{A||B}}$, $\mathbf{W}^{\text{A<B}}$, $\mathbf{W}^{\text{B<A}}$ or \mathbf{W}^{SEP} .

For example, let us consider the case $\mathbf{W}^{\text{A||B}}$. Theoretically, the distance between a process matrix W and the set of free process matrices $\mathbf{W}^{\text{A||B}}$ can be expressed by

$$\text{dist}(W, \mathbf{W}^{\text{A||B}}) = \min_{\widetilde{W} \in \mathbf{W}^{\text{A||B}}} \max \left\{ \|\sqrt{N}(W - \widetilde{W})\sqrt{N}\|_1 : N \in \text{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O) \right\}. \tag{55}$$

Analogously, for the sets $\mathbf{W}^{\text{A<B}}$, $\mathbf{W}^{\text{B<A}}$ and \mathbf{W}^{SEP} with the minimization condition $\min_{\widetilde{W} \in \mathbf{W}^{\text{A<B}}}$, $\min_{\widetilde{W} \in \mathbf{W}^{\text{B<A}}}$, $\min_{\widetilde{W} \in \mathbf{W}^{\text{SEP}}}$, respectively. Due to the results obtained from the previous section (see Program 1) and Slater theorem we are able to note the Eq. (55) to SDP problem presented in the Program 2. We use the SDP optimization via SCS solver^{39,40} with absolute convergence tolerance 10^{-8} and relative convergence tolerance 10^{-8} . The implementations of SDPs in the Julia language are available on GitHub⁴¹.

Semidefinite program for computation the distance between a process matrix W and Υ , which can be one of the set $\mathbf{W}^{\text{A||B}}$, $\mathbf{W}^{\text{A<B}}$, $\mathbf{W}^{\text{B<A}}$ or \mathbf{W}^{SEP} . Depending on the selected set we include additional constrains to SDP described by Eq. (16) for $\mathbf{W}^{\text{A||B}}$, Eq. (17) for $\mathbf{W}^{\text{A<B}}$ and $\mathbf{W}^{\text{B<A}}$ or Eq. (18) for \mathbf{W}^{SEP}

SDP calculating the distance between a process matrix W and the set Υ .	
minimize:	$4 \dim(\mathcal{A}_I) \dim(\mathcal{B}_I) \alpha - 2$
subject to:	$\mathbb{1}_{\mathcal{A}_O} \otimes Y_0 - \frac{\mathbb{1}_{\mathcal{A}_I \mathcal{A}_O}}{\dim(\mathcal{A}_I)} \otimes \text{tr}_{\mathcal{A}_I}(Y_0) + \mathbb{1}_{\mathcal{B}_O} \otimes Y_1 +$ $-\frac{\mathbb{1}_{\mathcal{B}_I \mathcal{B}_O}}{\dim(\mathcal{B}_I)} \otimes \text{tr}_{\mathcal{B}_I} Y_1 + \alpha \cdot \mathbb{1}_{\mathcal{A}_I \mathcal{A}_O \mathcal{B}_I \mathcal{B}_O} \geq \frac{1}{2} W,$ $\mathbb{1}_{\mathcal{A}_O} \otimes Y_0 - \frac{\mathbb{1}_{\mathcal{A}_I \mathcal{A}_O}}{\dim(\mathcal{A}_I)} \otimes \text{tr}_{\mathcal{A}_I}(Y_0) + \mathbb{1}_{\mathcal{B}_O} \otimes Y_1 +$ $-\frac{\mathbb{1}_{\mathcal{B}_I \mathcal{B}_O}}{\dim(\mathcal{B}_I)} \otimes \text{tr}_{\mathcal{B}_I} Y_1 + \alpha \cdot \mathbb{1}_{\mathcal{A}_I \mathcal{A}_O \mathcal{B}_I \mathcal{B}_O} \geq \frac{1}{2} W^*,$ $W^* \in \Upsilon,$ $Y_0 \in \text{Herm}(\mathcal{A}_I \otimes \mathcal{B}_I \otimes \mathcal{B}_O),$ $Y_1 \in \text{Herm}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I),$ $\alpha \in \mathbb{R}.$

Example. Let $\mathcal{A}_I = \mathcal{A}_O = \mathcal{B}_I = \mathcal{B}_O = \mathbb{C}^2$. Let us consider a causally non-separable process matrix comes from Ref.⁵ of the form

$$W^{\text{CNS}} = \frac{1}{4} \left[\mathbb{1}_{\mathcal{A}_I \mathcal{A}_O \mathcal{B}_I \mathcal{B}_O} + \frac{1}{\sqrt{2}} \left(\sigma_z^{\mathcal{A}_O} \sigma_z^{\mathcal{B}_I} \otimes \mathbb{1}_{\mathcal{A}_I \mathcal{B}_O} + \sigma_z^{\mathcal{A}_I} \sigma_x^{\mathcal{B}_I} \sigma_z^{\mathcal{B}_O} \otimes \mathbb{1}_{\mathcal{A}_O} \right) \right], \tag{56}$$

where $\sigma_x^{\mathcal{X}}, \sigma_z^{\mathcal{X}}$ are Pauli matrices on space $L(\mathcal{X})$. We have calculated the distance in trace norm between W^{CNS} and different subset of process matrices. Finally, we obtain

$$\text{dist}(W^{CNS}, W^{A||B}) \approx 1.00000001 \approx 1, \tag{57}$$

$$\text{dist}(W^{CNS}, W^{A \prec B}) \approx 0.7071068 \approx \frac{\sqrt{2}}{2}, \tag{58}$$

$$\text{dist}(W^{CNS}, W^{B \prec A}) \approx 0.7071068 \approx \frac{\sqrt{2}}{2}, \tag{59}$$

$$\text{dist}(W^{CNS}, W^{SEP}) \approx 0.2928932 \approx 1 - \frac{\sqrt{2}}{2}. \tag{60}$$

The numerical computations give us some intuition about the geometry of the set of process matrices. Those results are presented in Fig. 8. Moreover, by using W^{CNS} given by Eq. (56) it can be shown that the set of all causally non-separable process matrices is not convex. To show this fact, it suffices to observe that for every $\sigma_i, \sigma_j, \sigma_k, \sigma_l \in \{\sigma_x, \sigma_y, \sigma_z, \mathbb{1}\}$ the following equation holds

$$(\sigma_i \otimes \sigma_j \otimes \sigma_k \otimes \sigma_l) W^{CNS} (\sigma_i \otimes \sigma_j \otimes \sigma_k \otimes \sigma_l)^\dagger \in W^{CNS}. \tag{61}$$

Simultaneously, the average of the process matrices of the form Eq. (61) distributed uniformly states $\frac{1}{4} \mathbb{1}_{C^{16}}$, however $\frac{1}{4} \mathbb{1}_{C^{16}} \notin W^{CNS}$. It implies that the set W^{CNS} is not convex.

Convex cone structure theory

From geometrical point of view, we present an alternative way to derive of Eq. (26). It turns out that the task of process matrices discrimination is strictly connected with the convex cone structure theory. To keep this work self-consistent, the details of convex cone structure theory are presented in the Supplementary Materials.

Let \mathcal{V} be a finite dimensional real vector space with a proper cone $\mathcal{C} \subset \mathcal{V}$. A base B of the proper cone \mathcal{C} is a compact convex subset $B \subset \mathcal{C}$ such that each nonzero element $c \in \mathcal{C}$ has a unique representation in the form $c = \alpha \cdot b$, where $\alpha > 0$ and $b \in B$. The corresponding base norm in \mathcal{V} is defined by

$$\|x\|_B = \{\alpha + \beta, x = \alpha b_1 - \beta b_2, \alpha, \beta \geq 0, b_1, b_2 \in B\}. \tag{62}$$

From Ref.³⁷, Corollary 2 the author showed that the base norm can be written as

$$\|x\|_B = \sup_{\tilde{b} \in \tilde{B}} \|\tilde{b}^{1/2} x \tilde{b}^{1/2}\|_1, \tag{63}$$

where

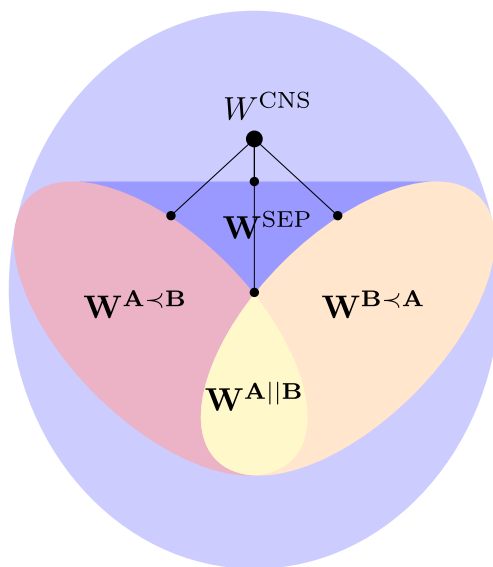


Figure 8. A schematic representation of the distances between W^{CNS} defined in Eq. (56) and the sets $W^{A||B}$, $W^{A \prec B}$, $W^{B \prec A}$ and W^{SEP} .

$$\widetilde{B} := \{Y \in \mathcal{C} : \text{tr}(XY) = 1, \forall X \in B\}. \tag{64}$$

Convex cone structure of process matrices set. Let \mathcal{V} be a Hilbert space given by

$$\mathcal{V} = \text{Herm}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O), \tag{65}$$

with proper cone

$$\mathcal{C} = \{W \in \mathcal{V} : W \in \text{Pos}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)\}. \tag{66}$$

Consider the linear subspace $\mathcal{S} \subset \mathcal{V}$ given by

$$\begin{aligned} \mathcal{S} = \{W \in \mathcal{V} : &_{B_I B_O} W =_{A_O B_I B_O} W, \\ &_{A_I A_O} W =_{A_O A_I B_O} W, \\ &_{A_O B_O} W =_{B_O} W +_{A_O} W - W\}, \end{aligned} \tag{67}$$

together with its proper cone $\mathcal{C}_\mathcal{S}$. Observe that if we fix trace of $W \in \mathcal{C}_\mathcal{S}$ such that $\text{tr}(W) = \dim(\mathcal{A}_O) \cdot \dim(\mathcal{B}_O)$, we achieve the set of all process matrices \mathbf{W}^{PROC} . And then, \mathbf{W}^{PROC} is a base of $\mathcal{C}_\mathcal{S}$.

Proposition 2 *Let \mathbf{W}^{PROC} be the set of process matrices. Then, the set $\widetilde{\mathbf{W}^{\text{PROC}}}$ is determined by*

$$\widetilde{\mathbf{W}^{\text{PROC}}} = \text{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O). \tag{68}$$

Proof To prove this proposition, we need to show that

$$\text{tr}(XW) = 1 \text{ for all } W \in \mathbf{W}^{\text{PROC}} \iff X \in \text{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O). \tag{69}$$

This equivalence, along with its proof, can be found eg. in Refs.^{29,44}. However, to keep this work self-consistent we present our modified version of their reasoning.

Let us first take $X \in \text{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$. Then, from Ref.²⁵, Lemma 1, we note

$$X = \sum_i \lambda_i A_i \otimes B_i, \tag{70}$$

where $A_i \in \text{NS}(\mathcal{A}_I \otimes \mathcal{A}_O)$, $B_i \in \text{NS}(\mathcal{B}_I \otimes \mathcal{B}_O)$ and $\lambda_i \in \mathbb{R}$ such that $\sum_i \lambda_i = 1$. From definition of process matrix and linearity we obtain

$$\text{tr}(XW) = \text{tr}\left(\sum_i \lambda_i (A_i \otimes B_i) W\right) = \sum_i \lambda_i \text{tr}((A_i \otimes B_i) W) = \sum_i \lambda_i = 1. \tag{71}$$

To prove opposite implication, let us take $W = \mathbb{1}_{\mathcal{A}_O} \otimes J$, where J is the Choi matrix of quantum channel $\Phi_J : L(\mathcal{B}_I \otimes \mathcal{B}_O) \rightarrow L(\mathcal{A}_I)$. Then, we have

$$1 = \text{tr}(WX) = \text{tr}((\mathbb{1}_{\mathcal{A}_O} \otimes J)X) = \text{tr}(J \text{tr}_{\mathcal{A}_O} X). \tag{72}$$

From Ref.⁴⁵, we have

$$\text{tr}_{\mathcal{A}_O} X = \mathbb{1}_{\mathcal{A}_I} \otimes P, \tag{73}$$

where $P \in \text{Pos}(\mathcal{B}_I \otimes \mathcal{B}_O)$. Similarly, if we take $W := \mathbb{1}_{\mathcal{B}_O} \otimes K$, where K is the Choi matrix of a quantum channel $\Phi_K : L(\mathcal{A}_I \otimes \mathcal{A}_O) \rightarrow L(\mathcal{B}_I)$, we obtain

$$\text{tr}_{\mathcal{B}_O} X = \mathbb{1}_{\mathcal{B}_I} \otimes P, \tag{74}$$

where $P \in \text{Pos}(\mathcal{A}_I \otimes \mathcal{A}_O)$. It implies that $X \in \text{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)$, which completes the proof. \square

Due to Proposition 2, and the fact that the base norm can be also written using the trace norm, like in Eq. (63), we immediately obtain the following corollary.

Corollary 3 *The base norm $\|\cdot\|_{\mathbf{W}^{\text{PROC}}}$ between two process matrices $W_1, W_2 \in \mathbf{W}^{\text{PROC}}$ can be expressed as*

$$\|W_1 - W_2\|_{\mathbf{W}^{\text{PROC}}} = \max\{\|\sqrt{N}(W_1 - W_2)\sqrt{N}\|_1 : N \in \text{NS}(\mathcal{A}_I \otimes \mathcal{A}_O \otimes \mathcal{B}_I \otimes \mathcal{B}_O)\}. \tag{75}$$

Conclusion and discussion

In this work, we studied the problem of single shot discrimination between process matrices. Our aim was to provide an exact expression for the optimal probability of correct distinction and quantify it in terms of the trace norm. This value was maximized over all Choi operators of non-signaling channels and and poses direct analogues to the Holevo–Helstrom theorem for quantum channels. In addition, we have presented an alternative

way to achieve this expression by using the convex cone structure theory. As a valuable by-product, we have also found the optimal realization of the discrimination task for process matrices that use such non-signalling channels. Additionally, we expressed the discrimination task as semidefinite programming (SDP). Due to that, we have created SDP calculating the distance between process matrices and we expressed it in terms of the trace norm. Moreover, we found an analytical result for discrimination of free process matrices. It turns out that the task of discrimination between free process matrices can be reduced to the task of discrimination between quantum states. Next, we consider the problem of discrimination for process matrices corresponding to quantum combs. We have studied which strategy, adaptive or non-signalling, should be used during the discrimination task. We proved that no matter which strategy you choose, the optimal probability of distinguishing two process matrices being a quantum comb is the same. So, it turned out that we do not need to use some unknown additional processing in this case. Finally, we discovered a particular class of process matrices having opposite causal order, which can be distinguished perfectly. This work paves the way toward a complete description of necessary and sufficient criterion for perfect discrimination between process matrices. Moreover, it poses a starting point to fully describe the geometry of the set of process matrices, particularly causally non-separable process matrices.

Data availability

The code used to generate numerical results analyzed during the current study is available in the Github repository, https://github.com/iitis/strategies_for_single_shot_discrimination_of_process_matrices.

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References

1. Bisio, A., Chiribella, G., D'Ariano, G. & Perinotti, P. Quantum networks: General theory and applications. *Acta Phys. Slovaca* **61**(3), 273–390 (2011).
2. Gödel, K. An example of a new type of cosmological solutions of Einstein's field equations of gravitation. *Rev. Mod. Phys.* **21**(3), 447 (1949).
3. Deutsch, D. & Lockwood, M. The quantum physics of time travel. *Sci. Am.* **270**(3), 68–74 (1994).
4. Gisin, N. Weinberg's non-linear quantum mechanics and supraluminal communications. *Phys. Lett. A* **143**(1–2), 1–2 (1990).
5. Oreshkov, O., Costa, F. & Brukner, Č. Quantum correlations with no causal order. *Nat. Commun.* **3**(1), 1–8 (2012).
6. Brukner, Č. Quantum causality. *Nat. Phys.* **10**(4), 259–263 (2014).
7. Bavaresco, J., Muraio, M. & Quintino, M. T. Strict hierarchy between parallel, sequential, and indefinite-causal-order strategies for channel discrimination. *Phys. Rev. Lett.* **127**(20), 200504 (2021).
8. Quintino, M. T. & Ebler, D. Deterministic transformations between unitary operations: Exponential advantage with adaptive quantum circuits and the power of indefinite causality. *Quantum* **6**, 679 (2022).
9. Bavaresco, J., Muraio, M. & Quintino, M. T. Unitary channel discrimination beyond group structures: Advantages of sequential and indefinite-causal-order strategies. *J. Math. Phys.* **63**(4), 042203 (2022).
10. Duan, R., Feng, Y. & Ying, M. Entanglement is not necessary for perfect discrimination between unitary operations. *Phys. Rev. Lett.* **98**(10), 100503 (2007).
11. D'Ariano, G. M., Presti, P. L. & Paris, M. G. Using entanglement improves the precision of quantum measurements. *Phys. Rev. Lett.* **87**(27), 270404 (2001).
12. Cao, T.-Q., Gao, F., Zhang, Z.-C., Yang, Y.-H. & Wen, Q.-Y. Perfect discrimination of projective measurements with the rank of all projectors being one. *Quantum Inf. Process.* **14**(7), 2645–2656 (2015).
13. Puchała, Z., Paweł, Ł., Krawiec, A. & Kukulski, R. Strategies for optimal single-shot discrimination of quantum measurements. *Phys. Rev. A* **98**(4), 042103 (2018).
14. Puchała, Z., Paweł, Ł., Krawiec, A., Kukulski, R. & Ozmaniec, M. Multiple-shot and unambiguous discrimination of von Neumann measurements. *Quantum* **5**, 425 (2021).
15. Wang, G. & Ying, M. Unambiguous discrimination among quantum operations. *Phys. Rev. A* **73**(4), 042301 (2006).
16. Krawiec, A., Paweł, Ł. & Puchała, Z. Discrimination of povms with rank-one effects. *Quantum Inf. Process.* **19**(12), 1–12 (2020).
17. Leditzky, F., Leung, D. & Smith, G. Quantum and private capacities of low-noise channels. In *2017 IEEE Information Theory Workshop (ITW)* 484–488 (IEEE, 2017).
18. Beale, S. J., Wallman, J. J., Gutiérrez, M., Brown, K. R. & Laflamme, R. Quantum error correction decoheres noise. *Phys. Rev. Lett.* **121**(19), 190501 (2018).
19. Christandl, M., König, R. & Renner, R. Postselection technique for quantum channels with applications to quantum cryptography. *Phys. Rev. Lett.* **102**(2), 020504 (2009).
20. Choi, M.-D. Completely positive linear maps on complex matrices. *Linear Algebra Appl.* **10**(3), 285–290 (1975).
21. Jamiołkowski, A. Linear transformations which preserve trace and positive semidefiniteness of operators. *Rep. Math. Phys.* **3**(4), 275–278 (1972).
22. Chiribella, G., D'Ariano, G. M. & Perinotti, P. Theoretical framework for quantum networks. *Phys. Rev. A* **80**(2), 022339 (2009).
23. Beckman, D., Gottesman, D., Nielsen, M. A. & Preskill, J. Causal and localizable quantum operations. *Phys. Rev. A* **64**(5), 052309 (2001).
24. Piani, M., Horodecki, M., Horodecki, P. & Horodecki, R. Properties of quantum nonsignaling boxes. *Phys. Rev. A* **74**(1), 012305 (2006).
25. Chiribella, G., D'Ariano, G. M., Perinotti, P. & Valiron, B. Quantum computations without definite causal structure. *Phys. Rev. A* **88**(2), 022318 (2013).
26. Nielsen, M. A. & Chuang, I. L. *Quantum Computation and Quantum Information* (Cambridge University Press, 2010).
27. Davies, E. B. & Lewis, J. T. An operational approach to quantum probability. *Commun. Math. Phys.* **17**(3), 239–260 (1970).
28. Chiribella, G., D'Ariano, G. M. & Perinotti, P. Quantum circuit architecture. *Phys. Rev. Lett.* **101**(6), 060401 (2008).
29. Araújo, M. *et al.* Witnessing causal nonseparability. *New J. Phys.* **17**(10), 102001 (2015).
30. Milz, S., Bavaresco, J. & Chiribella, G. Resource theory of causal connection. *Quantum* **6**, 788 (2022).
31. Oreshkov, O. & Giarmatzi, C. Causal and causally separable processes. *New J. Phys.* **18**(9), 093020 (2016).
32. Gour, G. Comparison of quantum channels by superchannels. *IEEE Trans. Inf. Theory* **65**(9), 5880–5904 (2019).
33. Jenčová, A. Generalized channels: Channels for convex subsets of the state space. *J. Math. Phys.* **53**(1), 012201 (2012).
34. Jenčová, A. Extremality conditions for generalized channels. *J. Math. Phys.* **53**(12), 122203 (2012).
35. Helstrom, C. *Quantum Detection and Estimation Theory, Series. Mathematics in Science and Engineering* Vol. 123 (Academic Press, 1976).

36. Watrous, J. *The Theory of Quantum Information* (Cambridge University Press, 2018).
37. Jenčová, A. Base norms and discrimination of generalized quantum channels. *J. Math. Phys.* **55**(2), 022201 (2014).
38. Gawron, P., Kurzyk, D. & Pawela, Ł. QuantumInformation.jl—A julia package for numerical computation in quantum information theory. *PLoS ONE* **13**, e0209358 (2018).
39. O'Donoghue, B., Chu, E., Parikh, N. & Boyd, S. Conic optimization via operator splitting and homogeneous self-dual embedding. *J. Optim. Theory Appl.* **169**, 1042–1068 (2016).
40. O'Donoghue, B., Chu, E., Parikh, N. & Boyd, S. SCS: Splitting Conic Solver, Version 3.2.1. <https://github.com/cvxgrp/scs> (2021).
41. Permanent Link to Code/Repository. https://github.com/iitis/strategies_for_single_shot_discrimination_of_process_matrices (Accessed 14 February 2023).
42. Watrous, J. Simpler semidefinite programs for completely bounded norms. Preprint at <http://arxiv.org/abs/1207.5726> (2012).
43. Afham, A., Kueng, R. & Ferrie, C. Quantum mean states are nicer than you think: Fast algorithms to compute states maximizing average fidelity. Preprint at <http://arxiv.org/abs/2206.08183> (2022).
44. Chiribella, G. & Ebler, D. Optimal quantum networks and one-shot entropies. *New J. Phys.* **18**(9), 093053 (2016).
45. Lewandowska, P., Kukulski, R. & Pawela, Ł. Optimal representation of quantum channels. In *International Conference on Computational Science* 616–626 (Springer, 2020).

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Author contributions

All the authors have equal contribution to the preparation of the paper. Z.P. proposed the main idea and made the analytical investigations presented in Supplementary Methods. Ł.P. performed the numerical analysis and wrote the SDP programs. P.L. made the analytical results presented in the main text and prepared the figures.

Competing interests

The authors declare no competing interests.

Additional information

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