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A laser pulse impactful on a half-space using the modified TPL G–N models

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This article aims to investigate the wave propagation of generalized thermoelastic half-plane under the effect of thermal loading due to laser pulse with and without energy dissipation. The normal mode method is proposed to solve the problem and get numerical results for the field quantities. The outcomes of the physical quantities have been illustrated graphically and reported to compare the simple Green–Naghdi II and III and their modified single-, dual-, and three-phase-lag models. The graphical outcomes indicate that the different types of Green–Naghdi models with thermal relaxations have great effects on the temperature, displacements, dilatation and stresses.

Coupling between mechanical and thermal fields have not occurred in the classical theory of thermoelasticity and so one needs more coupled and generalized theories. The coupled thermoelasticity theory of Biot¹ thinks about the trading of mechanical energy and the thermal energy but one still needs the generalized theories. One of the most important generalized thermoelasticity theories is the theory of Green–Naghdi (G–N)^{2–4}. This theory is a consistent one that considers elastic and thermal waves associated with the second sound. A lot of researchers dealt with various theoretical and practical features in thermoelasticity, in the context of the G–N models of type II or/and of type III.

Excitation of thermoelastic waves by a pulsed laser in a continuum is of incredible enthusiasm because of broad utilization of pulsed laser advancements in material handling and non-destructive recognizing and characterization. At the point when the continuum is illuminated with a laser pulse, assimilation of the laser pulse brings about a restricted temperature increment, which thus causes thermal extension and creates a thermoelastic wave in the medium. Deswal *et al.*⁵ studied the vibrations induced by a laser beam in the context of generalized magneto-thermoelasticity for isotropic and homogeneous elastic solids under G–N model in the x - z plane. Youssef and El-Bary⁶ derived the induced temperature and stress fields in an elastic half-space heated by a non-Gaussian laser beam with the pulse in the context of different coupled thermoelasticity theories. Othman *et al.*⁷ studied the rotation of initially stressed thermoelastic half-space with voids under thermal loading due to laser pulse in the context of G–N theory. Zenkour and Abouelregal⁸ investigated the vibration analysis of a nanobeam under a sinusoidal pulse varying heat in the context of a unified generalized nonlocal thermoelasticity theory with dual-phase-lag (DPL).

Othman and Tantawi⁹ investigated the impact of the gravitational field on a 2D thermoelastic solid affected by thermal loading because of a laser pulse. Abbas and Marin¹⁰ considered the problem of a 2D thermoelastic half-space by pulsed laser heating with regards to the generalized thermoelastic theory with one relaxation time. Ailawalia *et al.*¹¹ presented the 2D deformation under the impact of laser pulse heating in a thermo microstretch elastic medium at the interface of thermoelastic solid in the context of G–N theory. Othman and Marin¹² discussed the wave propagation of generalized thermoelastic half-space with voids under the impact of thermal loading because of a laser pulse with energy dissipation. Mondal *et al.*¹³ analyzed the effect of the laser pulse as a heat source utilizing a memory-dependent derivative with regards to three thermoelastic theories. Ailawalia and Singla¹⁴ dealt with the 2D deformation of laser pulse heating in a thermoelastic micro-elongated layer immersed in an infinite non-viscous fluid. Othman and Abd-Elaziz¹⁵ studied the impact of thermal loading because of a laser pulse in generalized thermoelastic half-space with voids in a DPL theory.

This article presents the temperature, displacements and stresses of a thermoelastic half-space under the impact of thermal loading because of a laser pulse. The material of the present thermoelastic half-space is homogeneous and isotropic and the medium itself is heated by a non-Gaussian laser beam with pulse duration. The

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normal mode method is proposed to obtain the numerical outcomes for the temperature, displacements, dilatation and stresses. These variables have been illustrated graphically by comparison between Green–Naghdi theory of both types II and III to show the advantages presented by the present modified models.

Different thermoelasticity models

In what follows we present a unified three-phase lag (TPL) Green–Naghdi heat conduction equation. Let the temperature change is small enough compared to the reference temperature, that is $\theta \rightarrow T_0$. So, the heat conduction equation can be simplified as (Zenkour^{16–21})

$$k\mathcal{L}_1(\nabla^2\theta) = \mathcal{L}_2(\rho C_e\theta + \gamma T_0 e) - \mathcal{L}_0(\rho Q_0). \quad (1)$$

In addition, the time differential operators $\mathcal{L}_i$ ($i=0, 1, 2$) are given by

$$\begin{aligned} \mathcal{L}_1 &= \left(1 + \sum_{n=1}^N \frac{\tau_\theta^n}{n!} \frac{\partial^n}{\partial t^n}\right) \frac{\partial}{\partial t} + \frac{\epsilon k^*}{k} \left(1 + \sum_{n=1}^N \frac{\tau_\theta^n}{n!} \frac{\partial^n}{\partial t^n}\right), \\ \mathcal{L}_0 &= \left(1 + \sum_{n=1}^N \frac{\tau_q^n}{n!} \frac{\partial^n}{\partial t^n}\right) \frac{\partial}{\partial t}, \quad \mathcal{L}_2 = \mathcal{L}_0 \frac{\partial}{\partial t}, \end{aligned} \quad (2)$$

in which ϵ is a dimensionless key number, may equal only to zero or one. Also, the thermal relaxation time parameters τ_q , τ_θ and τ_ϑ are the thermal memories with $0 \leq \tau_\vartheta < \tau_\theta < \tau_q$. Equation (1) is more general when N has different integers greater than zero. Some special cases may be obtained from the above relations as

- (i) TPL G–N III model ($\epsilon=1, N \geq 1$).
- (ii) DPL G–N III model ($\epsilon=1, \tau_\theta=0, N \geq 1$).
- (iii) SPL G–N III model ($\epsilon=1, \tau_\theta=\tau_\vartheta=0, N \geq 1$).
- (iv) DPL G–N II model ($\epsilon=0, N \geq 1$):

$$k \left(1 + \sum_{n=1}^N \frac{\tau_\theta^n}{n!} \frac{\partial^n}{\partial t^n}\right) (\nabla^2\theta) = \left(1 + \sum_{m=1}^N \frac{\tau_q^m}{m!} \frac{\partial^m}{\partial t^m}\right) \left[\frac{\partial}{\partial t} (\rho C_e\theta + \gamma T_0 e) - \rho Q_0 \right]. \quad (3)$$

- (v) SPL G–N II model ($\epsilon=0, \tau_\theta=0, N \geq 1$):

$$k(\nabla^2\theta) = \left(1 + \sum_{m=1}^N \frac{\tau_q^m}{m!} \frac{\partial^m}{\partial t^m}\right) \left[\frac{\partial}{\partial t} (\rho C_e\theta + \gamma T_0 e) - \rho Q_0 \right]. \quad (4)$$

- (vi) Simple G–N III model ($\epsilon=1, \tau_q=\tau_\theta=\tau_\vartheta=0$):

$$\left(k \frac{\partial}{\partial t} + k^*\right) \nabla^2\theta = \frac{\partial}{\partial t} \left[\frac{\partial}{\partial t} (\rho C_e\theta + \gamma T_0 e) - \rho Q_0 \right]. \quad (5)$$

- (vii) Simple G–N II model ($k \rightarrow 0, \epsilon=1, \tau_q=0$):

$$k^* \nabla^2\theta = \frac{\partial}{\partial t} \left[\frac{\partial}{\partial t} (\rho C_\vartheta\theta + \gamma T_0 e) - \rho Q_0 \right], \quad (6)$$

or

- (viii) Simple G–N II model ($\epsilon=0, \tau_q=\tau_\theta=0$):

$$k \nabla^2\theta = \frac{\partial}{\partial t} (\rho C_\vartheta\theta + \gamma T_0 e) - \rho Q_0. \quad (7)$$

It is to be noted that Eqs. (6) and (7) represent two forms of the simple G–N II model, the first is in terms of the rate of thermal conductivity k^* while the second is in terms of the heat conductivity coefficient k . A lot of investigators have dealt with the simple G–N II and III models while other investigators have dealt with the TPL G–N III model ($N=1$)^{22–41}. All these models are presented without the higher-order time derivatives as those presented in this study.

Basic equations

Consider a thermoelastic problem of a half-space medium as shown in Fig. 1 in with regards to the multi-dual-phase-lag theory. The present half-space is characterized in the region Ψ as follows:

$$\Psi = \{(x, y, z) \mid 0 \leq x < \infty, -\infty < y < \infty, z = 0\}. \quad (8)$$

Here, all variables will be depending on t, x and y . So, our analysis has been taken in the 2D xy -plane. The displacement vector can be taken in the form $\vec{u} = (u, v, 0)$, where u and v are the horizontal and vertical components. Thus, the displacements u_i will be

$$u_1 = u(x, y, t), \quad u_2 = v(x, y, t), \quad u_3 = 0. \quad (9)$$

Figure 1. Geometry of the thermoelastic half-space.

The equations of motion are expressed as

$$\sigma_{ij,j} + F_i = \rho \ddot{u}_i. \quad (10)$$

The constitutive equations will be simplified to

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e - \gamma\theta)\delta_{ij}. \quad (11)$$

For the present problem one can summarize the governing equations in the form

$$\begin{aligned} \mu \nabla^2 u + (\lambda + \mu) \frac{\partial e}{\partial x} - \gamma \frac{\partial \theta}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2}, \\ \mu \nabla^2 v + (\lambda + \mu) \frac{\partial e}{\partial y} - \gamma \frac{\partial \theta}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2}, \end{aligned} \quad (12)$$

$$k \mathcal{L}_1(\nabla^2 \theta) = \mathcal{L}_2(\rho C_e \theta + \gamma T_0 e) - \mathcal{L}_0(\rho Q_0). \quad (13)$$

The plate surface is lit up by the laser pulse given by the heat input

$$Q_0 = \frac{I_0 \gamma_0}{2\pi r^2} \exp\left(-\frac{y^2}{r^2} - \gamma_0 x\right) f_0(t). \quad (14)$$

The temporal profile $f(t)$ can be defined as

$$f_0(t) = \frac{t}{t_0^2} \exp\left(-\frac{t}{t_0}\right). \quad (15)$$

In the accompanying relations, it is advantageous to report the dimensionless variables in the form:

$$\begin{aligned} \{x', y', r'\} &= \frac{\{x, y, r\}}{\eta}, \quad \{u', v'\} = \frac{\lambda + 2\mu}{\eta \gamma T_0} \{u, v\}, \quad e' = \frac{\lambda + 2\mu}{\gamma T_0} e, \\ \{t', \tau_0', \tau_\theta', \tau_q'\} &= \frac{c_0}{\eta} \{t, \tau_0, \tau_\theta, \tau_q\}, \quad \gamma_0' = \eta \gamma_0, \quad \sigma_{ij}' = \frac{\sigma_{ij}}{\gamma T_0}, \quad \theta' = \frac{\theta}{T_0} \end{aligned} \quad (16)$$

where $(k^*)' = \frac{\eta k^*}{c_0 \rho}$, $Q_0' = \frac{c_0 \rho Q_0}{k T_0}$, $c_0^2 = \frac{\lambda + 2\mu}{\rho}$ and $\eta = \frac{k}{\rho c_0 C_e}$. All governing equations, with the above non-dimensions, are reduced to (dropping the dashed for comfort)

$$\begin{aligned} \sigma_{11} &= \frac{\partial u}{\partial x} + c_1 \frac{\partial v}{\partial y} - \theta, \\ \sigma_{12} &= c_1 \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \\ \sigma_{22} &= c_1 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \theta, \end{aligned} \quad (17)$$

$$\begin{aligned} \nabla^2 u + c_2 \frac{\partial e}{\partial x} - c_3 \frac{\partial \theta}{\partial x} &= c_3 \frac{\partial^2 u}{\partial t^2}, \\ \nabla^2 v + c_2 \frac{\partial e}{\partial y} - c_3 \frac{\partial \theta}{\partial y} &= c_3 \frac{\partial^2 v}{\partial t^2}, \end{aligned} \quad (18)$$

$$\mathcal{L}_1(\nabla^2\theta) = \mathcal{L}_2(\theta + c_4e) + \bar{Q}_0, \quad (19)$$

where

$$\begin{aligned} c_1 &= \frac{\lambda}{\lambda + 2\mu}, \quad c_2 = \frac{\lambda + \mu}{\mu}, \quad c_3 = \frac{\lambda + 2\mu}{\mu}, \quad c_4 = \frac{T_0\gamma^2}{c_0^2C_e\rho^2}, \\ \bar{Q}_0 &= \frac{I_0\gamma_0q_t}{2\pi r^2t_0^3} \exp\left(-\frac{y^2}{r^2} - \frac{t}{t_0}\right) \exp(-\gamma_0x), \\ q_t &= \sum_{n=0}^N \frac{\tau_q^n [t - (n+1)t_0]}{(-t_0)^n n!}. \end{aligned} \quad (20)$$

Solution of the problem

To obtain the total solutions of the physical amounts, the normal mode method is applied. The physical fields in this system are shown as

$$\{u, v, \theta, \sigma_{ij}\}(x, y, t) = \{u^*, v^*, \theta^*, \sigma_{ij}^*\}(x) \exp(\omega t + iby), \quad (21)$$

where $\omega = \omega_0 + i\omega_1$ in which ω_0 and ω_1 are constants, $i = \sqrt{-1}$, b represents the wave number in y -direction while the values of $u^*(x)$, $v^*(x)$, $\theta^*(x)$ and $\sigma_{ij}^*(x)$ are the amplitudes of the quantities $u(x)$, $v(x)$, $\theta(x)$ and $\sigma_{ij}(x)$, respectively.

Applying the normal mode method on Eqs. (18) and (19), we have the following system of three ordinary homogeneous differential equations:

$$(\mathcal{D}^2 - c_5)u^* + c_6v^* - c_7\mathcal{D}\theta^* = 0, \quad (22)$$

$$(\mathcal{D}^2 - c_8)v^* + c_9\mathcal{D}u^* - c_{10}\theta^* = 0, \quad (23)$$

$$(\mathcal{D}^2 - c_{11})\theta^* - c_{12}\mathcal{D}u^* - c_{13}v^* = g(y, t) \exp(-\gamma_0x), \quad (24)$$

where

$$\begin{aligned} c_5 &= \frac{b^2 + c_3\omega^2}{1 + c_2}, \quad c_6 = \frac{ibc_2}{1 + c_2}, \quad c_7 = \frac{c_3}{1 + c_2}, \quad c_8 = b^2(1 + c_2) + c_3\omega^2, \\ \{c_9, c_{10}\} &= ib\{c_2, c_3\}, \quad c_{11} = b^2 + \frac{\varpi_2}{\varpi_1}, \quad c_{12} = c_4\frac{\varpi_2}{\varpi_1}, \quad c_{13} = ibc_{12}, \quad \mathcal{D} = \frac{d}{dx}, \\ g(y, t) &= \frac{I_0\gamma_0q_t}{2\pi r^2t_0^3\varpi_1} \exp\left(-\frac{y^2}{r^2} - iby - \frac{1 + \omega t_0}{t_0}t\right), \end{aligned} \quad (25)$$

in which

$$\varpi_1 = \omega \left(1 + \sum_{n=1}^N \frac{\tau_\theta^n}{n!} \omega^n\right) + \frac{\epsilon k^*}{k} \left(1 + \sum_{n=1}^N \frac{\tau_\theta^n}{n!} \omega^n\right), \quad \varpi_2 = \omega^2 \left(1 + \sum_{n=1}^N \frac{\tau_q^n}{n!} \omega^n\right). \quad (26)$$

The system of differential Eqs. (22)–(24) may be given in a unified form as

$$(D^6 - B_2D^4 + B_1D^2 - B_0)\{u^*, v^*, \theta^*\}(x) = \{Q_1, Q_2, Q_3\} g(y, t) \exp(-\gamma_0x), \quad (27)$$

where

$$\begin{aligned} Q_1 &= -c_7\gamma_0^3 + (c_6c_{10} + c_7c_8)\gamma_0, \\ Q_2 &= (c_{10} - c_7c_9)\gamma_0^2 - c_5c_{10}, \\ Q_3 &= \gamma_0^4 - (c_6c_9 + c_5 + c_8)\gamma_0^2 + c_5c_8, \end{aligned} \quad (28)$$

and the coefficients B_i are given by

$$\begin{aligned} B_0 &= c_5(c_8c_{11} - c_{10}c_{13}), \\ B_1 &= c_5(c_8 + c_{11}) + c_6(c_9c_{11} + c_{10}c_{12}) + c_7(c_8c_{12} + c_9c_{13}) + c_8c_{11} - c_{10}c_{13}, \\ B_2 &= c_6c_9 + c_7c_{12} + c_5 + c_8 + c_{11}. \end{aligned} \quad (29)$$

The complete solutions of the system appeared in Eq. (27) of the considered physical quantities bound as $x \rightarrow \infty$ will be on the form

$$\{u^*, v^*, \theta^*\}(x) = \sum_{j=1}^3 \{H_j, H_j', H_j''\} e^{-\beta_j x} + \{\bar{Q}_1, \bar{Q}_2, \bar{Q}_3\} g(y, t) \exp(-\gamma_0 x), \quad (30)$$

where H_j, H_j' and H_j'' are the integration parameters and β_j ($j = 1, 2, 3$) are the +ve roots of the characteristic equations

$$\beta_j^6 - B_2 \beta_j^4 + B_1 \beta_j^2 - B_0 = 0. \quad (31)$$

The roots β_j are given respectively by

$$\begin{aligned} \beta_1 &= \frac{1}{\sqrt{6B_3}} \sqrt{2B_2 B_3 + 4(B_2^2 - 3B_1) + B_3^2}, \\ \beta_{2,3} &= \frac{1}{2\sqrt{3B_3}} \sqrt{4B_2 B_3 - 4(1 \pm i\sqrt{3})(B_2^2 - 3B_1) - (1 \mp i\sqrt{3})B_3^2}, \end{aligned} \quad (32)$$

in which

$$\begin{aligned} B_3^3 &= 108B_0 + 8B_2^3 - 36B_1 B_2 + 12B_4, \\ B_4^2 &= 81B_0^2 + 12B_1^3 - 54B_0 B_1 B_2 + 12B_0 B_2^3 - 3B_1^2 B_2^2. \end{aligned} \quad (33)$$

Also, the parameter $\bar{Q}_i$ in Eq. (30) can be represented as

$$\bar{Q}_i = \frac{Q_i}{\gamma_0^6 - B_2 \gamma_0^4 + B_1 \gamma_0^2 - B_0}, \quad i = 1, 2, 3. \quad (34)$$

The relations between the parameters H_j, H_j' and H_j'' can be obtained by using Eq. (30) into the equations of motion appeared in Eqs. (22) and (23):

$$\{H_j', H_j''\} = \{\alpha_j^1, \alpha_j^2\} H_j, \quad j = 1, 2, 3, \quad (35)$$

where

$$\alpha_j^1 = \frac{(c_{10} - c_7 c_9) \beta_j^2 + c_5 c_{10}}{\beta_j [c_6 c_{10} - c_7 (c_8 + \beta_j^2)]}, \quad \alpha_j^2 = \frac{\beta_j^4 + (c_5 + c_8 - c_6 c_9) \beta_j^2 + c_5 c_8}{\beta_j [c_6 c_{10} - c_7 (c_8 + \beta_j^2)]}, \quad j = 1, 2, 3. \quad (36)$$

Therefore, the displacements and temperature are given in their final form as

$$\begin{aligned} \{u, v, \theta\} &= \sum_{j=1}^3 H_j \{1, \alpha_j^1, \alpha_j^2\} \exp(\omega t - \beta_j x + i b y) \\ &\quad + \{\bar{Q}_1, \bar{Q}_2, \bar{Q}_3\} \bar{g}(y, t) \exp(-\gamma_0 x), \end{aligned} \quad (37)$$

where

$$\bar{g}(y, t) = \frac{I_0 \gamma_0 q_t}{2\pi r^2 t_0^3 \varpi_1} \exp\left(-\frac{y^2}{r^2} - \frac{t}{t_0}\right). \quad (38)$$

In addition, the stresses in their final form may be simplified as

$$\begin{aligned} \{\sigma_{11}, \sigma_{12}, \sigma_{22}\} &= \sum_{j=1}^3 H_j \{A_j^1, A_j^2, A_j^3\} \exp(\omega t - \beta_j x + i b y) \\ &\quad - \{\hat{Q}_1, \hat{Q}_2, \hat{Q}_3\} \bar{g}(y, t) \exp(-\gamma_0 x), \end{aligned} \quad (39)$$

where

$$\begin{aligned} A_j^1 &= -\beta_j + i b c_1 \alpha_j^1 - \alpha_j^2, \quad \hat{Q}_1 = \gamma_0 \bar{Q}_1 + \bar{Q}_3 + \frac{2c_1 y}{r^2} \bar{Q}_2, \\ A_j^2 &= c_1 (i b - \beta_j \alpha_j^1), \quad \hat{Q}_2 = c_1 \left(\frac{2y}{r^2} \bar{Q}_1 + \gamma_0 \bar{Q}_2 \right), \\ A_j^3 &= -c_1 \beta_j + i b \alpha_j^1 - \alpha_j^2, \quad \hat{Q}_3 = c_1 \gamma_0 \bar{Q}_1 + \bar{Q}_3 + \frac{2y}{r^2} \bar{Q}_2. \end{aligned} \quad (40)$$

Thermomechanical conditions

The boundary conditions on the surface of the half-space medium can be applied to get the parameters H_j ($j = 1, 2, 3$). The positive exponentials are taken boundless at infinity in this physical problem. Concerning the mechanical boundary conditions, we have:

		θ	u	$\bar{v}$	e	σ_{11}	σ_{12}	σ_{22}
	Simple	1.512372	1.591605	1.301214	2.757593	6.519950	1.235728	8.506570
SPL	N = 1	2.952816	1.706902	2.180816	2.363967	6.482731	1.371582	8.856741
	N = 2	3.365649	1.780184	2.523733	2.462019	6.687988	1.456905	9.142401
	N = 3	3.354514	1.794955	2.563222	2.534689	6.766222	1.473206	9.206851
	N = 4	3.332676	1.794690	2.554985	2.544888	6.773250	1.472594	9.207583
	N = 5	3.329662	1.794288	2.552766	2.544662	6.772398	1.472103	9.206117
	N = 6	3.329632	1.794242	2.552618	2.544470	6.772178	1.472051	9.205920
DPL	N = 1	1.772513	1.608112	1.461717	2.662211	6.486078	1.255494	8.543088
	N = 2	1.877064	1.627437	1.550205	2.668080	6.534818	1.275813	8.619095
	N = 3	1.871791	1.633587	1.560353	2.692088	6.566787	1.281716	8.648939
	N = 4	1.862517	1.633615	1.556018	2.697346	6.570842	1.281590	8.650457
	N = 5	1.861188	1.633413	1.554971	2.697398	6.570431	1.281372	8.649705
	N = 6	1.861193	1.633386	1.554917	2.697302	6.570297	1.281346	8.649577

Table 1. Effect of G–N II thermoelasticity theories on all quantities of the medium.

Figure 2. Effect of G–N models on the temperature θ of the medium.

(i) The traction load can be applied on the plane surface $x = 0$ and takes the value σ_0 in normal direction:

$$\sigma_{11}(0, y, t) = f(y, t) = \sigma_0 \exp(\omega t + iby). \quad (41)$$

(ii) The tangent traction is free

$$\sigma_{12}(0, y, t) = 0. \quad (42)$$

(iii) The thermal boundary on the surface $x = 0$ is uniform. That is

$$\theta(0, y, t) = \theta_0. \quad (43)$$

Therefore, using Eqs. (37)₃, (39)₁ and (39)₂ for θ , σ_{11} and σ_{12} , respectively, the parameters H_j can be determined by solving the following relations:

$$\begin{Bmatrix} H_1 \\ H_2 \\ H_3 \end{Bmatrix} = \begin{bmatrix} A_1^1 & A_2^1 & A_3^1 \\ A_1^2 & A_2^2 & A_3^2 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{bmatrix}^{-1} \begin{Bmatrix} \sigma_0 - \hat{Q}_1 g(y, t) \\ -\hat{Q}_2 g(y, t) \\ \theta_0 - \bar{Q}_3 g(y, t) \end{Bmatrix}. \quad (44)$$

Validation and applications

Some applicable examples will be presented to put into suggestion the impact of different models on the variable quantities. The material properties of the annular disk are mentioned according to the following values of parameters:

		θ	u	$\bar{v}$	e	σ_{11}	σ_{12}	σ_{22}
	Simple	1.250421	1.688300	1.367425	3.175144	7.078673	1.319450	9.003136
SPL	N=1	2.825331	1.852343	2.541629	3.028455	7.242929	1.524866	9.502878
	N=2	3.155114	1.933443	2.942543	3.210221	7.510762	1.621049	9.803943
	N=3	3.084529	1.945332	2.958523	3.285629	7.590957	1.631951	9.859016
	N=4	3.054645	1.944116	2.942560	3.292630	7.595473	1.629742	9.857132
	N=5	3.051885	1.943642	2.939768	3.291854	7.594193	1.629145	9.855498
	N=6	3.052016	1.943601	2.939672	3.291644	7.593959	1.629105	9.855321
DPL	N=1	2.417869	1.836434	2.323061	3.145307	7.315904	1.497349	9.482311
	N=2	2.627620	1.903557	2.628105	3.306441	7.553721	1.573455	9.737820
	N=3	2.553403	1.913119	2.629475	3.370931	7.623653	1.581172	9.785402
	N=4	2.526651	1.911881	2.613832	3.376771	7.627193	1.579016	9.783421
	N=5	2.524440	1.911445	2.611392	3.376033	7.625950	1.578488	9.781894
	N=6	2.524596	1.911409	2.611337	3.375843	7.625733	1.578456	9.781732
TPL	N=1	1.640052	1.713541	1.627332	3.071965	7.038972	1.353852	9.052592
	N=2	1.774255	1.739417	1.757544	3.093316	7.106089	1.382411	9.147667
	N=3	1.756790	1.746352	1.766877	3.123781	7.144858	1.388677	9.180943
	N=4	1.743602	1.746103	1.759818	3.128889	7.148787	1.388068	9.181799
	N=5	1.742150	1.745849	1.758455	3.128719	7.148177	1.387782	9.180894
	N=6	1.742212	1.745821	1.758414	3.128602	7.148027	1.387757	9.180764

Table 2. Effect of G–N III thermoelasticity theories on all quantities of the medium.

Figure 3. Effect of G–N models on the displacement u of the medium.

$\lambda = 7.76 \times 10^{10} \text{ Nm}^{-2}$, $\mu = 3.86 \times 10^{10} \text{ Nm}^{-2}$, $k = 386 \text{ Wm}^{-1} \text{ K}^{-1}$, $\rho = 8954 \text{ kg m}^{-3}$, $\alpha_t = 1.78 \times 10^{-5} \text{ K}^{-1}$, $C_e = 383.1 \text{ J kg}^{-1} \text{ K}^{-1}$, $T_0 = 293 \text{ K}$, $k^* = 1.5$, $I_0 = 10^2 \text{ J m}^{-2}$, $r = 0.2 \text{ } \mu\text{m}$, $\gamma_0 = 25 \text{ m}^{-1}$, $t_0 = 10 \text{ ns}$, $\theta_0 = 10$, $\sigma_0 = 1$.

For convenience, the real values of the field quantities have been adopted to represent the outcomes. Numerical results are obtained for $\omega = 1.9 + 2.9i$, $\tau_q = 0.2$, $\tau_\theta = 0.15$, $\tau_0 = 0.1$, $b = \frac{\pi}{3}$ and $t = 0.3$. Results of all variable field quantities of the half-space due to G–N II, SPL G–N II and DPL G–N II thermoelasticity theories are reported in Table 1 for a fixed $y = 2$ and different values of x . That is $\theta(1.5)$, $u(0.6)$, $\bar{v} = 10 \text{ v}(1.5)$, $e(0.5)$, $\sigma_{11}(0.7)$, $\sigma_{22}(0.7)$ and $\sigma_{12}(1.0)$. Similar results of the variable field quantities based on G–N III, SPL G–N III, DPL G–N III, and TPL G–N III thermoelasticity theories are reported in Table 2. Additional sample 1D and 2D graphs in the half-space are plotted in Figs. 1–10 to investigate the effect of all models on the field quantities. The results reported in both tables will serve as benchmarks for other investigators. It is concluded from these tables that:

- The simple G–N II yields the smallest temperature θ , displacements u and $\bar{v}$, stresses, σ_{22} and σ_{12} while it yields the greatest dilatation e .
- The modified G–N II and G–N III may be applied with a number of terms reaches 6. In fact, $N = 5$ may be enough to get accurate results.
- All modified, SPL G–N II, DPL G–N II, SPL G–N III, DPL G–N III, and TPL G–N III models yield smaller variable quantities comparing to the simple G–N II and G–N III models.
- For all modified SPL, DPL or TPL models the results are slightly decreasing with the increase in a number of terms. The decreasing amounts may be insensitive when $N \geq 5$.

Figure 4. Effect of G–N models on the displacement v of the medium.

Figure 5. Effect of G–N models on the dilatation e of the medium.

- For the modified G–N II and III, the DPL model gives quantities greater than those of the SPL model.
- For the modified G–N III, the TPL model gives quantities greater than those of the DPL model.

Now, Figs. 2–11 are presented as a sample to illustrate the effect of all models on the temperature, displacements, dilatation, and stresses along the x -axis of the medium. Figure 2 presents the temperature distribution as waves that begin with negative values and ends with zero as x increases. The single-phase-lag (SPL) Green–Naghdi II and III models give the temperature θ with the largest amplitude. However, the simple G–N II and III models give the temperature θ with the smallest amplitude. For G–N III, the dual-phase-lag (DPL) model yields temperature θ with amplitude intermediates those of the SPL and triple-phase-lag (TPL) models. Also, the TPL model yields temperature θ with amplitude intermediates those of the DPL and the simple ones. The relative errors between models increase at the peak points of the temperature wave.

Figures 3–5 present the distributions of the displacements u , v and the dilatation e along the x -axis of the medium. The displacement-waves begin with positive values and ends with zero as x increases while dilatation-wave begins with negative values and ends with zero as x increases. The SPL G–N II and III models give the displacements u , v and dilatation e with the largest amplitudes. However, the simple G–N II and III models give the displacements u , v and dilatation e with the smallest amplitudes. For G–N III, the DPL model yields displacements u , v and dilatation e with amplitudes intermediate those of the SPL and TPL models. Also, the TPL model yields displacements u , v and dilatation e with amplitudes intermediate those of the DPL and the simple ones. The relative errors between models increase at the peak points of the displacement and dilatation waves.

Figures 6–8 present the distributions of all stresses along the x -axis of the medium. The in-plane normal stress-waves σ_{11} begin with negative values while the in-plane longitudinal stress-waves σ_{22} begin positive values and the in-plane tangential stress-waves σ_{12} begin with zero values. All stresses vanish as x increases. The SPL G–N II and III models give stresses with the largest amplitudes. However, the simple G–N II and III models give stresses with the smallest amplitudes. For G–N III, the DPL model yields stresses with amplitudes intermediate

Figure 6. Effect of G-N models on the stress σ_{11} in the medium.

Figure 7. Effect of G-N models on the stress σ_{12} in the medium.

Figure 8. Effect of G-N models on the stress σ_{22} in the medium.

Figure 9. 3D distributions of the temperature of the medium using G-N III theory.

Figure 10. 3D distributions of displacements and dilatation of the medium using TPL G-N III theory.

Figure 11. 3D distributions of stresses in the medium using TPL G-N III theory.

those of the SPL and TPL models. Also, the TPL model yields stresses with amplitudes intermediate those of the DPL and the simple ones. The relative errors between models increase at the peak points of the stress waves.

Finally, Figs. 9–11 present the 3D distributions of all field quantities of the medium using G-N III theory. The maximum temperature due to the simple, DPL and TPL models occurs at the origin point (0, 0) while the minimum one occurs at the point (0, 2). The maximum displacements u , $\bar{v}$ and dilatation e occur at different positions when $y = 0$. The maximum normal σ_{11} and longitudinal σ_{22} stresses occur at different positions when $y = 2$ while the maximum tangential stress σ_{12} occurs when $y = 0$. The wave amplitude for all quantities is decreasing as x increases. These figures are very important to study the dependence of the physical quantities on the 2D components of the distance.

Conclusions

This article presents analytical solutions for generalized thermoelastic interaction with multi thermal relaxations on a half-space subjected thermal loading due to laser pulse. The nonhomogeneous basic equations of the mathematical model are derived. The surface of the half-space is taken to be traction free in the tangential direction with uniform heat and traction in the normal direction. The system of two differential coupled equations is solved

using the normal mode approach, and the temperature, displacements, dilatation, and stresses are obtained for the thermoelastic interaction of the medium. The modified Green and Naghdi theories of types II and III are presented to get novel and accurate models of single-, dual-, and three-phase-lag of multi terms. The third phase-lag is included in the Green and Naghdi theory. This process may help experimental scientists working in the area of computational wave propagation. Some results are tabulated to serve as benchmark results for future comparisons with other investigators. The reported and illustrated results show that the simple G–N II and III models yield the largest values of all field quantities. The single-phase-lag model gives the smallest values. However, the dual-phase-lag model yield results that intermediate those of the simple and single-phase-lag Green–Naghdi II models. Finally, the dual-phase-lag and the tree-phase-lag models yield results that intermediate those of the simple and single-phase-lag Green–Naghdi III models. In fact, one can easily see that the different models have great effects on all field quantities which supports the physical fact.

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References

1. Biot, M. Thermoelasticity and irreversible thermodynamics. *J. Appl. Phys.* **27**, 240–253 (1956).
2. Green, A. E. & Naghdi, P. M. A re-examination of the basic postulates of thermomechanics. *Proc. R. Soc. Lond. A.* **432**, 171–194 (1991).
3. Green, A. E. & Naghdi, P. M. On undamped heat waves in an elastic solid. *J. Therm. Stress.* **15**, 253–264 (1992).
4. Green, A. E. & Naghdi, P. M. Thermoelasticity without energy dissipation. *J. Elast.* **31**, 189–208 (1993).
5. Deswal, S., Sheoran, S. & Kalkal, K. K. A two-dimensional problem in magnetothermoelasticity with laser pulse under different boundary conditions. *J. Mech. Mater. Struct.* **8**(8–10), 441–459 (2013).
6. Youssef, H. M. & El-Bary, A. A. Thermoelastic material response due to laser pulse heating in context of four theorems of thermoelasticity. *J. Therm. Stresses* **37**(12), 1379–1389 (2014).
7. Othman, M. I. A., Zidan, M. E. M. & Hilal, M. I. M. The effect of initial stress on thermoelastic rotating medium with voids due to laser pulse heating with energy dissipation. *J. Therm. Stresses* **38**(8), 835–853 (2015).
8. Zenkour, A. M. & Abouelregal, A. E. The nonlocal dual phase lag model of a thermoelastic nanobeam subjected to a sinusoidal pulse heating. *Int. J. Comput. Meth. Eng. Sci. Mech.* **16**(1), 44–52 (2015).
9. Othman, M. I. A. & Tantawi, R. S. The effect of a laser pulse and gravity field on a thermoelastic medium under Green–Naghdi theory. *Acta Mech.* **227**(12), 3571–3583 (2016).
10. Abbas, I. A. & Marin, M. Analytical solution of thermoelastic interaction in a half-space by pulsed laser heating. *Phys. E* **87**, 254–260 (2017).
11. Ailawalia, P., Sachdeva, S. & Pathania, D. Laser pulse heating in thermo-microstretch elastic layer overlying thermoelastic half-space. *J. Appl. Phys. Sci. Int.* **7**(4), 178–192 (2017).
12. Othman, M. I. A. & Marin, M. Effect of thermal loading due to laser pulse on thermoelastic porous medium under G–N theory. *Results Phys.* **7**, 3863–3872 (2017).
13. Mondal, S., Pal, P. & Kanoria, M. Transient response in a thermoelastic half-space solid due to a laser pulse under three theories with memory-dependent derivative. *Acta Mech* **230**, 179–199 (2019).
14. Ailawalia, P. & Singla, A. A thermoelastic microelongated layer immersed in an infinite fluid and subjected to laser pulse heating. *Mech. Mech. Eng.* **23**, 233–240 (2019).
15. Othman, M. I. A. & Abd-Elaziz, E. M. The effect of thermal loading due to laser pulse in generalized thermoelastic medium with voids in dual-phase-lag model. *J. Therm. Stresses* **38**(9), 1068–1082 (2015).
16. Zenkour, A. M. Refined two-temperature multi-phase-lags theory for thermomechanical response of microbeams using the modified couple stress analysis. *Acta Mech.* **229**(9), 3671–3692 (2018).
17. Zenkour, A. M. Refined microtemperatures multi-phase-lags theory for plane wave propagation in thermoelastic medium. *Res. Phys.* **11**, 929–937 (2018).
18. Zenkour, A. M. Refined multi-phase-lags theory for photothermal waves of a gravitated semiconducting half-space. *Compos. Struct.* **212**, 346–364 (2019).
19. Zenkour, A. M. Effect of thermal activation and diffusion on a photothermal semiconducting half-space. *J. Phys. Chem. Solids* **132**, 56–67 (2019).
20. Zenkour, A. M. Magneto-thermal shock for a fiber-reinforced anisotropic half-space studied with a refined multi-dual-phase-lag model. *J. Phys. Chemist. Solids* **137**, 109213 (2020).
21. Zenkour, A. M. Wave propagation of a gravitated piezo-thermoelastic half-space via a refined multi-phase-lags theory, *Mech. Advanc. Mater. Struct.*, <https://doi.org/10.1080/15376494.2018.1533057> (2020).
22. Choudhuri, S. K. R. On a thermoelastic three-phase-lag model. *J. Therm. Stresses* **30**, 231–238 (2007).
23. Quintanilla, R. & Racke, R. A note on stability in three-phase-lag heat conduction. *Int. J. Heat Mass Transfer* **51**, 24–29 (2008).
24. Kar, A. & Kanoria, M. Generalized thermoelastic functionally graded orthotropic hollow sphere under thermal shock with three-phase-lag effect. *Eur. J. Mech. A/Solids* **1**, 1–11 (2009).
25. Mukhopadhyay, S. & Kumar, R. Effect of three phase lag on generalized thermoelasticity for an infinite medium with a cylindrical cavity. *J. Therm. Stresses* **32**, 1149–1165 (2009).
26. Mukhopadhyay, S., Kothari, S. & Kumar, R. On the representation of solutions for the theory of generalized thermoelasticity with three phase-lags. *Acta Mech.* **214**, 305–314 (2010).
27. Kumar, R. & Chawla, V. A study of plane wave propagation in anisotropic three-phase-lag and two-phase-lag model. *Int. Commun. Heat Mass Transfer* **38**, 1262–1268 (2011).
28. Miranville, A. & Quintanilla, R. A phase-field model based on a three-phase-lag heat conduction. *Appl. Math. Optim.* **63**, 133–150 (2011).
29. Banik, S. & Kanoria, M. Effects of three-phase-lag on two-temperature generalized thermoelasticity for infinite medium with spherical cavity. *Appl. Math. Mech. -Engl. Ed.* **33**(4), 483–498 (2012).
30. Das, P. & Kanoria, M. Magneto-thermo-elastic response in a perfectly conducting medium with three-phase-lag effect. *Acta Mech.* **223**(4), 811–828 (2012).
31. El-Karamany, A. S. & Ezzat, M. A. On the three-phase-lag linear micropolar thermoelasticity theory. *Eur. J. Mech. A/Solids* **40**, 198–208 (2013).
32. Othman, M. I. A. & Said, S. M. 2D problem of magneto-thermoelasticity fiber-reinforced medium under temperature dependent properties with three-phase-lag model. *Mecc.* **49**(5), 1225–1241 (2014).
33. Kumar, R., Kaur, M. & Rajvanshi, S. C. Reflection and transmission between two micropolar thermoelastic half-spaces with three-phase-lag model. *J. Eng. Phys. Thermophys.* **87**(2), 295–307 (2014).

34. Kumar, A. & Kumar, R. A domain of influence theorem for thermoelasticity with three-phase-lag model. *J. Therm. Stresses* **38**, 744–755 (2015).
35. Said, S. M. Influence of gravity on generalized magneto-thermoelastic medium for three-phase-lag model. *J. Comput. Appl. Math.* **291**, 142–157 (2016).
36. Biswas, S., Mukhopadhyay, B. & Shaw, S. Thermal shock response in magneto-thermoelastic orthotropic medium with three-phase-lag model. *J. Electromagnetic Waves Appl.* **31**(9), 879–897 (2017).
37. Othman, M. I. A. & Eraki, E. E. M. Generalized magneto-thermoelastic half-space with diffusion under initial stress using three-phase-lag model. *Mech. Based Design Struct. Mach.* **45**(2), 145–159 (2017).
38. Othman, M. I. A. & Abd-Elaziz, E. M. Effect of rotation on a micropolar magneto-thermoelastic medium with dual-phase-lag model under gravitational field. *Microsys. Technol.* **23**(10), 4979–4987 (2017).
39. Othman, M. I. A. & Eraki, E. E. M. Effect of gravity on generalized thermoelastic diffusion due to laser pulse using dual-phase-lag model. *Multidiscipline Model Mater. Struct.* **149**(3), 457–481 (2018).
40. Zenkour, A. M. & Kutbi, M. A. Multi thermal relaxations for thermodiffusion responses in a thermoelastic half-space. *Int. J. Heat Mass Transfer* **143**, 118568 (2019).
41. Mashat, D. S. & Zenkour, A. M. Modified DPL Green–Naghdi theory for thermoelastic vibration of temperature-dependent nanobeams. *Res. Phys.* **16**, 102845 (2020).

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Competing interests

The authors declare no competing interests.

Additional information

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