Spanning space

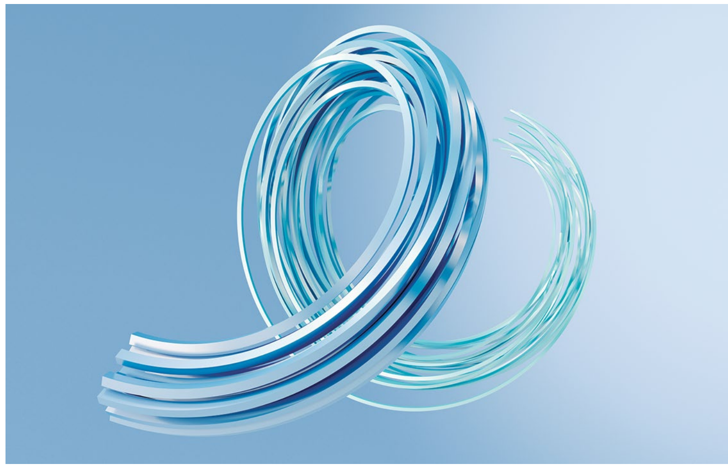
Solid angle is an ancient notion with modern relevance. A one-page primer by Ben Kravitz.

A solid angle, quantified by the dimensionless SI unit steradian (sr), measures how big an object appears to an observer. This concept can be traced back to the time of Archimedes, and the term appears, for example, in a geometry book by Emerson in 1794[1].

Solid angle relates to projective geometry[2]. Imagine you're outside on a sunny day. To shield the Sun from your eyes, you might hold up your hand. Of course, the Sun is much bigger than your hand, but it's also much farther away. So because shielding the Sun with your hand actually works, one can say that the Sun subtends a smaller solid angle than your hand. Adapting this idea, if a surface S is projected onto a unit sphere, then the surface area of that sphere that is covered by that projection is defined as the solid angle subtended by S. In the example of shielding your eyes from the sun, the sky is the sphere that completely surrounds you, and S is (the surface of) your hand.

In spherical coordinates, the solid angle Ω is defined as $\Omega = \frac{1}{r^2}\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\sin\theta \ \mathrm{d}\theta \ \mathrm{d}\phi$, where θ is the polar angle (latitude in geography or declination in astronomy), ϕ is the azimuthal angle (longitude or right ascension) and r is the radius of the sphere projected on. If integrating over the entire sphere ($\theta_1 = 0 \leq \theta \leq \theta_2 = \pi$ and $\phi_1 = 0 \leq \phi \leq \phi_2 = 2\pi$), for a unit sphere, one obtains $\Omega = 4\pi$ sr. In another definition (amounting to the same), $\Omega = A/r^2$, where A is the spherical surface area bounded by S. Note that the farther away an object, the smaller the solid angle it subtends; this is why shielding the Sun with your hand is effective, even though the Sun is over one million km across, and your hand isn't. If S is a circle (that is, S subtends a cone with its tip at the centre of the projection sphere), then $\Omega = \frac{2\pi}{r^2}(1-\cos\zeta)$, where ζ is the cone's opening angle.

The solid angle idea is useful for a variety of applications in electromagnetic radiation propagation and remote sensing[3]. For example, consider a satellite aimed at the Earth to measure reflectivity. If that reflectivity is measured at multiple wavelengths and multiple angles θ, then one can derive a great deal of information about the surfaces or objects being observed.



Credit: Getty/Westend61

Satellite detectors essentially measure radiance L, which is a measure of the radiant flux Φ propagating toward or away from a surface in a specified direction in units W m^{-2} sr^{-1}. Formally: $L = \Phi/\Omega A\cos\theta$, where A is the area of the ellipse formed by the observation cone with solid angle Ω, and θ is the observing angle ($\theta = 0$ corresponds to observations from directly overhead). Ω is likely to be fixed for any observing instrument, and from Ω and θ, A can be calculated. Then, by measuring various L and knowing the input radiant flux (in many cases the energy input from the Sun), one can calculate the reflectivity of the surface.

While 'developments' in calculating solid angles — there is no tool to physically measure them — are obviously not to be expected as it's a mathematical construction, the concept is relevant in present-day physical applications, including signal propagation or detection through a medium. (And, as physicists know, solid angle plays a key role in the derivation of Gauss's law, which describes the distribution of an electric charge to the resulting electric field.) Any transmitting antenna has an associated solid angle (the radiation intensity of the antenna is the power radiated per unit solid angle), which determines how narrowly its beam is focused — that is, how much of the antenna's power is aimed in a particular direction[4]. A recent application involves guided rays in fibre optics; the acceptance cone of the fibre, which is expressed using solid angle, describes the parameters under which signals will be guided by the fibre. Image synthesis and ray-tracing algorithms used in computer graphics (pictured) heavily rely on methodologies for uniform sampling of the solid angle subtended by a disc[5].

The fundamentals of solid angles can be understood from a first-year calculus course, and are unlikely to change. Yet new applications involving solid angles and their quantification are likely to continue to emerge in the coming years, sustaining the importance of this simple, ancient concept. ❐

Ben Kravitz

Atmospheric Sciences and Global Change Division, Pacific Northwest National Laboratory, Richland, WA, USA.

e-mail: ben.kravitz@pnnl.gov

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