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The non-Newtonian maxwell nanofluid flow between two parallel rotating disks under the effects of magnetic field

Ali Ahmadian¹, Muhammad Bilal², Muhammad Altaf Khan^{3,4✉} & Muhammad Imran Asjad⁵

The main feature of the present numerical model is to explore the behavior of Maxwell nanoliquid moving within two horizontal rotating disks. The disks are stretchable and subjected to a magnetic field in axial direction. The time dependent characteristics of thermal conductivity have been considered to scrutinize the heat transfer phenomena. The thermophoresis and Brownian motion features of nanoliquid are studied with Buongiorno model. The lower and upper disk's rotation for both the cases, same direction as well as opposite direction of rotation is investigated. The subsequent arrangement of the three dimensional Navier Stoke's equations along with energy, mass and Maxwell equations are diminished to a dimensionless system of equations through the Von Karman's similarity framework. The comparative numerical arrangement of modeled equations is further set up by built-in numerical scheme "boundary value solver" (Bvp4c) and Runge Kutta fourth order method (RK4). The various physical constraints, such as Prandtl number, thermal conductivity, magnetic field, thermal radiation, time relaxation, Brownian motion and thermophoresis parameters and their impact are presented and discussed briefly for velocity, temperature, concentration and magnetic strength profiles. In the present analysis, some vital characteristics such as Nusselt and Sherwood numbers are considered for physical and numerical investigation. The outcomes concluded that the disk stretching action opposing the flow behavior. With the increases of magnetic field parameter M the fluid velocity decreases, while improving its temperature. We show a good agreement of the present work by comparing with those published in literature.

List of symbols

ε	Thermal conductivity
Λ	Pressure gradient parameter
g	Transform azimuthal velocity
q	Heat flux
σ	Fluid electric conductivity
η	Dimensionless variable
Rd	Radiation parameter
Ω_1	Rotation rate of lower disk
D_B	Brownian diffusion coefficient
$K(T)$	Variable thermal conductivity
ρ	Fluid density
λ_1	Relaxation time parameter
Nb	Brownian motion parameter
D_B	Brownian diffusion coefficient
V	Kinematic viscosity
T	Temperature

¹Institute of Industry Revolution 4.0, The National University of Malaysia, 43600 UKM Bangi, Selangor, Malaysia. ²Department of Mathematics, City University of Science and Information Technology, Peshawar, Pakistan. ³Informetrics Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam. ⁴Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam. ⁵Department of Mathematics, University of Management and Technology, Lahore, Pakistan. ✉email: muhammad.altaf.khan@tdtu.edu.vn

D_T	Coefficient of thermophoretic diffusion
Sc	Schmidt number
T_1	Temperature of Lower disk
Bt	Batclor number
Φ, Θ	Transform concentration & temperature
r, θ, z	Cylindrical coordinate system
C_p	Specific heat
M	Magnetic parameter
P	Pressure
Ω	Rotation parameter
d	Vertical distance between disks
J	Current density
B_0	Strength of magnetic field
Ω_2	Rotation rate of upper disk
∇	Nabla
Pr	Prandtl number
C	Concentration
β_1	Deborah number
Nt	Thermophoresis parameter
D_T	Thermophoretic diffusion coefficient.
μ	Dynamic viscosity
$\mathbf{V}$	Velocity vector
k	Thermal conductivity
Re	Reynolds number
T_2	Temperature of upper disk
Re_m	Magnetic Reynolds number
f, f'	Transform axial and radial velocity
u, v, w	Velocity components

The study of the fluid flow on the surface of rotating disk has got great attentions around the globe from the researcher's due to its many applications in practical problems. Electric power generating system, rotating machinery, co rotating turbines, chemical process and computer storage, in the field of aerodynamics engineering, geothermal industry, for lubrication purposes, over the surface of rotating disk the fluid flow is widely applicable. Von Karman's¹ examined the solution of Navier stoke's equations by considering an appropriate transformation. Further, he used the fluid flow over the rotating frame for the first time. The Von Karman's problem and its solution numerically have been discussed by Cochran². Also, he used two series expansion by solving the limitation in the Von Karman's work. Sheikholeslami et al.³ used numerical technique for the solution of nanofluid flow over an inclined rotating disk. During the rotation of the disk, Millsaps and Pahlhausen⁴ studied the heat transport characteristic. The electric field in radial direction has been considered by Turkyilmazoglu⁵, where the heat transfer phenomena in magnetohydro-dynamic (MHD) fluid flow has been investigated. Under the transverse magnetic field influence, Khan et al.⁶ considered the non-Newtonian Powell-Eyring fluid over the rotating disk surface. The entropy generation due to porosity of rotating disk in MHD flow has been investigated by Rashidi et al.⁷. Hayat et al.⁸ scrutinized the transfer of heat with viscous nanoliquid among two stretchable rotating sheets. The thermal conductivity that depends on temperature in Maxwell fluid over a rotating disk has been studied by Khan et al.⁹. Batchelor¹⁰ was the first researcher, who discussed the fluid flow between the gaps of the rotating frame. The influence of blowing with wall transpiration, suction and mixed convection has investigated by Yan and Soong¹¹. Recently Shuaib et al.¹². studied the fractional behavior of fluid flow through a flexible rotating disk with mass and heat characteristics.

The attention of researcher's is increasing towards nanofluid studies day by day due to its many applications in technology that binging facilities in many industrial process of heat transfer. The applications of nanofluid are in drugs delivery, power generation, micromanufacturing process, metallurgical sectors, and thermal therapy, etc. Choi¹³ is a researcher who worked for the first time on nanofluid, where he considered it for cooling and coolant purpose in technologies. He found from his work that in a base fluid (water, oil and blood, etc.) by adding the nanoparticles, the heat transfer of thermal conductivity becomes more effective. Using the idea of Choi's idea, many researchers investigated and obtained results using the nanofluids^{14,15}. A concentric circular pipe with slip flow has been discussed in Turkyilmazoglu¹⁶. By using finite element method (FEM), Hatami et al.¹⁷ finds the solution for the heat transfer in nanofluid with free natural convective in a circular cavity. The Cattaneo-Christov heat flux and thermal radiation for an unsteady squeezing MHD flow has been considered by Ganji and Dogonchi¹⁸. They considered the heat of transfer of the nanofluid among two plates. Dilan et al.¹⁹ studied nanofluids effective viscosity based on suspended nanoparticles. A carbon nanotubes based multifunctional hybrid nanoliquid has been considered by Rossella²⁰. The influence of SWCNTs on human epithelial tissues is studied by Kaiser et al.²¹. Hussanan et al.²² examined the Oxide nanoparticles for the enhancement of energy in engine nanofluids, kerosene oil and water. Saeed et al.²³ examined nanofluid to improve the heat transfer rate and reduce time for food processing in the industry. Some recent studies related to heat and mass transfer through nanofluids are examined by many researchers^{24–28}.

To study the behavior, impact and properties of magnetic field over viscous fluids is known as MHD. Salt water, plasmas and electrolytes are the examples of magnetofluids. In the present era, the researchers and investigators are taking very keen interest in this field. A lot of work has been done in this area. The tectonic applications of

MHD in engineering, chemistry, physics, industrial tackle and in many other fields, for instance, pumps, bearings, MHD generators and boundary layer control are contrived by the intercourse of conducting fluid and magnetic field. In affiliation with these applications, the work of numerous explores has been deliberated. The most essential and consequential challenge is the hydro magnetic behavior of boundary layers with the magnetic field transversely along the moving surfaces or fixed surfaces. Hannes Alfén²⁹ was the first one to innovate the MHD field. In 1970 he received the Nobel Prize in physics because of his innovation in MHD field. In Medical Sciences the applications of MHD fluid flow in distinguishable configuration pertinent to human body parts are very fascinating and tectonic in the scientific area. The important applications of MHD in peristaltic flow, pulsatile flow, simple flow and drug delivery are explored by Rashidi *et al.*³⁰. The numerical solution has been presented by Nadeem *et al.*³¹ for the nanoparticles with different base fluids with slip and MHD effect. Khatsayuk *et al.*³² has explored the numerical simulation of MHD vortex technology and its verification is also ensured. The main of letters portrays casting principle into the electromagnetic mold to invoke small diameter ingots³³. Deng and W. M. Liu *et al.*^{34–37} have presented the numerical and theoretical analysis in a rotating Bose–Einstein of the quantized vortices condensate with modulated interaction in anharmonic and harmonic potentials. They further scrutinized the nonlinear matter of the quasi-2D Bose–Einstein condensates with nonlinearity in the harmonic potential. They concluded that all of the Bose–Einstein condensates have discrete energies with an arbitrary number of localized non-linear matter waves, which are the exact solutions of the mathematical Gross–Pitaevskii equation.

Our inspiration of the present work is to analyze and model the Maxwell nano liquid flow within two stretchable coaxially rotating disks. The second priority is to initiate three dimensional Maxwell equation along with the Navier stokes equation for such type of flow and set up an arrangement for temperature, concentration, velocity and magnetic strength profile. For comparative results the built-in numerical scheme bvp4c and RK4 are opting. We have extended the idea of Ahmed *ET al.*³⁸ and portrayed this mathematical model. The commitments flow factors on velocity, temperature, concentration, pressure and magnetic strength profile are studied and via graphical and in tabulated form. In the next section, the problem will be formulated and discussed.

Mathematical formulation of the problem. We assumed the nanoliquid steady motion within, the two horizontal parallel rotating disks. The disks are stretchable and subjected to magnetic field B_0 in axial direction. The upper disk is considered at a constant position $z = d$, while the lower disk is at $z = 0$. The stretching rate and velocity during rotation are (S_1, Ω_1) , while stretching rate and rotation velocity of upper disk are (S_2, Ω_2) . The concentration and temperature of the lower and upper disk are respectively given by (C_1, C_2) and (T_1, T_2) . The geometry of the considered problem is shown in Fig. 1. The governing equation of nanofluid flows are^{9,39}

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \quad (1)$$

$$\begin{aligned} u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{u^2}{r} = & -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left(2 \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial}{\partial r} \left(\frac{\partial w}{\partial z} \right) + \frac{2}{r} \frac{\partial u}{\partial r} - \frac{2u}{r^2} \right) \\ & - \lambda_1 \left(u^2 \frac{\partial^2 u}{\partial r^2} + w^2 \frac{\partial^2 u}{\partial z^2} + 2uw \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial z} \right) - \frac{2uv}{r} \frac{\partial v}{\partial r} - \frac{2vw}{r} \frac{\partial v}{\partial z} + \frac{uv^2}{r^2} + \frac{v^2}{r} \frac{\partial u}{\partial r} \right) \\ & - \frac{\sigma^2}{B_0} \rho (u + w \lambda_1 u_2), \end{aligned} \quad (2)$$

$$\begin{aligned} u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} - \frac{uv}{r} = & \nu \left(2 \frac{\partial^2 v}{\partial r^2} + \frac{v}{r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) - \lambda_1 \left(u^2 \frac{\partial^2 v}{\partial r^2} + w^2 \frac{\partial^2 v}{\partial z^2} \right) \\ & + 2uw \frac{\partial}{\partial r} \left(\frac{\partial v}{\partial z} \right) - \frac{2uv}{r} \frac{\partial v}{\partial r} - \frac{2vw}{r} \frac{\partial u}{\partial z} - 2 \frac{u^2 v}{r^2} + \frac{v^2}{r} \frac{\partial v}{\partial r} - \frac{\sigma^2}{B_0} \rho (v + w \lambda_1 v_2), \end{aligned} \quad (3)$$

$$\begin{aligned} u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = & -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left(\frac{\partial}{\partial r} \left(\frac{\partial u}{\partial z} \right) + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial r} + 2 \frac{\partial^2 w}{\partial z^2} \right) \\ & - \lambda_1 \left(u^2 \frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 v}{\partial z^2} + 2uw \frac{\partial}{\partial r} \left(\frac{\partial w}{\partial z} \right) + \frac{v^2}{r} \frac{\partial w}{\partial r} \right), \end{aligned} \quad (4)$$

$$\begin{aligned} (\rho c_p)_f \left(u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} \right) = & \frac{k(T)}{r} \frac{\partial T}{\partial r} + \frac{\partial}{\partial r} \left(k(T) \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial r} \left(k(T) \frac{\partial T}{\partial z} \right) \\ & + (\rho c_p)_p \left[D_B \left(\frac{\partial T}{\partial z} \frac{\partial C}{\partial z} + \frac{\partial T}{\partial r} \frac{\partial C}{\partial r} \right) + \frac{D_T}{T_2} \left\{ \left(\frac{\partial T}{\partial z} \right)^2 + \left(\frac{\partial T}{\partial r} \right)^2 \right\} \right], \end{aligned} \quad (5)$$

$$u \frac{\partial C}{\partial r} + w \frac{\partial C}{\partial z} = D_B \left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial z^2} \right) + \frac{D_B}{T_2} \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right), \quad (6)$$

$$-w \frac{\partial Br}{\partial z} - Br \frac{\partial w}{\partial z} + u \frac{\partial Bz}{\partial z} + Bz \frac{\partial u}{\partial z} + \frac{1}{\sigma \mu_2} \left(\frac{\partial^2 Br}{\partial r^2} + \frac{\partial^2 Br}{\partial z^2} + \frac{1}{r} \frac{\partial Br}{\partial r} - \frac{Br}{r^2} \right) = 0, \quad (7)$$

Figure 1. Geometry of the problem.

$$-u \frac{\partial B\theta}{\partial r} - B\theta \frac{\partial u}{\partial r} + v \frac{\partial Br}{\partial r} + Br \frac{\partial v}{\partial r} + v \frac{\partial Bz}{\partial z} + Bz \frac{\partial v}{\partial z} - w \frac{\partial B\theta}{\partial z} - B\theta \frac{\partial w}{\partial z} + \frac{1}{\sigma \mu_2} \left(\frac{\partial^2 B\theta}{\partial r^2} + \frac{\partial^2 B\theta}{\partial z^2} + \frac{1}{r} \frac{\partial B\theta}{\partial r} - \frac{B\theta}{r^2} \right) = 0, \quad (8)$$

$$w \frac{\partial Br}{\partial r} + Br \frac{\partial w}{\partial r} + \frac{1}{r} wBr - u \frac{\partial Bz}{\partial r} + Bz \frac{\partial u}{\partial r} - \frac{1}{r} uBz + \frac{1}{\sigma \mu_2} \left(\frac{\partial^2 Bz}{\partial r^2} + \frac{\partial^2 Bz}{\partial z^2} + \frac{1}{r} \frac{\partial Bz}{\partial r} \right) = 0, \quad (9)$$

where T represent the fluid temperature. The nanofluid heat capacity and base fluid specific heat are $(\rho C_p)_{nf}$ and $(\rho C_p)_f$ respectively. The heat flux q is defined as

$$q = -\nabla T k(T), \quad (10)$$

In which variable thermal conductivity $k(T)$ can be written as⁹

$$k(T) = k_\infty \left(1 + \varepsilon \frac{T - T_2}{T_1 - T_2} \right). \quad (11)$$

ε Is the parameter of variable thermal conductivity and k_∞ is the fluid thermal conductivity. The boundary conditions are:

$$\left. \begin{aligned} u = s_1 r, \quad v = \omega_1 r, \quad w = 0, \quad T = T_1, \quad C = C_1, \quad Br = 0, \quad Bz = 0 \quad \text{at } z = 0 \\ u = s_2 r, \quad v = \omega_2 r, \quad w = 0, \quad T = T_2, \quad C = C_2, \quad Br = \frac{dM_0}{2R}, \quad Bz = -\alpha M_0, \quad \text{at } z = d. \end{aligned} \right\} \quad (12)$$

Transformation. The transformation, which are adopted to make the system of PDE dimensionless are as follow³⁸:

$$\left. \begin{aligned} u &= r\Omega_1 f'(\eta), \quad v = r\Omega_1 g(\eta), \quad w = -2d\Omega_1 f(\eta), \quad \eta = \frac{z}{d} \\ p &= \rho\Omega_1 v(P(\eta) + \frac{1}{2} \frac{r^2}{d^2}), \quad \Theta(\eta) = \frac{T - T_1}{T_1 - T_2}, \quad \phi(\eta) = \frac{C - C_1}{C_1 - C_2} \\ Br &= r\Omega_1 M_0 M'(\eta), \quad B\theta = r\Omega_1 M_0 N(\eta), \quad Bz = M_0 \sqrt{(2\nu_f \Omega) M(\eta)}. \end{aligned} \right\} \quad (13)$$

The required dimensionless form of the system of differential equations given in Eqs. (1–9) are:

$$f''' = \frac{Re((f')^2 - g^2 - 2ff'') - Re(4ff'f'' - 4fgg') + MRe(f' - 2\beta_1 ff'') - \Lambda}{1 - 4Re\beta_1 f^2} \quad (14)$$

$$g'' = \frac{-2Re(fg' - f'g) - Re\beta_1(4ff'g' + 4ff''g) + MRe(g - 2\beta_1 fg')}{1 - 4Re\beta_1 f^2} \quad (15)$$

$$P' = -2f'' - Re(4ff' - 8\beta_1 f^2 f''), \quad (16)$$

$$\Theta'' = \frac{-2RePrf\Theta' - \varepsilon\Theta'^2 - PrNb\Theta'\Phi' + PrNt\Theta'^2}{1 + \varepsilon\Theta} \quad (17)$$

$$\Phi'' = -2ReScf\Phi' - \frac{Nt}{Nb}\Theta'', \quad (18)$$

$$M'''' = -2ReBt(Mf''' + f''M' - fM''' - M''f'), \quad (19)$$

$$N'' = 2ReBt(Mg' - fN'), \quad (20)$$

with condition

$$\left. \begin{aligned} f(0) &= 0, \quad f'(0) = S_1, \quad g(0) = 1, \quad P(0) = 1, \quad \Theta(0) = 1, \quad \Phi(0) = 1, \quad M'(0) = 0, \quad N(0) = 0, \\ \text{at } \eta &= 0 \\ f(1) &= 0, \quad f'(1) = S_2, \quad g(1) = \Omega, \quad \Theta(1) = 0, \quad \Phi(1) = 0, \quad M'(1) = 1, \quad N(1) = 1, \\ \text{at } \eta &= d \end{aligned} \right\} \quad (21)$$

The magnetic field M , Deborah number β_1 , lower and upper disks stretching parameters S_1 and S_2 , parameter of Brownian motion Nb , Reynolds number Re , thermophoresis parameter Nt and Schmidt number Sc are defined as:

$$\begin{aligned} M &= \frac{\sigma B_0^2}{\rho\Omega_1}, \quad \beta_1 = \lambda_1 \Omega_1, \quad S_1 = \frac{s_1}{\Omega_1}, \quad S_2 = \frac{s_2}{\Omega_2}, \quad Nb = \frac{D_B(C_1 - C_2)(\rho c_p)_p}{\nu(\rho c_p)_f}, \\ Nt &= \frac{D_B(T_1 - T_2)(\rho c_p)_p}{\nu T_2(\rho c_p)_f}, \quad Re = \frac{\Omega_1 d^2}{\nu}, \quad Sc = \frac{\nu}{D_B}. \end{aligned} \quad (22)$$

Sherwood and Nusselt numbers. The mass and rate of heat transfer for both disks can be illustrated as³⁸:

$$\begin{aligned} Sh_{r1} &= -\frac{h}{k(C_1 - C_2)} \left(\frac{\partial C}{\partial z} \right), \quad Nu_{r1} = -\frac{h}{k(T_1 - T_2)} \left(\frac{\partial T}{\partial z} \right), \quad \text{at } z = 0, \\ Sh_{r2} &= -\frac{h}{k(C_1 - C_2)} \left(\frac{\partial C}{\partial z} \right), \quad Nu_{r2} = -\frac{h}{k(T_1 - T_2)} \left(\frac{\partial T}{\partial z} \right), \quad \text{at } z = d. \end{aligned} \quad (23)$$

The dimensionless form of Sherwood and Nusselt numbers can be written as

$$\begin{aligned} Sh_{r1} &= -\Phi'(0), \quad Nu_{r1} = -\Theta'(0), \\ Sh_{r2} &= -\Phi'(1), \quad Nu_{r2} = -\Theta'(1). \end{aligned} \quad (24)$$

Figure 2. β_1 impact on axial $f(\eta)$, radial $f'(\eta)$ and azimuthal velocity $g(\eta)$, temperature $\Theta(\eta)$ and pressure profile $P(\eta)$, for S_2 , when $S_1 = 0.0$. dashed lines for $\Omega = 0.5$ and lines for $\Omega = -0.5$.

Graphical interpretation

Results and discussions. The governing equations of Non-Newtonian Maxwell nanofluid flow problem has been solved numerically using bvp4c scheme after using Karman's scaling approach. In this section the results are illustrated through tables and Figures to visualize the impact of different physical constraints on velocity, pressure, concentration, temperature and magnetic strength profile. Both cases of disks rotation, same ($\Omega = 0.5$) and in opposite direction ($\Omega = -0.5$) of rotation has been sketched in Figs. 2, 3, 4, 5, 6, 7, 8. The entire calculation has been performed by keeping the values of constraints as $Re = 4.0$, $M = 0.3$, $Nb = Nt = 0.3$, $\beta_1 = 0.2$, $S_1 = S_2 = 0.4$, $\varepsilon = 0.1$ and $Sc = 3.0$.

Figure 3. M impact on an axial $f(\eta)$, radial $f'(\eta)$ and azimuthal velocity $g(\eta)$ and temperature profile $\Theta(\eta)$, for S_2 , when $S_1 = 0.0$. dashed lines for $\Omega = 0.5$ and lines for $\Omega = -0.5$.

Figure 2a–e are plotted, in order to illustrate the influence of Deborah number β_1 on axial velocity profile $f(\eta)$, radial $f'(\eta)$ and azimuthal velocity $g(\eta)$, temperature $\Theta(\eta)$ and pressure profile $P(\eta)$ respectively. The fluid behaves as a solid substance with high Deborah number β_1 shown in Fig. 2a. That's why axial velocity reduces with the increases of β_1 . The fluid with low Deborah number possess less elastic property and vice versa illustrated in Fig. 2b,c. So the radial velocity and azimuthal velocity reduces with the improvement of β_1 . The fluid temperature rises with β_1 shown in Fig. 2d. The pressure profile of fluid decline with the rising values of Deborah number β_1 Fig. 2e.

Figure 3a–d demonstrate the behavior of axial velocity profile $f(\eta)$, radial $f'(\eta)$, azimuthal velocity $g(\eta)$ and the temperature $\Theta(\eta)$ versus magnetic parameter M . The axial velocity and radial velocity decline with the effects of magnetic parameter M see Fig. 3a,b. Because the magnetic field creates some resistive forces, which oppose the fluid velocity and as a result axial and radial velocity reduces. The same trend has been received of azimuthal velocity via M Fig. 3c. By the enhancement of magnetic strength on the fluid flow generate friction, which produces some amount of heat and as a result the average temperature of the fluid increases which is given in Fig. 3d. The dominance of Reynolds number against axial velocity, radial and azimuthal velocity is elaborated in Fig. 4a–c. Figure 4d elaborated to observe that the temperature field decline with the rising credit of Reynolds number (Re). The pressure profile of fluid also decline with the rising values of Reynolds number Fig. 4e.

The two different cases for S_2 , when the lower disk stretching rate is ($S_1 = 0$) and when it is ($S_1 = 0.5$) have been sketched in Figs. 5a,b and 6a,b. In both cases the axial and radial velocity of fluid decreases with the improving values of S_2 . While in azimuthal velocity an opposite seen has been observed, because by increasing stretching rate S_2 the kinematics energy of fluid increases which enhanced the azimuthal velocity $g(\eta)$ illustrated in Figs. 5c and 6c. Figures 5d and 6d are sketched to observe the upper disk stretching impact versus temperature profile, while keeping the lower disk stretching rate ($S_1 = 0$) and ($S_1 = 0.5$) respectively. When the disk stretch the fluid particle above the disk surface get some space and become relaxed for a while, as a result their temperature reduce, which causes the average temperature of fluid to reduce.

Figure 4. Re impact on axial $f(\eta)$, radial $f'(\eta)$ and azimuthal velocity $g(\eta)$, temperature profile $\Theta(\eta)$ and pressure profile $P(\eta)$, for S_2 , when $S_1 = 0.0$. dashed lines for $\Omega = 0.5$ and lines for $\Omega = -0.5$.

Figure 7a,b is drawn in order to reveal the impact of the parameters ε and Pr which represent respectively thermal conductivity and Prandtl number on temperature field $\Theta(\eta)$. From Fig. 7a, it is obvious that by increasing the thermal conductivity parameter ε , the temperature field will improve. Figure 7b demonstrate the inverse relation of Prandtl number Pr versus temperature profile, physically large Prandtl fluid have less thermal diffusivity while less Prandtl fluid have always high thermal diffusivity, that's why the temperature field and Prandtl number has inverse relation. Figure 7c,d are plotted to examine the influence of thermophoresis parameter Nt and Schmidt number Sc on $\Phi(\eta)$. The mass transfer rate reduces with the improvement of both thermophoresis parameter Nt and Schmidt number Sc .

The dominant behavior of Batchlor number Bt versus magnetic field has been illustrated in Fig. 8a. When Batchlor number is large, a less current will generates high induced magnetic field $M(\eta)$, while the opposite trend has been observed with the magnetic Reynolds number on magnetic field in Fig. 8b. The enhancement of Reynolds number reduces the magnetic fields $M(\eta)$.

Figure 5. S_2 impact on axial $f(\eta)$, radial $f'(\eta)$ and azimuthal velocity $g(\eta)$ and temperature profile $\Theta(\eta)$, for S_2 , when $S_1 = 0.0$. dashed lines for $\Omega = 0.5$ and lines for $\Omega = -0.5$.

Table 1 shows the comparison of our work with that in Turkyilmazoglu¹², Ahmed et al.³⁸ and Rogers and Lance⁴⁰ for different values of rotation parameter Ω , in case when $S_1 = S_2 = 0$. Table 2 is displayed for numerical outcomes of Reynolds number and rotation parameter Ω , while keeping the upper plate stretching rate $S_2 = 0$ and lower plate $S_1 = 0.5$. The results in Table 2 are also compared with published work¹². For the validity of the results two well-known best numerical approaching techniques Runge Kutta order four method and boundary value solver are compared in Table 3. The numerical outputs for Sherwood number Sh_{r1} and Nusselt number Nu_{r1} at lower disk are plotted in Table 3. By varying Prandtl number, thermal conductivity, magnetic field, Reynolds number, thermophoresis and upper disk stretching parameters, the Nusselt number for lower Nu_{r1} and upper disks Nu_{r2} are also calculated. In Table 4, the Nusselt number for lower Nu_{r1} and upper disks Nu_{r2} are calculated by varying Prandtl number, thermal conductivity, magnetic field, Reynolds number, thermophoresis and upper disk stretching parameters.

Conclusion

The present numerical model is intended to explore the behavior of Non-Newtonian (Maxwell) nanoliquid moving within two stretchable rotating disks subjected to axial magnetic field. The disks are separated from each other by fixed distance. The time dependent characteristics of thermal conductivity have been considered to scrutinize the heat transfer phenomena. The thermophoresis and Brownian motion features of nanoliquid are studied with Buongiorno model. The system of equations is solved numerically through Runge Kutta order four method and bvp4c. The concluded outputs are listed as:

- The rising credit of thermophoresis and Brownian motion positively affects the temperature field.
- It is examined that by varying the upper disk stretching, the axial flow changes its behavior to upper form lower disk.
- A significant change in tangential velocity and slight enhancement in temperature profile are observed with the rising values of upper disk stretching rate.
- The temperature field is enhanced with the variation in thermal conductivity and magnetic field parameters.
- The transfer of mass and heat rate is inclined at the lower disk surface with the Schmidt number.
- When the upper disk stretching rate become zero, the heat transport rate decline at lower disk surface, while incline at upper disk with the parameter ε (thermal conductivity).
- The radial, axial and azimuthal velocity decreases while temperature field increases with varying of β_1 (Deborah number).

Figure 6. S_2 impact on axial $f(\eta)$, radial $f'(\eta)$ and azimuthal velocity $g(\eta)$ and temperature profile $\Theta(\eta)$, for S_2 , when $S_1=0.5$. dashed lines for $\Omega=0.5$ and lines for $\Omega=-0.5$.

Appendix

Solution methodology. For the solution of the model numerically, we convert the high order system into a system of first order system of differential equations which can be easily solved by the method of Runge–Kutta order four schemes. In order to convert the system the following scales are considered:

$$\begin{aligned} \chi_1 &= f, \chi_2 = f', \chi_3 = f'', \chi_4 = g, \chi_5 = g', \chi_6 = P, \chi_7 = \theta, \chi_8 = \theta', \chi_9 = \phi, \chi_{10} = \phi', \\ M &= \chi_{11}, M' = \chi_{12}, M'' = \chi_{13}, M''' = \chi_{14}, N = \chi_{15}, N' = \chi_{16}. \\ \chi_1' &= \chi_2, \chi_2' = \chi_3, \\ \chi_3' &= \frac{Re(\chi_2^2 - \chi_4^2 - 2\chi_1\chi_3) - Re(4\chi_1\chi_2\chi_3 - 4\chi_1\chi_4\chi_5) + M_1Re(\chi_2 - 2\beta_1\chi_1\chi_3) - \Lambda}{1 - 4Re\beta_1\chi_1^2}, \\ \chi_4' &= \chi_5, \\ \chi_5' &= \frac{-2Re(\chi_1\chi_5 - \chi_2\chi_4) - Re\beta_1(4\chi_1\chi_2\chi_5 - 4\chi_1\chi_4\chi_3) + M_1Re(\chi_4 - 2\beta_1\chi_1\chi_5) - \Lambda}{1 - 4Re\beta_1\chi_1^2}, \\ \chi_6' &= -2\chi_3 - Re(4\chi_1\chi_2 - 8\beta_1\chi_1^2\chi_3), \chi_7' = \chi_8 \\ \chi_8' &= \frac{-2RePr\chi_1\chi_8 - \varepsilon\chi_8^2 - PrNb\chi_8\chi_{10} + PrNt\chi_8^2}{1 + \varepsilon\chi_7}, \\ \chi_9' &= \chi_{10}, \\ \chi_{10}' &= -2ReSc\chi_1\chi_{10} - \frac{Nt}{Nb}\chi_8', \\ \chi_{11}' &= \chi_{12}, \chi_{12}' = \chi_{13}, \chi_{13}' = \chi_{14}, \\ \chi_{14}' &= -2ReBt(\chi_{11}\chi_{13}' + \chi_3\chi_{12} - \chi_1\chi_{14} - \chi_{13}\chi_2), \\ \chi_{15}' &= \chi_{16}, \chi_{16}' = -2ReBt(\chi_{11}\chi_5 - \chi_1\chi_{16}). \end{aligned} \quad (25)$$

Figure 7. ϵ and Pr impact on temperature profile $\Theta(\eta)$, while Nt and Sc on concentration profile, for S_2 , when $S_1 = 0.0$. dashed lines for $\Omega = 0.5$ and lines for $\Omega = -0.5$.

Figure 8. Bt and Re_m impact on magnetic strength profile $M(\eta)$ for S_2 , when $S_1 = 0.0$. dashed lines for $\Omega = 0.5$ and lines for $\Omega = -0.5$.

Ω	-1.0	-0.8	-0.3	0.0	0.50
$F''(0)$					
Ref. ⁴⁰	0.06667000	0.08384000	0.10385000	0.09987000	0.06653000
Ref. ¹²	0.06667313	0.08384206	0.10385088	0.09987221	0.06653419
Ref. ³⁸	0.06667358	0.08384164	0.10385000	0.09987146	0.06653400
Present	0.06667723	0.08384354	0.10386000	0.09988248	0.06653500
$-G'(0)$					
Ref. ⁴⁰	2.00094000	1.80258000	1.30432000	1.00438000	0.50251000
Ref. ¹²	2.00094215	1.80257847	1.30432355	1.00437756	0.50251351
Ref. ³⁸	2.00094200	1.80257800	1.30432300	1.00437700	0.50251350
Present	2.00095250	1.80259000	1.30433300	1.00438600	0.50251750
Λ					
Ref. ⁴⁰	0.19993000	0.17184000	0.20636000	0.29924000	0.57458000
Ref. ¹²	0.19992538	0.17185642	0.20635721	0.29923645	0.57457342
Ref. ³⁸	0.19992651	0.17185728	0.20635898	0.29923784	0.57457377
Present	0.19997752	0.17186723	0.20635981	0.29923843	0.57457499

Table 1. For various values of rotation parameter Ω the comparison of $-G'(0)$, $-F''(0)$ and Λ has been shown for the case when $S_2 = S_1 = 0$.

Re	Ω	$F''(0)$		$-G'(0)$	
		Ref. ¹²	Present	Ref. ¹²	Present
0	-0.5	-2.00000007	-2.00000000	1.50000000	1.50000000
10	-0.5	-1.60562889	-1.60563754	3.40116128	3.40117328
0	0.0	-2.00000007	-2.00000000	1.00000000	1.00000000
10	0.0	-1.44561724	-1.44561896	2.56217438	2.56218932
0	0.5	-2.00000007	-2.00000000	0.50000000	0.50000000
10	0.5	-1.89459839	-1.89459945	1.50020105	1.50022800

Table 2. The comparison of $-G'(0)$, $-F''(0)$ and Λ for different values of Re and Ω in case, when $S_1 = 0.5$, $S_2 = 0.0$.

Sh_{r1}			Nu_{r1}		
Pr	Bvp4c	RK4	Sc	Bvp4c	RK4
2.0	0.8718294	0.8718295	2.0	1.557996	1.557995
3.0	0.7999895	0.7999895	2.5	1.631221	1.631220
4.0	0.7290099	0.7290098	3.0	1.696949	1.696949
5.0	0.6632382	0.6632380	3.5	1.785639	1.785639

Table 3. The comparison of RK4 and Bvp4c for Sherwood Sh_{r1} and Nusselt number Nu_{r1} at the lower disk, when $M = 1.2$, $Sc = 3.0$, $Pr = 3.0$, $Nt = Nb = 0.3$, $Re = 4.0$, $\beta_1 = 0.2$, $S_1 = S_2 = 0.5$.

The transform conditions are:

$$\left. \begin{aligned} \chi_1(0) = 0, \quad \chi_2(0) = S_1, \quad \chi_4(0) = 1, \quad \chi_6(0) = 1, \quad \chi_7(0) = 1, \quad \chi_9(0) = 1, \quad \chi_{12}(0) = 0, \quad \chi_{15}(0) = 0, \\ \text{at } \eta = 0 \\ \chi_1(1) = 0, \quad \chi_2(1) = S_2, \quad \chi_4(1) = \Omega, \quad \chi_7(1) = 0, \quad \chi_9(1) = 0, \quad \chi_{12}(1) = 1, \quad \chi_{15}(1) = 1, \\ \text{at } \eta = d \end{aligned} \right\} \quad (26)$$

S_2	M	Re	Pr	ε	Nt	Nu_{r1}	Nu_{r2}
0.0	2.0	6.0	2.5	0.5	0.2	0.969396	1.444753
0.3						0.829980	1.906798
0.6						0.715537	2.419063
0.0	1.0					0.996962	1.393135
	1.5					0.982571	1.422293
	2.0					0.969396	1.444753
	2.0	0.0				0.722765	1.828593
		3.0				0.867816	1.586953
		6.0				0.922574	1.379949
		6.0	1.0			0.957529	1.3392912
			3.0			0.982961	1.486556
			5.0			0.997661	1.569729
			2.0	0.0		1.149913	1.069889
				0.3		0.926457	1.335418
				0.6		0.947561	1.498999
				0.5	0.2	0.969396	1.444753
					0.4	0.848575	1.682722
					0.6	0.739949	0.938282

Table 4. The Nusselt numbers at lower Nu_{r1} and upper Nu_{r2} disks respectively, when $S_1 = 0.5$, $\Omega = 0.5$, $\beta_1 = 0.2$, $Sc = 1.0$, $Nb = 0.3$.

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Author contributions

M. B and M.A.K. wrote the original manuscript and obtained the theoretical as well as the numerical solutions. A. A and M.I.A verified the results. M. B. M. A. K, A. A. and M. I. revised the results and approved the final draft of the work.

Competing interests

The authors declare no competing interests.

Additional information

Correspondence and requests for materials should be addressed to M.A.K.

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