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# Controllable symmetry breaking solutions for a nonlocal Boussinesq system

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The generalized Boussinesq equation is a useful model to describe the water wave. In this paper, with the coupled Alice-Bob (AB) systems, the nonlocal Boussinesq system can be obtained via the parity and time reversal symmetry reduction. By introducing an extended Bäcklund transformation, the symmetry breaking rogue wave, symmetry breaking soliton and symmetry breaking breather solutions for a nonlocal Boussinesq system are obtained through the derived Hirota bilinear form. The residual symmetry and finite symmetry transformation of the nonlocal AB-Boussinesq system are also studied.

Recently, Lou proposed various intrinsic models named the Alice-Bob systems (ABSs) to describe two-place physical problems[1,2]. The technical assumption was so-called $\hat{P}$-$\hat{T}$-$\hat{C}$ principle with $\hat{P}$ (the parity), $\hat{T}$ (time reversal) and $\hat{C}$ (charge conjugation). These ABSs mainly involved the following integrable systems, such as the Korteweg-de Vries (KdV), the nonlinear Schrödinger (NLS) and the Ablowitz-Kaup-Newell-Segur (AKNS), the modified KdV (MKdV), the sine-Gordon (sG), the Kadomtsev-Petviashvili (KP) and the Toda lattice systems. As a result, the $\hat{P}$-$\hat{T}$-$\hat{C}$ invariant multiple soliton solutions were explicitly presented for the above ABSs. In fact, various physically important models possess $\hat{P}$-$\hat{T}$-$\hat{C}$ invariance and these symmetries are important in many mathematical and physical fields. Up to now, researchers at least have studied the following works through the $\hat{P}$-$\hat{T}$-$\hat{C}$ principle: for a nonlocal AB-KdV system, its exact solutions including $P_sT_d$ invariant and $P_sT_d$ symmetric breaking solutions were given through different methods[3]. These solutions possess rich structures, such as single soliton, infinitely many singular soliton, cnoidal wave and the interaction of the different coherent structures. Nonlocal integrable peakon equations were also constructed and shown to have peakon solutions through this principle[4]. A new elegant form of the N-soliton solutions of the AB-KdV system which derived from the nonlinear inviscid dissipative and barotropic vorticity equation in a $\beta$-plane channel was obtained[5]. This system was applied to the two correlated **monopole** blocking events which was responsible for the snow disaster in the winter of 2007/2008 happened in Southern China. On the basis of the Lie point symmetry, the nonlocal symmetry of the AB-mKdV system was gained and the explicit solution of the system was presented[6]. By using the Darboux transformation, some types of shifted parity and time reversal symmetry breaking solutions, including such localized structures as one-soliton, two-soliton, and rogue wave solutions were explicitly depicted for the AB-mKdV system[7]. This methodology also induced a large number of nonlinear systems, including some integrable ones, such as the NLS, (potential) KdV, (potential) KP and sG systems from nothing (trivial $0 = 0$ equation) by introducing the new idea: consistent correlated bang and other suitable "Dao" which were implied by some symmetric or antisymmetric operators[8]. From a two-vortex interaction model established from the nonlinear inviscid dissipative and barotropic vorticity equation in a $\beta$-plane channel, a general nonlocal variable coefficient KdV (VCKdV) equation with shifted parity and delayed time reversal was derived through the multi-scale expansion method[9]. A special approximate analytical solution of the original two-vortex interaction model was given to describe two correlated dipole blocking events with a lifetime. Exact solutions of a new general nonlocal modified KdV equation were obtained including periodic waves, kink waves, solitary waves, kink- and/or anti-kink-cnoidal periodic wave interaction solutions, which can be utilized to describe various two-place and time-delayed correlated events[10]. The two-place nonlocal integrable models were systematically extended to multi-place nonlocal integrable (and nonintegrable) nonlinear models by means of the discrete symmetry reductions of the coupled local systems[11]. Especially, various two-place and four-place nonlocal NLS systems and KP equations were obtained. An integrable two-place nonlocal real system with three different nonlocal properties was derived which one nonlocality was related to a special nonlocal KdV system, while the other two were associated with the nonlocal Boussinesq

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systems[12]. The multisoliton solutions for the nonlocal KdV and Boussinesq systems were investigated and the possible prohibitions on multisoliton solutions were also discussed.

Now, we focus our attention on the following generalized Boussinesq equation

$$u_{tt} + \alpha u_{xx} + \beta(u^2)_{xx} + \gamma u_{xxxx} = 0, \tag{1}$$

which was first introduced by Boussinesq in 1871 to describe the propagation of long waves in shallow water[13]. This Boussinesq Eq. (1) is a soliton equation solvable by inverse scattering which arises in several other physical applications including one-dimensional nonlinear lattice-waves, vibrations in a nonlinear string and ion sound waves in a plasma[14–17]. Also, the soliton-cnoidal wave interaction solutions of the Boussinesq Eq. (1) was described when $\alpha = \gamma = -1$, $\beta = -3$[18]. The above Boussinesq equation with minus fourth-order term had multiple soliton solutions, whereas the model with the plus fourth-order term had multiple complex soliton solutions when the second-order derivative $u_{xx}$ was deleted and $\beta = \pm 1$[19]. Rational solutions of this equation and applications to rogue waves was shown when $\alpha = -\beta = 1$, $\gamma = -\frac{1}{3}$[20]. The bilinear transformation method was proposed to find the rogue wave solutions for the above generalized Eq. (1)[21]. Two exponential-type integrators were proposed and analyzed for the "good" Boussinesq equation, i.e. $\alpha = \beta = -1$, $\gamma = 1$ with rough initial data[22]. The Lie symmetry analysis was applied and some special solitons such as dark, singular and periodic solitons was devoted[23].

The next several sections, we focus on the AB-Boussinesq system of the generalized Eq. (1). In Section 2, a nonlocal AB-Boussinesq system is constructed and its **bilinear form** is written through an extended Bäcklund transformation. In Section 3, the symmetry breaking rogue wave, symmetry breaking soliton and symmetry breaking breather solutions are presented through the derived **Hirota bilinear form**. Starting from a modified Bäcklund transformation, the residual symmetry of the system is studied by introducing two auxiliary variables in Section 4. Therefore, the finite symmetry transformation is obtained through solving the initial value problem. Some conclusions are given in the final section.

## Methods

### A nonlocal AB-Boussinesq system and its bilinear form.
In this section, the model of AB system is applied into the Boussinesq equation. Based on the principle of the AB system referred in[1,2], when substituting $u = \frac{1}{2}(A + B)$ into (1), the nonlocal AB-Boussinesq system is

$$A_{tt} + B_{tt} + \alpha(A_{xx} + B_{xx}) + \beta(A_x + B_x)^2 + \beta(A + B)(A_{xx} + B_{xx}) + \gamma(A_{xxxx} + B_{xxxx}) = 0, \tag{2}$$

which can be split into two equatuions

$$A_{tt} + \alpha A_{xx} + \beta(A + B)A_{xx} + \frac{1}{2}\beta(A_x + B_x)^2 + \gamma A_{xxxx} + G(A, B) = 0, \tag{3a}$$

$$B_{tt} + \alpha B_{xx} + \beta(A + B)B_{xx} + \frac{1}{2}\beta(A_x + B_x)^2 + \gamma B_{xxxx} - G(A, B) = 0, \tag{3b}$$

where $B$ is related to $A$ by $B = \hat{P}_s\hat{T}_d A = A(-x + x_0, -t + t_0)$ ($\hat{P}_s\hat{T}_d$ expresses parity with a shift and time reversal with a delay). $G(A, B)$ is an arbitrary function of $A$ and $B$, but should be $\hat{P}_s\hat{T}_d$ invariant. Although there are infinite formulas satisfying this $G(A, B)$ in Eq. (3), we take $G(A, B) = 0$ for simplicity, and Eq. (3) are reduced to the following AB-Boussinesq system

$$A_{tt} + \alpha A_{xx} + \beta(A + B)A_{xx} + \frac{1}{2}\beta(A_x + B_x)^2 + \gamma A_{xxxx} = 0, \tag{4a}$$

$$B_{tt} + \alpha B_{xx} + \beta(A + B)B_{xx} + \frac{1}{2}\beta(A_x + B_x)^2 + \gamma B_{xxxx} = 0. \tag{4b}$$

Now, we introduce an extended Bäcklund transformation

$$A = \frac{6\gamma}{\beta}(\ln f)_{xx} + b(\ln f)_x, \quad B = \frac{6\gamma}{\beta}(\ln f)_{xx} - b(\ln f)_x, \tag{5}$$

with $b$ being an arbitrary constant and $f \equiv f(x, t)$ is an undetermined function of $x$ and $t$. When $b = 0$, Eq. (5) becomes one normal Bäcklund transformation. Substituting Eq. (5) into Eq. (4), the bilinear form can be derived as follows

$$(D_t^2 + \alpha D_x^2 + \gamma D_x^4)(f \cdot f) = 0, \tag{6}$$

where $D_t^2, D_x^2$ and $D_x^4$ are the bilinear derivative operators defined by[24,25]

$$D_x^m D_t^n(f \cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \times f(x, t)g(x', t')|_{x'=x, t'=t}. \tag{7}$$

According to the properties of bilinear operator $D$, Eq. (6) is equal to

$$ff_{tt} - f_t^2 + \alpha(ff_{xx} - f_x^2) + \gamma(ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2) = 0, \tag{8}$$

which happens to be a bilinear form of Eq. (1).

## Symmetry breaking rogue wave, soliton and breather solutions of the AB-Boussinesq system

In this section, we turn our attention to the **Hirota bilinear form** (6) of the nonlocal AB-Boussinesq system (4) to derive the symmetry breaking rogue wave, symmetry breaking soliton and symmetry breaking breather solutions.

### Symmetry breaking rogue wave solutions.
In nonlinear science, especially for some nonlinear integrable systems which as one of the essential topics, the theoretical study of rogue waves has gotten more and more attention in recent years. Rogue waves, also known as freak waves, monster waves, extreme waves, or hundred year waves, are relatively large and spontaneous ocean surface waves, which are a serious threat even to large ships and ocean liners[26,27]. These phenomena are ubiquitous in nature and can appear in a variety of different contexts such as reported in liquid Helium, nonlinear optics and microwave cavities[28].

For describing the symmetry breaking rogue wave solutions, we seek solutions of Eq. (4) in the form

$$A_n = \frac{6\gamma}{\beta}(\ln f_n)_{xx} + b(\ln f_n)_x, \quad B_n = \frac{6\gamma}{\beta}(\ln f_n)_{xx} - b(\ln f_n)_x, \quad n \geq 1, \tag{9}$$

where $f_n \equiv f_n(x, t)$ is a polynomial of degree $\frac{1}{2}n(n+1)$ in $X^2$ and $T^2$ described by

$$f_n = \sum_{m=0}^{n(n+1)/2} \sum_{j=0}^{m} a_{j,m}\left(x - \frac{x_0}{2}\right)^{2j}\left(t - \frac{t_0}{2}\right)^{2(m-j)} \tag{10}$$

with total degree $\frac{1}{2}n(n+1)$ and the constants $a_{j,m}\left(j = 0, 1, \ldots, m, m = 0, 1, \ldots, \frac{n(n+1)}{2}\right)$ are determined by equating powers of $X$ and $T$[20]. Utilizing the **Hirota bilinear form** (6), we derive the following polynomials

$$f_1 = 1 - \frac{\alpha}{3\gamma}X^2 - \frac{\alpha^2}{3\gamma}T^2, \tag{11a}$$

$$f_2 = 1 - \frac{\alpha^3 X^6}{1875\gamma^3} + \frac{\alpha^2 \Delta_1 X^4}{1875\gamma^3} + \frac{\alpha \Delta_2 X^2}{1875\gamma^3} - \frac{\alpha^6 T^6}{1875\gamma^3} + \frac{17\alpha^4 T^4}{1875\gamma^2} - \frac{19\alpha^2 T^2}{75\gamma}, \tag{11b}$$

$$\begin{aligned}
f_3 = {} & 1 + \frac{9\alpha^6 X^{12}}{878826025\gamma^6} - \frac{18\alpha^5 \Delta_3 X^{10}}{878826025\gamma^6} + \frac{27\alpha^4 \Delta_4 X^8}{175765205\gamma^6} \\
& - \frac{12\alpha^3 \Delta_5 X^6}{175765205\gamma^6} - \frac{3\alpha^2 \Delta_6 X^4}{175765205\gamma^6} - \frac{6\alpha \Delta_7 X^2}{878826025\gamma^6} \\
& + \frac{9\alpha^{12} T^{12}}{878826025\gamma^6} - \frac{522\alpha^{10} T^{10}}{878826025\gamma^5} + \frac{7803\alpha^8 T^8}{175765205\gamma^4} \\
& - \frac{68484\alpha^6 T^6}{25109315\gamma^3} + \frac{40143\alpha^4 T^4}{717409\gamma^2} - \frac{870\alpha^2 T^2}{847\gamma},
\end{aligned} \tag{11c}$$

where

$$\begin{aligned}
\Delta_1 &= 25\gamma - 3\alpha^2 T^2, \\
\Delta_2 &= 125\gamma^2 + 90\alpha^2\gamma T^2 - 3\alpha^2 T^4, \\
\Delta_3 &= 49\gamma - 3\alpha^2 T^2, \\
\Delta_4 &= 49\gamma^2 - 46\alpha^2\gamma T^2,
\end{aligned} \tag{12a}$$

$$\begin{aligned}
\Delta_5 &= 3773\gamma^3 - 2793\alpha^2\gamma^2 T^2 + 231\alpha^4\gamma T^4 - 3\alpha^6 T^6, \\
\Delta_6 &= 1037575\gamma^4 + 132300\alpha^2\gamma^3 T^2 - 22470\alpha^4\gamma^2 T^4 \\
& \quad + 876\alpha^6\gamma T^6 - 9\alpha^8 T^8,
\end{aligned} \tag{12b}$$

$$\begin{aligned}
\Delta_7 &= 79893275\gamma^5 - 848925\alpha^2\gamma^4 T^2 \\
& \quad - 22050\alpha^4\gamma^3 T^4 - 53130\alpha^6\gamma^2 T^6 \\
& \quad + 855\alpha^8\gamma T^8 - 9\alpha^{10} T^{10},
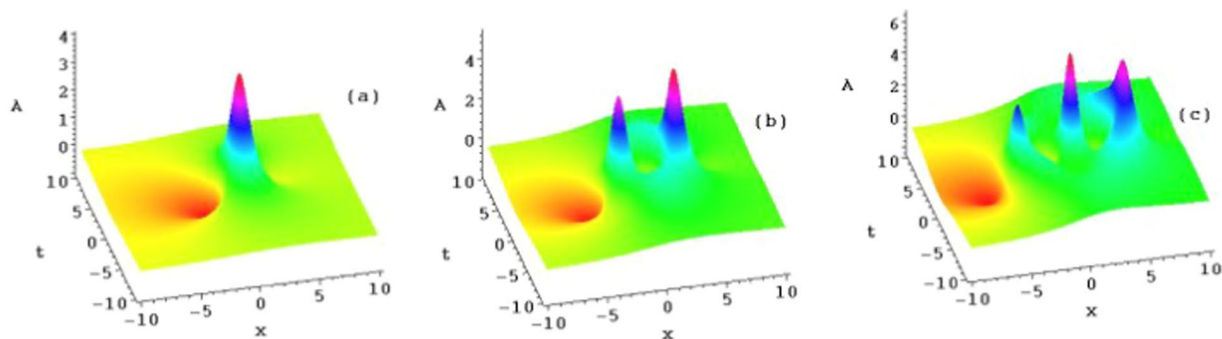\end{aligned} \tag{12c}$$

**Figure 1.** Plots of the symmetry breaking rogue wave solutions $A_n$, for $n = 1, 2, 3$, of the AB-Boussinesq system.
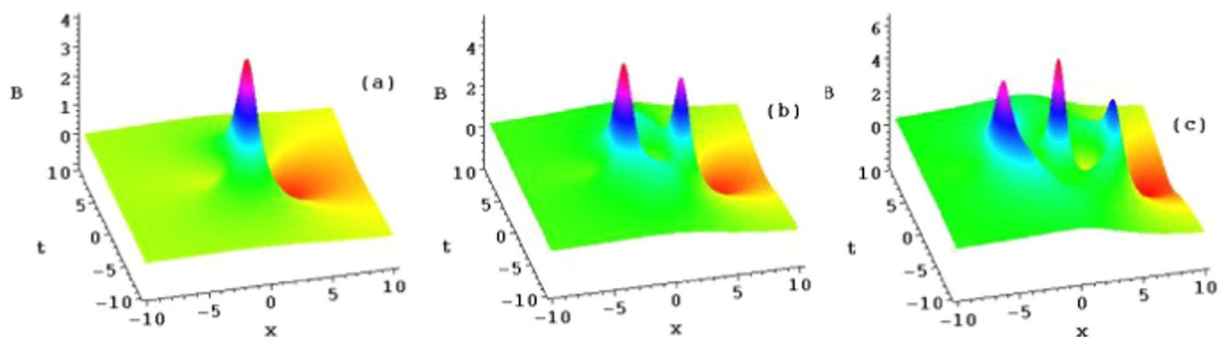


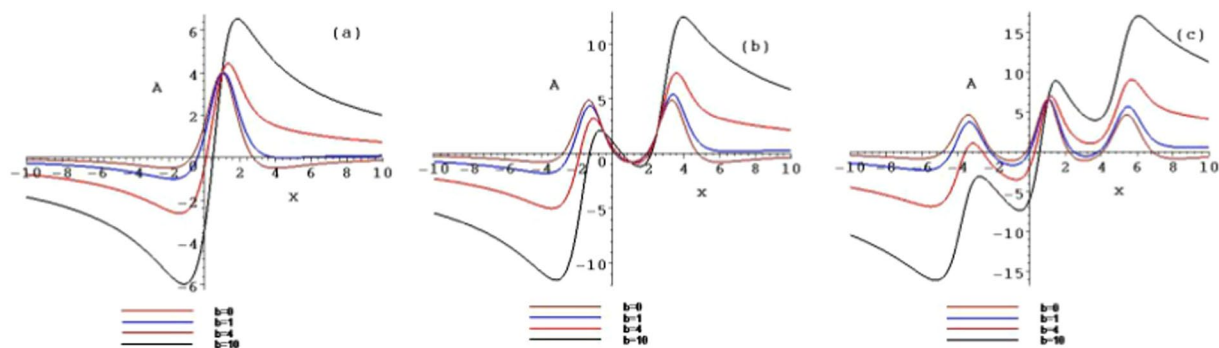**Figure 2.** Plots of the symmetry breaking rogue wave solutions $B_n$, for $n = 1, 2, 3$, of the AB-Boussinesq system.



**Figure 3.** The contour plots of the symmetry breaking rogue wave solutions $A_n$, for $n = 1, 2, 3$, with $b = 0$, $b = 1$, $b = 4$ and $b = 10$ at the time $t = 1$.

with

$$X = x - \frac{x_0}{2}, \quad T = t - \frac{t_0}{2}. \tag{13}$$

The following four sets of figures (Figs. 1, 2, 3 and 4) are presented for the purposes of illustration the solutions $u_1$, $u_2$ and $u_3$. Figure 1 shows three ranks of the symmetry breaking rogue wave solutions $A_n(n = 1, 2, 3)$ in Eq. (9) of the AB-Boussinesq system when the constants are $b = 1$, $\alpha = 1$, $\beta = \gamma = -1$ and $x_0 = t_0 = 2^{1-3,29}$. Figure 2 depicts the reversal structures of Fig. 1 by another sets of solutions $B_n(n = 1, 2, 3)$ (which are $\hat{P}_s\hat{T}_d$ symmetries of the solutions $A_n(n = 1, 2, 3)$ in Eq. (9)). These structures all possess two types of the rogue waves, namely, the bright and dark ones at the same time. This corresponds to the phenomenon that the shifted parity and delayed time reversal are applied to describe two-place events. We should mention that all the anti-symmetric solutions of Eq. (9) lead to the $\hat{P}_s\hat{T}_d$ symmetry breaking solutions of Eq. (4).
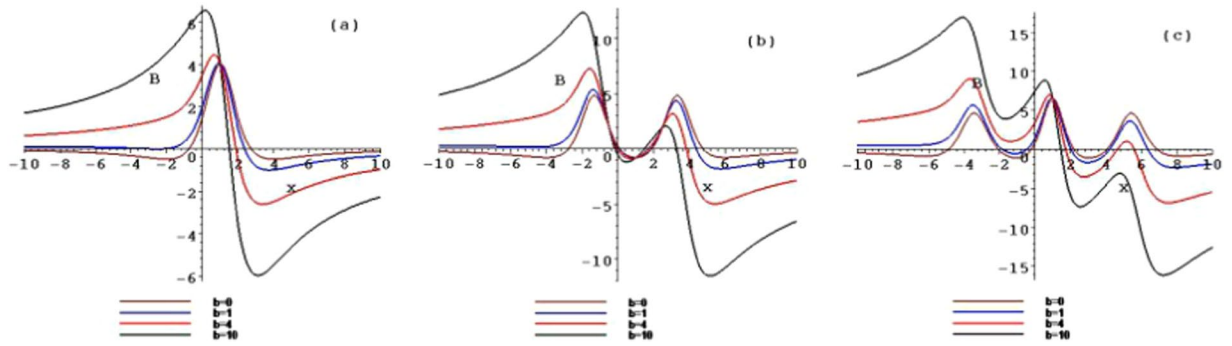
**Figure 4.** The contour plots of the symmetry breaking rogue wave solutions $B_n$, for $n = 1, 2, 3$, with $b = 0$, $b = 1$, $b = 4$ and $b = 10$ at the time $t = 1$.

When selecting different parameter values $b$, the different structures of the above rogue waves can be described. Figures 3 and 4 show the contour plots of the symmetry breaking rogue waves in the $(x, A/B)$-plane, taking $b = 0$, $b = 1$, $b = 4$ and $b = 10$ at the time $t = 1$, respectively. This parameter $b$ can impact the wave shape. With the increase of $b$ from 0 to 10, the amplitude is amplified, while the distance between the peaks is elongated. The parameter $b$ plays an important role of stretching the rogue waves along with the $x$ axis.

**Symmetry breaking soliton solutions.** Based on the bilinear form (6) of the AB-Boussinesq system, the symmetry breaking multisoliton solutions of Eq. (4) can be written as a summation of some special functions[1,2,12]. For example, after choosing $\alpha = 0$, $\beta = \gamma = -1$, we can take

$$
\begin{aligned}
f_n &= \sum_{\{\nu\}} K_{\{\nu\}} \cosh\left(\sum_{i=1}^{n} \nu_i \xi_i\right), \\
\xi_i &= k_i\left(x - \frac{x_0}{2}\right) + 2\delta_i k_i^2\left(t - \frac{t_0}{2}\right), \\
\nu_i^2 &= \delta_i^2 = 1,
\end{aligned}
\tag{14}
$$

where the summation of $\{\nu\} = \{\nu_1, \nu_2, \ldots, \nu_n\}$, and $k_i(i = 1, 2, \ldots, n)$, $x_0$, $t_0$ are arbitrary constants, while

$$
K_{\{\nu\}} = \prod_{i<j}^{n} a_{ij}, \quad a_{ij}^2 = 2k_i^2 + 2k_j^2 - k_i k_j(\delta_i \delta_j + 3\nu_i \nu_j).
\tag{15}
$$

**Example 3.1** Symmetry breaking line soliton solution

We take

$$
f_1 = \cosh(\xi_1), \quad \xi_1 = k_1\left(x - \frac{x_0}{2}\right) + 2\delta_1 k_1^2\left(t - \frac{t_0}{2}\right),
\tag{16}
$$

with $n = 1$. By substituting Eq. (16) into Eq. (5), one can get the symmetry breaking line soliton solution of Eq. (4)

$$
A_1 = 6k_1^2 + bk_1 \tanh(\xi_1) - 6k_1^2 \tanh^2(\xi_1),
\tag{17a}
$$

$$
B_1 = 6k_1^2 - bk_1 \tanh(\xi_1) - 6k_1^2 \tanh^2(\xi_1).
\tag{17b}
$$

Because of the existence of tanh term which makes $B_1 = \hat{P}_s \hat{T}_d A_1$ or $A_1 = \hat{P}_s \hat{T}_d B_1$, the line soliton solution (17) of the AB-Boussinesq system (4) is $\hat{P}_s \hat{T}_d$ symmetric breaking solution.

Figure 5(a,b) are two corresponding symmetric breaking structures when the constants taken as $b = k_1 = 1$, $\delta_1 = -1$, $x_0 = t_0 = 2$. Figure 5(c,d) are the contour plots of these structures in the $(x, A/B)$-plane, taking $b = 0$, $b = 1$, $b = 4$ and $b = 10$ at the time $t = 1$, respectively. As the parameter $b$ increases, the line soliton pairs trend to two symmetry breaking kinks.

**Example 3.2** Interaction of the symmetry breaking line solitons

The interaction of the symmetry breaking line soliton solution of Eq. (4) can be expressed as
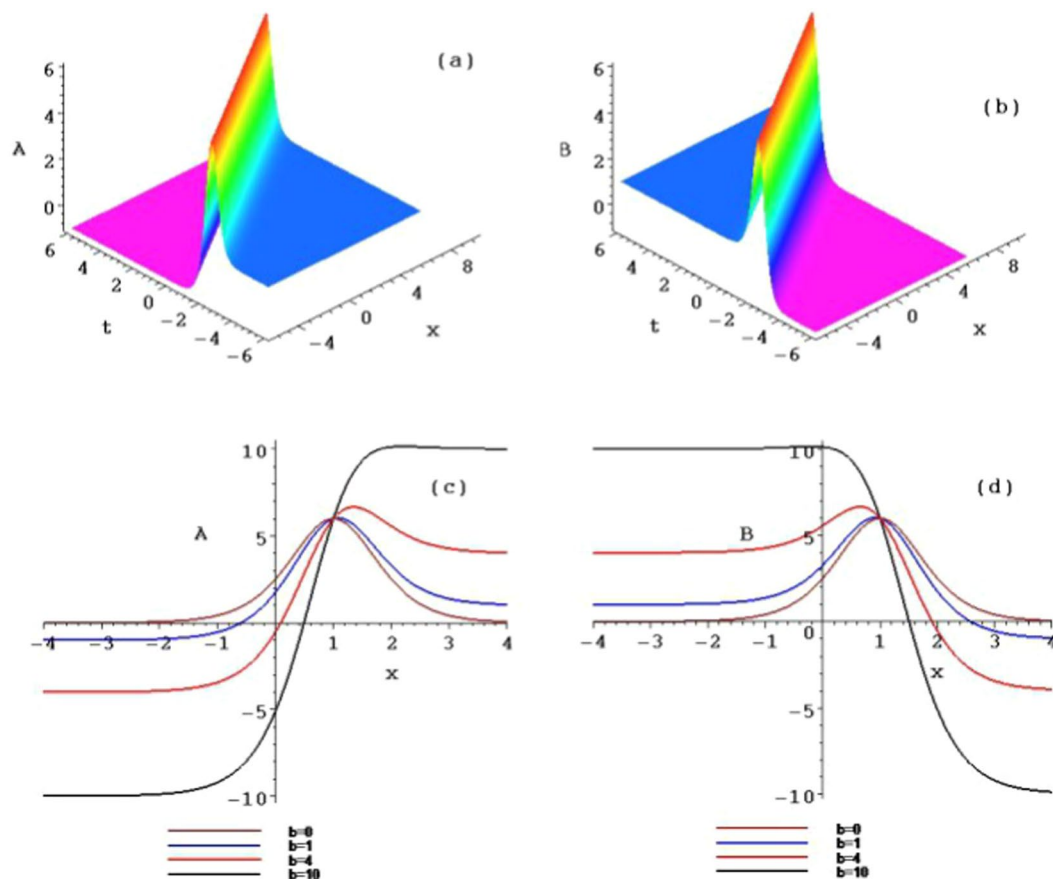
**Figure 5.** Plots of the symmetry breaking line solitons of $A_1$ and $B_1$ of the solution (17).

$$A_2 = 6(\ln f_2)_{xx} + b(\ln f_2)_x, \quad B_2 = \hat{P}_s \hat{T}_d A_2, \tag{18}$$

with

$$
\begin{aligned}
f_2 &= \delta_+ \sqrt{2k_1^2 + 2k_2^2 - k_1 k_2 (\delta_1 \delta_2 + 3)} \cosh(\xi_1 + \xi_2) \\
&\quad + \delta_- \sqrt{2k_1^2 + 2k_2^2 - k_1 k_2 (\delta_1 \delta_2 - 3)} \cosh(\xi_1 - \xi_2), \\
\delta_+^2 &= \delta_-^2 = 1,
\end{aligned}
\tag{19}
$$

and

$$\xi_i = k_i \left( x - \frac{x_0}{2} \right) + 2\delta_i k_i^2 \left( t - \frac{t_0}{2} \right), \quad i = 1, 2.$$

When $k_2 = k_1$, $\delta_2 = -\delta_1$, Eq. (19) is simplified in the form

$$
\begin{aligned}
f_2 &= \delta_+ \sqrt{4k_1^2 + k_1^2(\delta_1^2 - 3)} \cosh\left[ 2k_1 \left( x - \frac{x_0}{2} \right) \right] \\
&\quad + \delta_- \sqrt{4k_1^2 + k_1^2(\delta_1^2 + 3)} \cosh\left[ 4\delta_1 k_1^2 \left( t - \frac{t_0}{2} \right) \right],
\end{aligned}
\tag{20}
$$

This is the parity and time reversal invariant form of Eq. (16). For example, taking the constants $b = k_1 = \delta_1 = \delta_2 = \delta_+ = \delta_- = 1$, $k_2 = -1$, $x_0 = t_0 = 2$, two corresponding interactions of the symmetry breaking line solitons are presented (Fig. 6). Figures 7 and 8 are the contour plots of these structures in the $(x, A/B)$-plane, taking $b = 0$, $b = 1$, $b = 4$ and $b = 10$ at different times $t = -2$, 1 and 3, respectively. As the parameter $b$ and time $t$ increases, the interactions of the symmetry breaking line soliton solutions of Eq. (4) also trend to two symmetry breaking kinks.

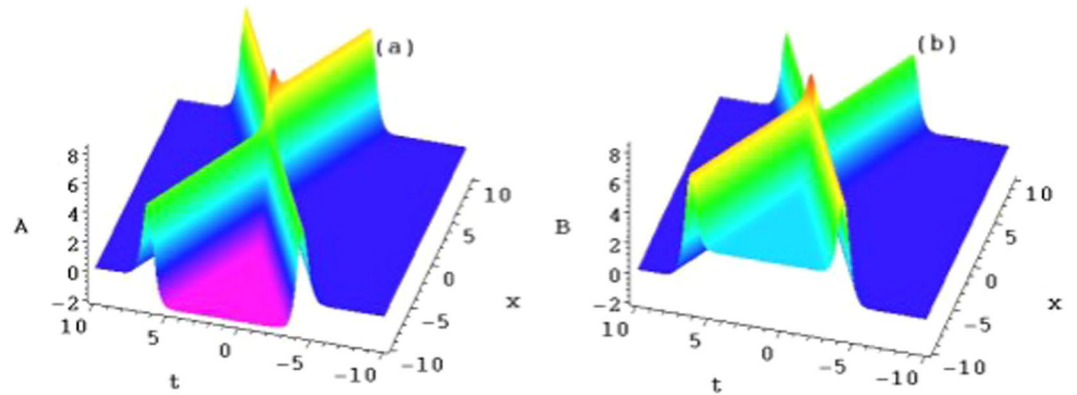**Example 3.3** Interaction of the triple symmetry breaking solitons

**Figure 6.** The interactions of the symmetry breaking line solitons of $A_2$ and $B_2$ of the solution (18).
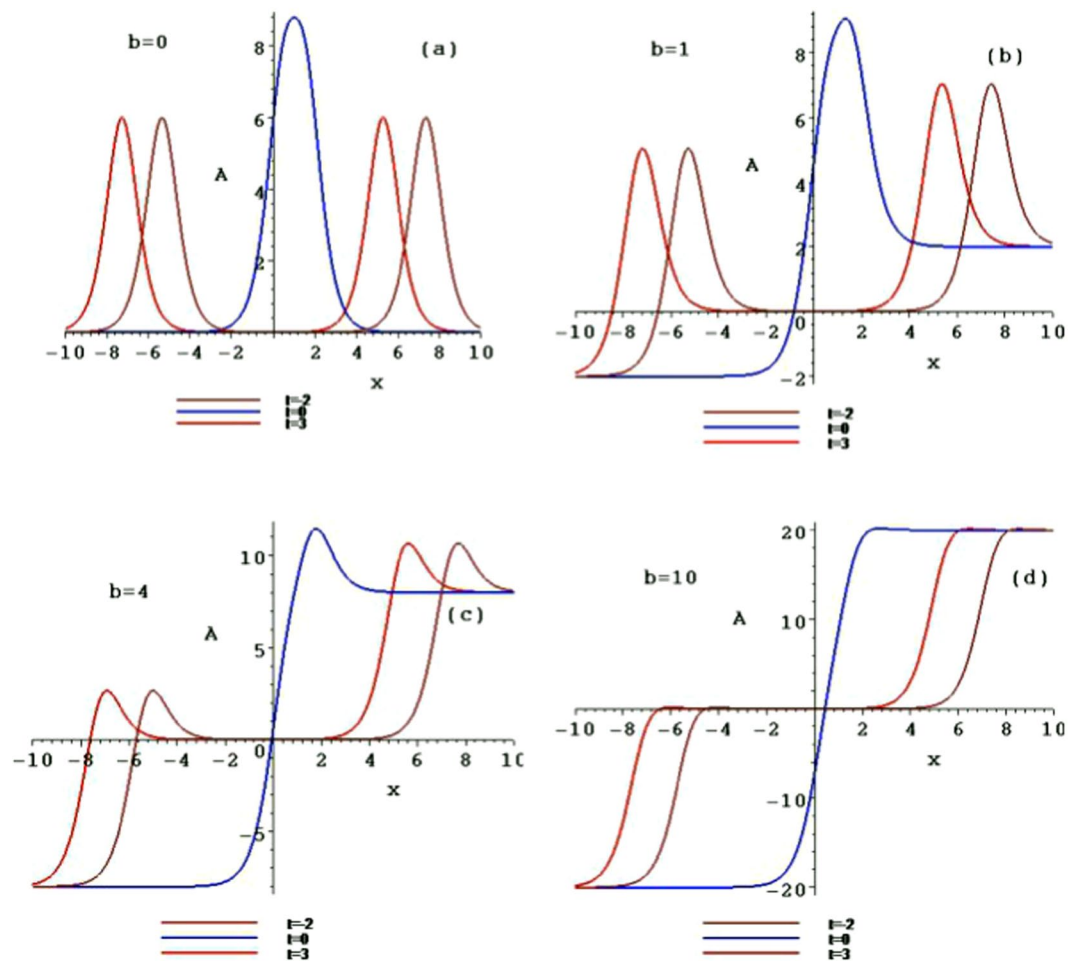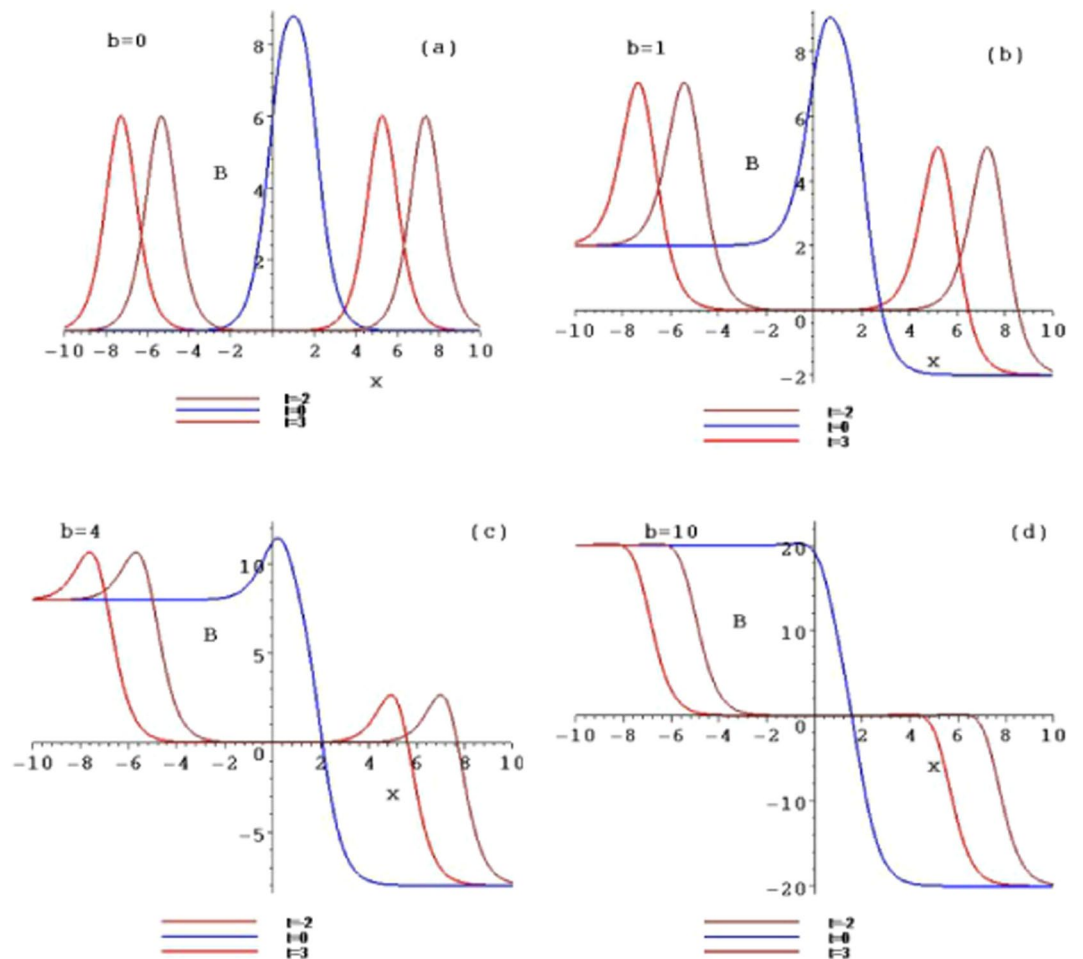


**Figure 7.** The contour plots of the interaction of the symmetry breaking line solitons $A_2$ with $b = 0$, $b = 1$, $b = 4$ and $b = 10$ at different times $t = -2$, 1 and 3, respectively.

For $n = 3$, the solution (14) possesses the following form

$$
\begin{aligned}
f \equiv f_3 = \ & K_{\{\}}\cosh(\xi_1 + \xi_2 + \xi_3) \\
& + K_{\{1\}}\cosh(\xi_1 - \xi_2 - \xi_3) \\
& + K_{\{2\}}\cosh(\xi_1 - \xi_2 + \xi_3) \\
& + K_{\{3\}}\cosh(\xi_1 + \xi_2 - \xi_3),
\end{aligned}
\tag{21}
$$

**Figure 8.** The contour plots of the interaction of the symmetry breaking line solitons $B_2$ with $b = 0$, $b = 1$, $b = 4$ and $b = 10$ at different times $t = -2$, 1 and 3, respectively.

where

$$K_{\{\}} = a_{12}^+ a_{13}^+ a_{23}^+, \quad K_{\{1\}} = a_{12}^- a_{13}^- a_{23}^+, \quad K_{\{2\}} = a_{12}^- a_{13}^+ a_{23}^-, \quad K_{\{3\}} = a_{12}^+ a_{13}^- a_{23}^-, \tag{22a}$$

$$\begin{aligned} a_{ij}^{\pm} &= \sqrt{2k_i^2 + 2k_j^2 - k_i k_j (\delta_i \delta_j \pm 3)}, \\ \xi_i &= k_i\left(x - \frac{x_0}{2}\right) + 2\delta_i k_i^2\left(t - \frac{t_0}{2}\right), \\ \delta_i^2 &= 1, \quad (i = 1, 2, 3). \end{aligned} \tag{22b}$$

Therefore, the interaction of the triple symmetry breaking soliton solution of Eq. (4) can be expressed as

$$A_3 = \frac{6\gamma}{\beta}(\ln f_3)_{xx} + b(\ln f_3)_x, \quad B_3 = A_3(-x + x_0, -t + t_0). \tag{23}$$

Figure 9 is two corresponding interactions of the triple symmetry breaking solitons when taking the constants $b = k_1 = \delta_1 = \delta_2 = \delta_3 = 1$, $k_2 = -1$, $k_3 = -1.1$, $x_0 = t_0 = 2$, and the parameters $\alpha = 0$, $\beta = \gamma = -1$. Figures 10 and 11 are the contour plots of these structures in the $(x, A/B)$-plane, taking $b = 0$, $b = 1$, $b = 4$ and $b = 10$ at different times $t = -2$, 1 and 3, respectively.

From these presented results of Examples 3.1–3.3, we have a different conclusion from the previous, that is "the multisoliton solutions with odd numbers and the multisoliton solutions with even numbers but with pursuant interactions are prohibited"[12].

**Symmetry breaking breather solution.** Based on the bilinear form (6) of the AB-Boussinesq system, the symmetry breaking breather solution of Eq. (4) can also be expressed as

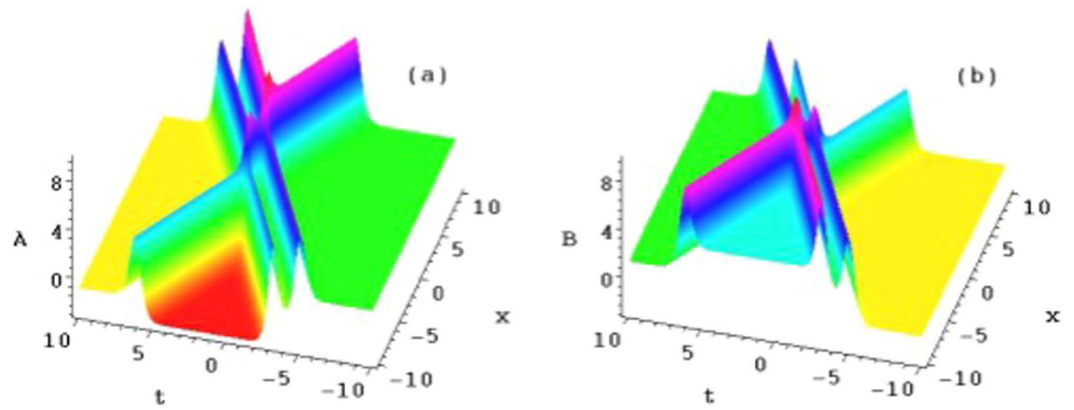$$A = \frac{6\gamma}{\beta}(\ln f)_{xx} + b(\ln f)_x, \quad B = A(-x + x_0, -t + t_0), \tag{24}$$

**Figure 9.** The interactions of the triple symmetry breaking solitons of $A_3$ and $B_3$ of the solution (23).
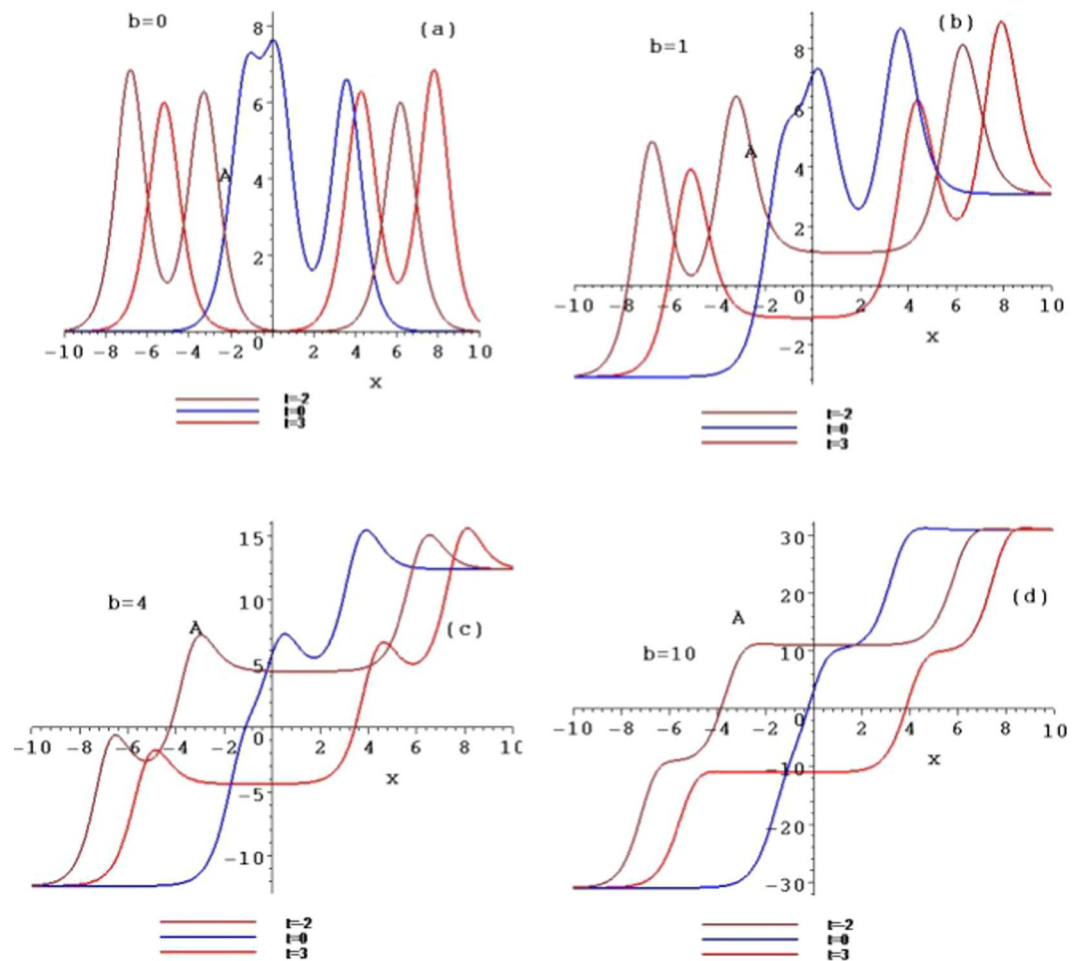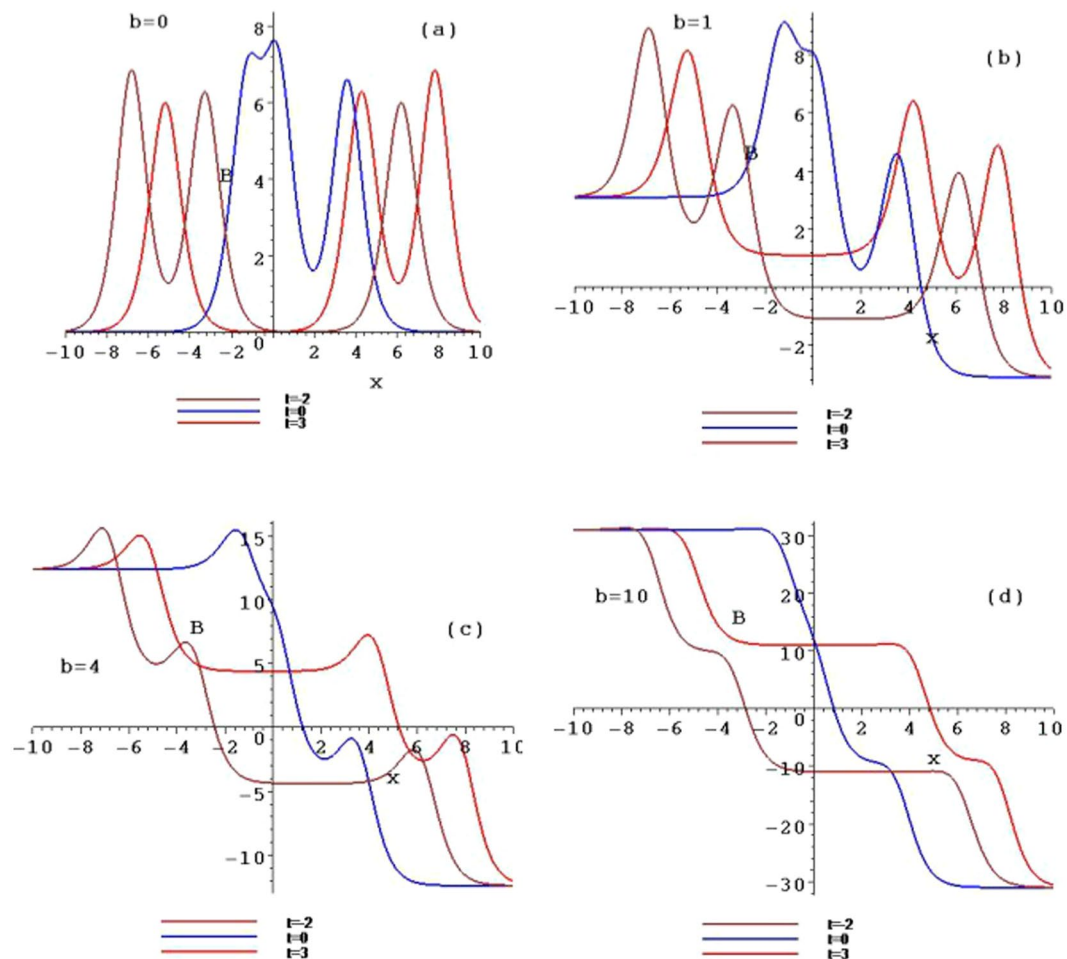


**Figure 10.** The contour plots of the interaction of the triple symmetry breaking solitons $A_3$ with $b = 0$, $b = 1$, $b = 4$ and $b = 10$ at different times $t = -2$, 1 and 3, respectively.

when the bilinear function

$$f = 2\sqrt{b_1}\cosh\left\{b_3\left[a_1\left(x - \frac{x_0}{2}\right) + a_2\left(t - \frac{t_0}{2}\right)\right]\right\}$$
$$+ b_2\cos\left\{b_4\left[a_3\left(x - \frac{x_0}{2}\right) + a_4\left(t - \frac{t_0}{2}\right)\right]\right\},$$

(25)

**Figure 11.** The contour plots of the interaction of the triple symmetry breaking solitons $B_3$ with $b = 0$, $b = 1$, $b = 4$ and $b = 10$ at different times $t = -2$, 1 and 3, respectively.

and the coefficients

$$a_2 = -\frac{\sqrt{2}\,a_1 a_3 b_4(\alpha + 2\gamma R_2)}{R_3},$$

$$a_4 = \frac{\sqrt{2}\,R_3}{2b_4},$$

$$b_1 = \frac{b_2^2 R_3^2 [R_3^2 - 2a_3^2 b_4^2 (4a_3^2 b_4^2 \gamma - \alpha)]}{8a_1^2 b_3^2 [R_3^2(\alpha + 4a_1^2 b_3^2 \gamma) + 2a_3^2 b_4^2(\alpha + 2\gamma R_2)^2]}, \tag{26}$$

and

$$R_1 = a_1^2 b_3^2 + a_3^2 b_4^2,$$
$$R_2 = a_1^2 b_3^2 - a_3^2 b_4^2,$$
$$R_3 = \sqrt{R_2\alpha + (R_2^2 - 4a_1^2 a_3^2 b_3^2 b_4^2)\gamma + R_1\sqrt{(\alpha^2 + 2R_2\alpha\gamma + R_1^2\gamma^2)}}\,. \tag{27}$$

## Residual symmetry of the AB-Boussinesq system

Recently, it was found that the residue in Painlevé truncated expansion corresponds to a nonlocal symmetry for the Painlevé integrable system and then such type of symmetry was referred to residual symmetry. This residual symmetry was localized to Lie point symmetry by introducing suitable prolonged system[30].

From the leading term analysis of the AB-Boussinesq system (4), the Bäcklund transformation is extended as follows

$$A = \frac{6\gamma}{\beta}(\ln f)_{xx} + b(\ln f)_x + \tilde{A}, \quad B = \frac{6\gamma}{\beta}(\ln f)_{xx} - b(\ln f)_x + \tilde{B}, \tag{28}$$

with

$$\tilde{A} = \frac{3\gamma f_{xx}^2}{2\beta f_x^2} - \frac{2\gamma f_{xxx}}{\beta f_x} - \frac{f_t^2}{2\beta f_x^2} - \frac{\alpha}{2\beta} + p,$$

$$\tilde{B} = \frac{3\gamma f_{xx}^2}{2\beta f_x^2} - \frac{2\gamma f_{xxx}}{\beta f_x} - \frac{f_t^2}{2\beta f_x^2} - \frac{\alpha}{2\beta} - p, \tag{29}$$

and $p \equiv p(x, t)$ is a constraint function to be determined later. The Schwarzian structure has two consistent conditions

$$C_t + CC_x + \gamma S_x = 0, \tag{30}$$

$$p_{xx} = -\frac{b}{6\gamma}CC_x - \frac{bf_{xx}}{2f_x}S - \frac{bf_{xx}^3}{4f_x^3} - \frac{b}{3}S_x, \tag{31}$$

where the variable quantities

$$C \equiv \frac{f_t}{f_x}, \quad S \equiv \frac{f_{xxx}}{f_x} - \frac{3f_{xx}^2}{2f_x^2}. \tag{32}$$

In other words, the coupled equations with the nonlocal symmetry functions $f$ and $p$ are

$$f_{tt} = \frac{f_{xx}f_t^2}{f_x^2} + \frac{4\gamma f_{xx}f_{xxx}}{f_x} - \frac{3\gamma f_{xx}^3}{f_x^2} - \gamma f_{xxxx}, \tag{33}$$

$$p_{xx} = \frac{5bf_{xx}f_{xxx}}{6f_x^2} - \frac{bf_{xxxx}}{3f_x} - \frac{af_t f_{xt}}{6\gamma f_x^2} + \frac{bf_t^2 f_{xx}}{6\gamma f_x^3} - \frac{bf_{xx}^3}{2f_x^3}. \tag{34}$$

After introducing two auxiliary functions $g \equiv g(x, t)$ and $h \equiv h(x, t)$, which should obey the rules

$$g = f_x, \quad h = g_x, \tag{35}$$

The enlarged AB-Boussinesq system of Eq. (4) according to Eqs. (4), (29), (33) and (34) given by

$$A_{tt} + \alpha A_{xx} + \beta(A + B)A_{xx} + \frac{1}{2}\beta(A_x + B_x)^2 + \gamma A_{xxxx} = 0, \tag{36a}$$

$$B_{tt} + \alpha B_{xx} + \beta(A + B)B_{xx} + \frac{1}{2}\beta(A_x + B_x)^2 + \gamma B_{xxxx} = 0, \tag{36b}$$

$$\alpha f_x^2 - 2\beta(p - A)f_x^2 + \gamma(4f_x f_{xxx} - 3f_{xx}^2) + f_t^2 = 0, \tag{36c}$$

$$\alpha f_x^2 + 2\beta(p + B)f_x^2 + \gamma(4f_x f_{xxx} - 3f_{xx}^2) + f_t^2 = 0, \tag{36d}$$

$$f_{tt}f_x^2 - f_t^2 f_{xx} + \gamma(3f_{xx}^3 + f_x^2 f_{xxxx} - 4f_x f_{xx}f_{xxx}) = 0, \tag{36e}$$

$$bf_t(f_t f_{xx} - f_x f_{xt}) + \gamma(5bf_x f_{xx}f_{xxx} - 2bf_x^2 f_{xxxx} - 6f_x^3 p_{xx} - 3bf_{xx}^3) = 0, \tag{36f}$$

Under the transformations

$$
\begin{aligned}
A &= A + \epsilon\sigma^A, \\
B &= B + \epsilon\sigma^B, \\
f &= f + \epsilon\sigma^f, \\
g &= g + \epsilon\sigma^g, \\
h &= h + \epsilon\sigma^h, \\
p &= p + \epsilon\sigma^p,
\end{aligned}
\tag{37}
$$

with the infinitesimal parameter $\epsilon$, the linearized system of the enlarged AB-Boussinesq Eq. (36) reads

$$\sigma_{tt}^A + \alpha\sigma_{xx}^A + \beta[(\sigma_x^A + \sigma_x^B)(A_x + B_x) + (\sigma^A + \sigma^B)A_{xx} + (A + B)\sigma_{xx}^A] + \gamma\sigma_{xxxx}^A = 0, \tag{38a}$$

$$\sigma_{tt}^B + \alpha\sigma_{xx}^B + \beta[(\sigma_x^A + \sigma_x^B)(A_x + B_x) + (\sigma^A + \sigma^B)B_{xx} + (A + B)\sigma_{xx}^B] + \gamma\sigma_{xxxx}^B = 0, \tag{38b}$$

$$f_t\sigma_t^f + \alpha f_x\sigma_x^f - \beta f_x(\sigma^P f_x + 2p\sigma_x^f - 2A\sigma_x^f - \sigma^A f_x) + \gamma(2f_x\sigma_{xxx}^f + 2f_{xxx}\sigma_x^f - 3f_{xx}\sigma_{xx}^f) = 0, \tag{38c}$$

$$f_t\sigma_t^f + \alpha f_x\sigma_x^f + \beta f_x(\sigma^P f_x + 2p\sigma_x^f + 2B\sigma_x^f + \sigma^B f_x) + \gamma(2f_x\sigma_{xxx}^f + 2f_{xxx}\sigma_x^f - 3f_{xx}\sigma_{xx}^f) = 0, \tag{38d}$$

$$\begin{aligned}
&f_x^2\sigma_{tt}^f + 2f_xf_{tt}\sigma_x^f - f_t^2\sigma_{xx}^f - 2f_tf_{xx}\sigma_t^f \\
&+ \gamma[(9f_{xx}^2 - 4f_xf_{xxx})\sigma_{xx}^f + (2f_xf_{xxxx} - 4f_{xx}f_{xxx})\sigma_x^f \\
&+ f_x(f_x\sigma_{xxxx}^f - 4f_{xx}\sigma_{xxx}^f)] = 0,
\end{aligned} \tag{38e}$$

$$\begin{aligned}
&\alpha[(f_xf_{xt} - 2f_tf_{xx})\sigma_t^f + f_t(f_x\sigma_{xt}^f - f_t\sigma_{xx}^f + f_{xt}\sigma_x^f)] \\
&+ \gamma[(6f_x^2\sigma_{xx}^p + 2bf_x\sigma_{xxxx}^f - 5bf_{xx}\sigma_{xxx}^f)f_x \\
&+ (4bf_xf_{xxxx} - 5bf_{xx}f_{xxx} + 18f_x^2p_{xx})\sigma_x^f \\
&+ (9bf_{xx}^2 - 5bf_xf_{xxx})\sigma_{xx}^f] = 0,
\end{aligned} \tag{38f}$$

$$\sigma_x^f - \sigma^g = 0, \quad \sigma_x^g - \sigma^h = 0. \tag{38g}$$

Thus, the derived residual symmetry $\{\sigma^A, \sigma^B\}$ has the local solution

$$\begin{aligned}
\sigma^A &= bg + \frac{6\gamma h}{\beta}, \\
\sigma^B &= -bg + \frac{6\gamma h}{\beta}, \\
\sigma^f &= -f^2, \\
\sigma^g &= -2fg, \\
\sigma^h &= -2(fh + g^2), \\
\sigma^p &= bg.
\end{aligned} \tag{39}$$

Meanwhile, by solving the following initial value problem

$$\begin{aligned}
\frac{\partial\hat{A}(\epsilon)}{\partial\epsilon} &= \frac{6\gamma}{\beta}\hat{H}(\epsilon) + b\hat{G}(\epsilon), \\
\frac{\partial\hat{B}(\epsilon)}{\partial\epsilon} &= \frac{6\gamma}{\beta}\hat{H}(\epsilon) - b\hat{G}(\epsilon), \\
\frac{\partial\hat{F}(\epsilon)}{\partial\epsilon} &= -\hat{F}^2(\epsilon),
\end{aligned} \tag{40a}$$

$$\begin{aligned}
\frac{\partial\hat{G}(\epsilon)}{\partial\epsilon} &= -2\hat{F}(\epsilon)\hat{G}(\epsilon), \\
\frac{\partial\hat{H}(\epsilon)}{\partial\epsilon} &= -2[\hat{G}^2(\epsilon) + \hat{F}(\epsilon)\hat{H}(\epsilon)], \\
\frac{\partial\hat{P}(\epsilon)}{\partial\epsilon} &= b\hat{G}(\epsilon),
\end{aligned} \tag{40b}$$

$$\hat{A}(0) = A, \quad \hat{B}(0) = B, \quad \hat{F}(0) = f, \quad \hat{G}(0) = g, \quad \hat{H}(0) = h, \quad \hat{P}(0) = p, \tag{40c}$$

one can achieve the finite transformation theorem.

**Theorem** If $\{u, v, f, g, h, p, h\}$ is a solution of the extended system (4), (29), (33) and (34), so is $\{\hat{A}(\epsilon), \hat{B}(\epsilon), \hat{F}(\epsilon), \hat{G}(\epsilon), \hat{H}(\epsilon), \hat{P}(\epsilon)\}$ with

$$\hat{A}(\epsilon) = A + \frac{b\epsilon g}{1 + \epsilon f} + \frac{6\gamma \epsilon h}{\beta(1 + \epsilon f)} - \frac{6\gamma \epsilon^2 g^2}{\beta(1 + \epsilon f)^2},$$

$$\hat{B}(\epsilon) = B - \frac{b\epsilon g}{1 + \epsilon f} + \frac{6\gamma \epsilon h}{\beta(1 + \epsilon f)} - \frac{6\gamma \epsilon^2 g^2}{\beta(1 + \epsilon f)^2}, \tag{41a}$$

$$\hat{F}(\epsilon) = \frac{f}{1 + \epsilon f},$$

$$\hat{G}(\epsilon) = \frac{g}{(1 + \epsilon f)^2},$$

$$\hat{H}(\epsilon) = \frac{h}{(1 + \epsilon f)^2} - \frac{2\epsilon g^2}{(1 + \epsilon f)^3},$$

$$\hat{P}(\epsilon) = \frac{b\epsilon g}{1 + \epsilon f} + p. \tag{41b}$$

where $\epsilon$ is an arbitrary group parameter.

## Summary and Conclusion

It is believed that there are some really novel phenomena in two-place nonlocal systems compared with local one. In this article, we studied the nonlocal Boussinesq equation coupled with AB systems. Furthermore, After introducing the bilinear variable transformation, the different expressions of function in the Bäcklund transformation are obtained. By choosing special parameters, the abundant solutions, including symmetry breaking rogue wave, soliton and breather solutions of the AB-Boussinesq system, are discussed in detail. At last, with the derived modified Bäcklund transformation, the residual symmetry of the nonlocal Boussinesq **system** is researched by introducing two auxiliary variables, and the finite symmetry transformation is obtained through solving the initial value problem. In brief, the variable coefficient with shifted parity and delayed time reversal in nonlocal AB-Boussinesq system is discussed, from which abundant of controllable symmetry breaking solutions are illustrated by changing parameters of events A and B. It is valuable to the nonlocal AB symmetric system analysis and the associated applications with mathematical models into more real physical systems for further study.

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## Author contributions

Zhengyi Ma and Weiping Cao proposed the idea and wrote the main manuscript text and Jinxi Fei prepared all figures and calculated the equations. All authors reviewed the manuscript.

## Competing interests

The authors declare no competing interests.

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