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Exact solution of an exciton energy for a monolayer medium

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We present exact solutions of an energy spectrum of 2-interacting particles in which they seem to be relativistic fermions in 2 + 1 space-time dimensions. The 2 × 2 spinor equations of 2-interacting fermions through general central potential were separated covariantly into the relative and center of mass coordinates. First of all, the coupled first order differential equations depending on radial coordinate were derived from 2 × 2 spinor equations. Then, a second order radial differential equation was obtained and solved for Coulomb interaction potential. We apply our solutions to exciton phenomena for a free-standing monolayer medium. Since we regard exciton as isolated 2-interacting fermions in our model, any other external effect such as substrate was eliminated. Our results show that the obtained binding energies in our model are in agreement with the literature. Moreover, the decay time of an exciton was found out spontaneously in our calculations.

Since its invention of monolayer form[1], graphene has attracted noticeable interest due to its extraordinary optical[2], electrical[3] and structural[4] properties apart from its bulk form of graphite especially. Nowadays, another family of atomically thin materials, transition metal dichalcogenides (TMDs), have came into interest[5].

As direct-gap semiconductors[6], the monolayer members of TMD family (WS_2, WSe_2, MoS_2, $MoSe_2$ etc.) have a strong potential in optoelectronic applications[7]. In semiconductors, promoting of an electron from valance to conduction band leaving a hole behind creates an exciton by photo-excitation which especially finds applications in monolayer TMDs due to higher binding energies than in that of bulk semiconductors[8,9] The spatial distance of these two particles is a result of screening effect of surrounding medium through Coulomb interaction[10]. These excitons can be used in variety of applications[11,12]. For example, extending exciton lifetimes in monolayer TMDs give possibility of energy storage in excitonic dark states even at room temperatures[13].

Although there are much experimental studies focusing on measurement of exciton binding energies for monolayer TMDs[14,15] and theoretical approaches for prediction of energy spectra of excitonic states[16–20] the obtained results and predictions vary in the range of 0.3–1 eV[9,14,15]. Some research groups explain the differences by the substrate effects in which the monolayer TMD sample rely on it[21]. In addition, triangular lattice model was developed in order to explain non-hydrogen-like behavior of these quasi-particles[22]. In the traditional approach; the exciton is regarded as a single particle where the electron is attracted by the Coulomb potential of an hole where screening effect gives possibility of excitonic states.

On the other hand, exploring of relativistic dynamics of interacting particles has been very attractive in Quantum Field Theory (QFT). Although these topics have been studied for a long time, there are still much efforts to explain significant differences between experimental and theoretical results[23]. For example, relativistic two-body equation was written first by Eddington and Gaunt in 1929[24]. In addition, a new relativistic Two Body Equation (TBE) was introduced short after[25] which is relatively straightforward for interacting spin-1/2 particles since it includes approximate Darwin Lagrangian as an interaction term and 2-free Dirac Hamiltonian. This effective Hamiltonian fails either the velocity of particles or distance between particles is very high. Therefore, it covers actually weak coupling approximation.

Another research group obtained a different formulation of relativistic equation[26] in the Quantum Field Theory (QFT) which satisfies the physics in[25] up to the non-relativistic regime. Since they found negative solutions for binding energy, their equation was not valid as a bound-state solution of 2-interacting particles due to relative time approach.

Since we focus on the exciton interaction energy, bound-state equation of relativistic 2-Dirac particles is very important[27,28]. However, the interaction potential was written phenomenologically in these studies. A complete equation which is quite similar to[27,28], starting from Quantum Electro-Dynamics (QED) and using Lagrange

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formalism assumed as a first principal in theoretical physics has been derived. This equation consists of most general electric and magnetic potentials[29]. Therefore, its solution gives us a well-known spectra similar to hydrogen-like atoms in all order (α^6) even though it could not be solved completely[30].

One can use equations in[29] for solving of 2-body problem in $(2+1)$ dimensions due to axial symmetry in Coulomb-like potentials and to obtain especially binding energy of electron-hole coupling (exciton) in monolayer materials. Although evaluating equations in $(3+1)$ space-time is sometimes more complicated[23,31–33], published results studied in $(2+1)$ dimensions[34,35] indicate that obtained results in $(2+1)$ dimensions are very close to results obtained in $(3+1)$ dimensions. Therefore, the direct solution of an exciton binding energy would be perfect regardless of surrounding medium. In this current work, exciton is regarded as 2 interacting oppositely charged fermions. Moreover, the solutions were obtained in $2+1$ dimensions for a free-standing monolayer medium in order to eliminate any other external effects such as substrate, surrounding medium. For these purposes, two-body problem is exactly solved in $2+1$ dimensions, firstly. Then, exciton binding energies and decay times are obtained in terms of spin and energy level of interacting particles as follows.

Two-body Dirac Equation in $(2+1)$ Dimensions

We first start from two-body Dirac equation in $(3+1)$ dimensions[29] and fully covariant separation of center of mass and relative variable for acquiring energy spectrum of 2-interacting particles.

$$\left\{\left[\gamma_\mu^{(1)} i\hbar\left(\partial_\mu^{(1)} - e_1 A_\mu^{(2)}\right) - m_1 c\right] \otimes \gamma_0^{(2)} + \gamma_0^{(1)} \otimes \left[\gamma_\mu^{(2)} i\hbar\left(\partial_\mu^{(2)} - e_2 A_\mu^{(1)}\right) - m_2 c\right]\right\}\phi(\mathbf{X}_1, \mathbf{X}_2) = 0 \tag{1}$$

Since the Dirac matrices have been chosen as $\gamma_0 = \sigma^3$, $\gamma_1 = i\sigma^1$ and $\gamma_2 = i\sigma^2$ satisfying Dirac algebra and two-body Dirac spinor is defined by Eq. 2 as follows;

$$\phi(\mathbf{X}_1, \mathbf{X}_2) = \begin{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \\ \begin{pmatrix} D_3 \\ D_4 \end{pmatrix} \end{pmatrix} \tag{2}$$

Equation 1 is reduced to $(2+1)$ dimensions. Thus, we could obtain a coupled equation set from Eq. 1 where $D_i = D_i(\mathbf{r}, \mathbf{R}, R_0)$. The explicit form of D_i is given by the following expression.

$$D_i = \Psi_i(r)\Phi(R)e^{-iwR_0},\ \Phi(R) = e^{-i\mathbf{k}.\mathbf{R}}$$

Here, D_i's where i = 1, 2, 3, 4 are the spinor components and we use relative (r) and center of mass (R) coordinates explicitly. Then, we introduce the $(2+1)$-dimensional center of mass and relative variables with the following expressions.

$$\begin{aligned} R &= \frac{1}{M}(m_1 X_1 + m_2 X_2) \\ r &= X_1 - X_2 \\ M &= m_1 + m_2 \\ \mathbf{X}_1 &= (x^{(1)}, y^{(1)}, t^{(1)}) \\ \mathbf{X}_2 &= (x^{(2)}, y^{(2)}, t^{(2)}) \end{aligned} \tag{3}$$

Here, m_1 and m_2 are the masses of particles where X_1 and X_2 are the magnitudes of position vectors of corresponding particles in $2+1$ dimensions, respectively. Writing positions of particles in terms of center of mass and relative coordinates

$$\begin{aligned} X_1 &= \frac{m_2}{M}\left(r + \frac{M}{m_2}R\right) \\ X_2 &= \frac{m_1}{M}\left(-r + \frac{M}{m1}R\right) \end{aligned} \tag{4}$$

and partial derivative of X_1 and X_2 give us the following equations.

$$\begin{aligned}\frac{\partial^{(1)}}{\partial X_\mu} &= \frac{\partial}{\partial r_\mu} + \frac{m_1}{M}\frac{\partial}{\partial R_\mu}\\ \frac{\partial^{(2)}}{\partial X_\mu} &= -\frac{\partial}{\partial r_\mu} + \frac{m_2}{M}\frac{\partial}{\partial R_\mu}\\ \frac{\partial^{(1)}}{\partial X_\mu} + \frac{\partial^{(2)}}{\partial X_\mu} &= \frac{\partial}{\partial R_\mu}\\ \frac{\partial}{\partial r_\mu} &= \frac{1}{M}\left(m_2\frac{\partial^{(1)}}{\partial X_\mu} - m_1\frac{\partial^{(2)}}{\partial X_\mu}\right)\end{aligned} \tag{5}$$

where $\mu = 0,\ 1,\ 2$ and

$$\partial_0 = \partial_t/c, \quad \partial_1 = \partial_x, \quad \partial_2 = \partial_y.$$

For central interaction; the components of vector potential are taken as follows $A_0 = V(\mathbf{x}_1 - \mathbf{x}_2)$, $A_1 = 0$, $A_2 = 0$.

Derivation of Radial Equations

We can apply a method of separation of variables in terms of relative and center of mass coordinates provided that a central potential is taken into account; $V \equiv V(\mathbf{r})$.

In our calculations, we assume that the center of mass does not carry a momentum because we take k as $k = 0$. Therefore, one can write set of equation in terms of relative coordinates.

In the second part of calculations, we reorganized the coupled equation set by inserting Eq. 5 and definition of D_i, we obtained the following equations.

$$\left(\frac{w}{c} - \frac{Mc}{\hbar} - iV_t(r)\right)\Psi_1 + \left(\partial_{r_1} - i\partial_{r_2}\right)\Psi_2 - \left(\partial_{r_1} - i\partial_{r_2}\right)\Psi_3 = 0, \tag{6}$$

$$\left(-\frac{w}{c} + \frac{\Delta mc}{\hbar} + iV_t(r)\right)\Psi_2 + \left(\partial_{r_1} + i\partial_{r_2}\right)\Psi_1 + \left(\partial_{r_1} - i\partial_{r_2}\right)\Psi_4 = 0, \tag{7}$$

$$\left(-\frac{w}{c} - \frac{\Delta mc}{\hbar} + iV_t(r)\right)\Psi_3 - \left(\partial_{r_1} - i\partial_{r_2}\right)\Psi_4 - \left(\partial_{r_1} + i\partial_{r_2}\right)\Psi_1 = 0, \tag{8}$$

$$\left(\frac{w}{c} + \frac{Mc}{\hbar} - iV_t(r)\right)\Psi_4 - \left(\partial_{r_1} + i\partial_{r_2}\right)\Psi_3 + \left(\partial_{r_1} + i\partial_{r_2}\right)\Psi_2 = 0 \tag{9}$$

In order to write the coupled equation set in an explicit form, we use both

$$\left(\partial_{r_1} - i\partial_{r_2}\right) = e^{-i\Phi}\left(-\frac{i}{r}\partial_\Phi + \partial_r\right) \tag{10}$$

and

$$\left(\partial_{r_1} + i\partial_{r_2}\right) = e^{i\Phi}\left(\frac{i}{r}\partial_\Phi + \partial_r\right) \tag{11}$$

definitions and obtained the following explicit equation set in polar coordinates. In addition, we multiply the Eq. 6 by $e^{i\Phi}$ and the Eq. 9 by $e^{-i\Phi}$ on the left side. For the Eqs 7 and 8, the phase terms ($e^{i\Phi}$ and $e^{-i\Phi}$) are required to be positioned in front of the operators paving the way that an additional term is included in front of the corresponding operator. By conducting these, we end up with the following equations;

$$\begin{aligned}&\left(\frac{w}{c} - \frac{Mc}{\hbar} - iV_t(r)\right)\Psi_1(r,\Phi)e^{i\Phi} + \left(-\frac{i}{r}\partial_\Phi + \partial_r\right)\Psi_2(r,\Phi)\\ &\quad - \left(-\frac{i}{r}\partial_\Phi + \partial_r\right)\Psi_3(r,\Phi) = 0\end{aligned} \tag{12}$$

$$\begin{aligned}&\left(-\frac{w}{c} + \frac{\Delta mc}{\hbar} + iV_t(r)\right)\Psi_2(r,\Phi) + \left(\frac{i}{r}\partial_\Phi + \frac{1}{r} + \partial_r\right)\Psi_1(r,\Phi)e^{i\Phi}\\ &\quad + \left(-\frac{i}{r}\partial_\Phi + \frac{1}{r} + \partial_r\right)\Psi_4(r,\Phi)e^{-i\Phi} = 0\end{aligned} \tag{13}$$

$$\left(-\frac{w}{c}-\frac{\Delta mc}{\hbar}+iV_t(r)\right)\Psi_3(r,\Phi)-\left(-\frac{i}{r}\partial_\Phi+\frac{1}{r}+\partial_r\right)\Psi_4(r,\Phi)e^{-i\Phi}-\left(\frac{i}{r}\partial_\Phi+\frac{1}{r}+\partial_r\right)\Psi_1(r,\Phi)e^{i\Phi}=0 \tag{14}$$

$$\left(\frac{w}{c}+\frac{Mc}{\hbar}-iV_t(r)\right)\Psi_4(r,\Phi)e^{-i\Phi}-\left(\frac{i}{r}\partial_\Phi+\partial_r\right)\Psi_3(r,\Phi)+\left(\frac{i}{r}\partial_\Phi+\partial_r\right)\Psi_2(r,\Phi)=0 \tag{15}$$

where $\Psi_g(r,\Phi)=\Psi_g(r)e^{ij_g\Phi}(g=1,2,3,4)$.

Since the Φ is a cyclic coordinate, we derive a coupled first order 4-differential equations by using this property. Then, multiplying the Eq. 12 by $\left(\frac{w}{c}+\frac{Mc}{\hbar}-iV_t(r)\right)$, the Eq. 15 by $\left(\frac{w}{c}-\frac{Mc}{\hbar}-iV_t(r)\right)$, the Eq. 13 by $\left(-\frac{w}{c}-\frac{\Delta mc}{\hbar}+iV_t(r)\right)$ and the Eq. 14 by $\left(-\frac{w}{c}+\frac{\Delta mc}{\hbar}+iV_t(r)\right)$ on the left side of equations cause in a new definition set indicated as follows.

Derivation of Complete 2nd Order Equation

By pursuing above mathematical procedures and substituting $\Psi_g(r,\Phi)$, we obtained the following equations for general central potentials.

$$(U(r)^2-M_1^2)\Psi^+(r)+2\left(M_1\frac{j}{r}+U(r)\frac{d}{dr}\right)\Psi_0(r)=0 \tag{16}$$

$$(U(r)^2-M_1^2)\Psi^-(r)+2\left(U(r)\frac{j}{r}+M_1\frac{d}{dr}\right)\Psi_0(r)=0 \tag{17}$$

$$(U(r)^2-M_2^2)\Psi_0(r)+2\left(M_2\frac{j}{r}\right)\Psi^-(r)-2M_2\frac{d}{dr}\Psi^+(r)=0 \tag{18}$$

$$(U(r)^2-M_2^2)\Psi_0(r)+2\left(U(r)\frac{j}{r}\right)\Psi^-(r)-2U(r)\frac{d}{dr}\Psi^+(r)=0 \tag{19}$$

where $M_1=\frac{Mc}{\hbar}$, $M_2=\frac{\Delta mc}{\hbar}$, and $U(r)=\left(\frac{w}{c}-iV_t(r)\right)$. From Eqs 16 and 17, we rewrite $\Psi^+(r)$ and $\Psi^-(r)$ in terms of $\Psi_0(r)$.

$$\Psi^+(r) = -\frac{2}{(U(r)^2-M_1^2)}\left(M_1\frac{j}{r}+U(r)\frac{d}{dr}\right)\Psi_0(r),$$
$$\Psi^-(r) = -\frac{2}{(U(r)^2-M_1^2)}\left(U(r)\frac{j}{r}r+M_1\frac{d}{dr}\right)\Psi_0(r)$$

Inserting these expressions into the Eq. 19, we obtain a radial second order differential equation which contains general central potential.

Solution for a Coulomb Interaction

Since the exciton is composed of an electron and hole, the interaction between them can be written in terms of Coulomb potential. In addition, these particles (an electron and a hole) posses very small masses. Therefore, we can neglect the M_1^2 and M_2^2 in our equations.

$$\frac{d^2}{dr^2}\Psi_0(r)-\left[\frac{U'(r)}{U(r)}\right]\frac{d}{dr}\Psi_0(r)+\left[\frac{U(r)^2}{4}-\frac{j^2}{r^2}-\frac{2jU'(r)M_1}{U(r)^2}-\frac{jM_1}{U(r)r^2}\right]\Psi_0(r)=0 \tag{20}$$

Here we defined a new dimensionless variable; $z=\frac{iwr}{2\alpha_c}$ for $V_t(r)=-2\frac{\alpha}{r}$. Then, the solution of simplified Eq. 20 becomes a HeunC(α_1, β, γ, δ, η, z) function. In this function, frequency relation is found by using $\delta=-\left(n+\frac{\beta+\gamma+2}{2}\right)\alpha_1$ expression. The following parameters;

n	j	E (eV)	Decay Time (ps)
1	−1/2	(0.214 + i 0.214)	0.01
	1/2	(0.214 − i 0.214)	
2	−1/2	(0.105 + i 0.105)	0.02
	1/2	(0.105 − i 0.105)	
	3/2	(0.370 − i 0.370)	0.005

Table 1. Interaction energies for an exciton. (The masses of electron and hole (m_1 and m_2) are taken to be equal as $m_1 = m_2 = 9.10938 \times 10^{-31}$ kg).

$$\begin{aligned}
\alpha_1 &= 2\alpha, \\
\beta &= -2\sqrt{\alpha^2 + j^2}, \\
\gamma &= \frac{2}{(w)}\sqrt{-4iNc^2 j\alpha + (w)^2}, \\
\delta &= 2\alpha^2 \\
2\alpha^2 &+ 2\left(n - \sqrt{\alpha^2 - j^2} + \frac{\sqrt{-4iN\alpha c^2 j + w^2}}{w}\right)\alpha = 0
\end{aligned}$$

yields

$$\begin{aligned}
w_{(n,j)} &= \pm 2\sqrt{iNc^2\alpha j}\,[2\sqrt{\alpha^2 - j^2}\,\alpha + 2\sqrt{\alpha^2 - j^2}\,n - 2\alpha^2 \\
&\quad - 2\alpha n - j^2 - n^2 + 2\sqrt{\alpha^2 - j^2} - 2\alpha - 2n]^{-1/2}
\end{aligned} \tag{21}$$

where $N = \frac{Mc}{\hbar}$, $M = m_1 + m_2$.

The explicit expression of $w_{(n,j)}$ allows us to obtain interaction energy of an exciton. This expression covers actually all the energy levels (n) where the particles might be and spin value of interacting particles. For clarity and quantification, a detailed table (Table 1) is composed of these variables.

Discussion

Our calculations are very close to results found in literature[15]. Since our model exclude any other environmental effects such as substrate or crystal structure, we found a little bit lower value for an exciton binding energy. The experimental measurements of an exciton binding energy are based on the photo-excitation of an electron from a valance to a conduction band leaving a hole behind. While the interaction between an electron and a hole is totally attractive, the screening effect allows an exciton to be created. Since we only use the masses of the electron and the hole in our calculations, the calculated value for exciton energy (0.214 eV) is valid in all 2-dimensional systems. Moreover, we presume that the differences between exciton binding energies in the literature (even for the same samples-WS_2[14,15]) stem from surrounding environment such as substrate or crystal structure. Moreover, we think that the origin of the differences in experimental results of exciton binding energy is due to screening effect depending on the substrate or the crystal structure. Therefore, experimental measurements include this screening energy. Another interesting result of our calculations is that the interaction energy of 2-fermions possesses imaginary part. By using this imaginary part, one can calculate the decay time (τ) of an exciton. Based on the definition in $\tau \propto \frac{1}{w_{(n,j)}}$, we found relaxation time as 0.01 ps for n = 1 condition.

Conclusion

In this work, we found a general definition of interaction energy for 2-interacting fermions. First, we had examined two Dirac particles interacting with their central potential before we derived two-body Dirac equation in (2 + 1) space-time geometry. For solution of this equation, we separated center of mass and relative coordinates by using explicit form of the equation. Then, we obtained 1^{st} order coupled radial differential equation set. Since the masses of an electron and a hole are very small (m_1 and m_2), we neglected M_1^2 and M_2^2 values in our equations. The solution of 2-interacting fermions in our model indicates that one can obtain general definition for an interaction energy including quantum numbers. We apply our model to an exciton and found its binding energy which is very close to results found in literature. Moreover, since we work in 2 + 1 dimensions, our results do not include any other effects for a monolayer medium. Finally, we calculated the decay time of an exciton (~ps) directly because the interaction energy in our calculations possesses imaginary part.

References

1. Geim, A. K. & Novoselov, K. S. The rise of graphene. *Nat. Mater.* **6**, 183–191 (2007).
2. Bonaccorso, F., Sun, Z., Hasan, T. & Ferrari, A. C. Graphene photonics and optoelectronics. *Nat.Photonics* **4**, 611–622 (2010).
3. Castro Neto, A. H., Guinea, F., Peres, N. M. R., Novoselov, K. S. & Geim, A. K. The electronic properties of graphene. *Rev. Mod. Phys.* **81**, 109–162 (2009).
4. Singh, V. *et al*. Graphene based materials: Past, present and future. *Prog. Mater. Sci.* **56**, 1178–1271 (2011).
5. Wang, Q. H., Kalantar-Zadeh, K., Kis, A., Coleman, J. N. & Strano, M. S. Electronics and optoelectronics of two-dimensional transition metal dichalcogenides. *Nat. Nanotechnol.* **7**, 699–712 (2012).

6. Mak, K. F., Lee, C., Hone, J., Shan, J. & Heinz, T. F. Atomically thin MoS_2: A new direct-gap semiconductor. *Phys. Rev. Lett.* **105**, 136805 (2010).
7. Mak, K. F. & Shan, J. Photonics and optoelectronics of 2D semiconductor transition metal dichalcogenides. *Nat. Photonics* **10**, 216–226 (2016).
8. Novoselov, K. *et al*. Two-dimensional atomic crystals. *Proc. The Natl. Acad. Sci. The United States Am.* **102**, 10451–10453 (2005).
9. Hanbicki, A. T., Currie, M., Kioseoglou, G., Friedman, A. L. & Jonker, B. T. Measurement of high exciton binding energy in the monolayer transition-metal dichalcogenides WS_2 and WSe_2. *Solid State Commun.* **203**, 16–20 (2015).
10. Singh, J. *Theory of Excitons*, 1–45. Excitation Energy Transfer Processes in Condensed Matter: Theory and Applications (Springer US, 1994).
11. Nozik, A. J. *et al*. Semiconductor quantum dots and quantum dot arrays and applications of multiple exciton generation to third-generation photovoltaic solar cells. *Chem. Rev.* **110**, 6873–6890 (2010).
12. Zhao, W.-W. *et al*. Exciton-plasmon interactions between cds quantum dots and ag nanoparticles in photoelectrochemical system and its biosensing application. *Anal. Chem.* **84**, 5892–5897 (2012).
13. Tseng, F., Simsek, E. & Gunlycke, D. Using dark states for exciton storage in transition-metal dichalcogenides. *J. Physics-Condensed Matter* **28**, 034005 (2016).
14. Ye, Z. *et al*. Probing excitonic dark states in single-layer tungsten disulphide. *Nature* **513**, 214–218 (2014).
15. Chernikov, A. *et al*. Exciton binding energy and nonhydrogenic rydberg series in monolayer WS_2. *Phys. Rev. Lett.* **113**, 076802 (2014).
16. Ramasubramaniam, A. Large excitonic effects in monolayers of molybdenum and tungsten dichalcogenides. *Phys. Rev. B* **86**, 115409 (2012).
17. Shi, H., Pan, H., Zhang, Y.-W. & Yakobson, B. I. Quasiparticle band structures and optical properties of strained monolayer MoS_2 and WS_2. *Phys. Rev. B* **87**, 155304 (2013).
18. Berkelbach, T. C., Hybertsen, M. S. & Reichman, D. R. Theory of neutral and charged excitons in monolayer transition metal dichalcogenides. *Phys. Rev. B* **88**, 045318 (2013).
19. Van der Donck, M., Zarenia, M. & Peeters, F. M. Excitons and trions in monolayer transition metal dichalcogenides: A comparative study between the multiband model and the quadratic single-band model. *Phys. Rev. B* **96**, 035131 (2017).
20. Van der Donck, M., Zarenia, M. & Peeters, F. M. Excitons, trions, and biexcitons in transition-metal dichalcogenides: Magnetic-field dependence. *Phys. Rev. B* **97**, 195408 (2018).
21. Komsa, H.-P. & Krasheninnikov, A. V. Effects of confinement and environment on the electronic structure and exciton binding energy of MoS_2 from first principles. *Phys. Rev. B* **86**, 241201 (2012).
22. Gunlycke, D. & Tseng, F. Triangular lattice exciton model. *Phys. Chem. Chem. Phys.* **18**, 8579–8586 (2016).
23. Whitney, J. F. & Crater, H. W. Baryon spectrum analysis using dirac's covariant constraint dynamics. *Phys. Rev. D* **89**, 014023 (2014).
24. Van Alstine, P. & Crater, H. W. A tale of three equations: Breit, eddington—gaunt, and two-body dirac. *Foundations of Phys.* **27**, 67–79 (1997).
25. Breit, G. The effect of retardation on the interaction of two electrons. *Phys. Rev.* **34**, 553–573 (1929).
26. Salpeter, E. E. & Bethe, H. A. A relativistic equation for bound-state problems. *Phys. Rev.* **84**, 1232–1242 (1951).
27. Kemmer, N. Field theory of nuclear interaction. *Phys. Rev.* **52**, 906–910 (1937).
28. Fermi, E. & Yang, C. N. Are mesons elementary particles? *Phys. Rev.* **76**, 1739–1743 (1949).
29. Barut, A. O. & Komy, S. Derivation of nonperturbative relativistic two-body equations from the action principle in quantumelectrodynamics. *Fortschritte der Physik/Progress of Phys.* **33**, 309–318 (1985).
30. Barut, A. O. & Unal, N. Radial equations for the relativistic two-fermion problem with the most general electric and magnetic potentials. *Fortschritte der Physik/Progress of Phys.* **33**, 319–332 (1985).
31. Crater, H. W. & Van Alstine, P. Two-body dirac equations for meson spectroscopy. *Phys. Rev. D* **37**, 1982–2000 (1988).
32. Aydin, Z. Z. & Yilmazer, A. U. On the relativistic two-fermion problem. *J. Phys. G: Nucl. Phys.* **14**, 1345 (1988).
33. Berman, O. L., Kezerashvili, R. Y. & Ziegler, K. Coupling of two dirac particles. *Phys. Rev. A.* **87**, 042513 (2013).
34. Dong, S.-H. & Ma, Z.-Q. Exact solutions to the dirac equation with a coulomb potential in $2+1$ dimensions. *Phys. Lett. A* **312**, 78–83 (2003).
35. Witten, E. $2+1$ dimensional gravity as an exactly soluble system. *Nucl. Phys. B* **311**, 46–78 (1988).

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Author Contributions

A.G. performed the calculation procedure. Y.S. and R.S. suggested the problem and discussed the physics behind the interaction mechanism. A.G. and R.S. analyzed the results. All authors equally contributed to preparation of the manuscript.

Additional Information

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