

## ARTICLE OPEN

## Mirrored entanglement witnesses

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Entanglement detection, which signifies the task of distinguishing entangled states from separable states, can be generally performed by realizing entanglement witnesses via local measurements on a single-copy level and classical communication, and are known to be experimenter friendly. We introduce a framework of constructing mirrored entanglement witnesses by showing that an experimental observable is twice as effective since it generally provides bounds from above and below for separable states. Differently stated, a pair of witnesses, mirrored witnesses, exists for the characterization of the bounds, which are two faces of one observable. We show how to generally construct those witnesses and provide examples for bipartite and multipartite systems. We also show that both mirrored witnesses can be improved by introducing nonlinearities, by which a larger set of entangled states can be certified.

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## INTRODUCTION

Revealing properties of an unknown quantum state of a physical system is of fundamental importance in quantum information theory and its application. By quantum state tomography, the full knowledge of a given physical system can be determined after a series of quantum measurements, which is, however, generally an expensive and cost-inefficient procedure demanding experimental resources. Moreover, one is often only interested in the question of whether a multipartite state is entangled or not and what type of entanglement is present, for instance bipartite or genuinely multipartite entanglement. Entanglement witnesses (EWs), particular observables, offer the solution with currently feasible technologies. EWs typically require less experimental setups to unambiguously verify the existence of entanglement and even additional information on the type of entanglement may be revealed. This concept relies on the fact that the set of separable states is convex, i.e. EWs correspond to hyperplanes separating some entangled states from the set of separable states in the Hilbert–Schmidt space.<sup>1</sup> In high-dimensional and multipartite systems also the entanglement structure itself becomes of interest which has a nested convex structure.<sup>2–4</sup> In the recent years, there has been much progress in the theory and experimental realization of EWs for multipartite entangled states (see e.g. refs. <sup>5–7</sup>).

EWs exhibit numerous advantages. They are observables on a single-copy level and are factored into local observables, namely no entanglement resources are needed to realize EWs. These properties make them suited for experimental investigations. A major drawback is that a priori information about entangled states to be detected is needed to be at hand in advance. This can be explained in Fig. 1 as follows. Suppose that an entangled state  $\tau$  is detected by a witness  $W$ , which however does not detect states  $\rho$  and  $w$ . The witness  $W$  can be optimized to construct a finer one  $W^{(-)}$ , that can detect a state  $\rho$  but not yet a state  $w$  at the other side. Therefore, in order to detect a state  $w$ , a priori information about the state is needed such that a relevant witness  $W^{(+)}$  can be

constructed. Or, to detect a state  $\rho$ , one may also need a proper optimization technique to lift  $W$  to  $W^{(-)}$ .<sup>8</sup>

In this work, we establish the framework of *mirrored EWs* that can construct a pair of EWs from an EW, and show that both of the EWs can perform detecting entangled states by realizing a single observable. In other words, an EW  $W^{(+)}$  has its “twin” one denoted by  $W^{(-)}$ , called a mirrored witness, such that the pair of EWs can be realized by a single observable. In Fig. 1, this can be illustrated that EWs  $W^{(+)}$  and  $W^{(-)}$  serve the characterization of lower and upper bounds of an observable  $A$  for separable states. Then, once an observable  $A$  is realized, entangled states are detected from violations of the lower or the upper bounds, which are equivalent to entanglement detection by  $W^{(+)}$  or  $W^{(-)}$ , respectively. Note that both a state  $\rho$ , which is not detected by  $W^{(+)}$ , and a state  $w$ , not by  $W^{(-)}$ , can be detected by realizing a single observable  $A$ . Hence, introducing mirrored EWs, one can double the capability of detecting entangled states. We show that mirrored EWs can be constructed from upper and lower bounds of an observable for separable states and also present how mirrored EWs are related to the structural physical approximation. Examples are presented for the illustration of mirrored EWs. Finally, we investigate the construction of nonlinear EWs from mirrored witnesses and show that mirrored EWs can be improved by both subtraction and addition of nonlinear polynomials.

Let us review the basic properties of entanglement witnesses. Throughout, let  $W$  denote an EW such that

$$\forall \sigma \in S_{\text{sep}}, \text{tr}[W\sigma] \geq 0, \text{ and } \exists \rho \notin S_{\text{sep}}, \text{tr}[W\rho] < 0, \quad (1)$$

where  $S_{\text{sep}}$  is the set of separable states. Assuming bipartite systems, an EW can be factorized into local measurements, i.e.,  $W = \sum_i c_i M_i^{(A)} \otimes N_i^{(B)}$  for some constants  $\{c_i\}$  and positive operator-valued measure (POVM)  $\{M_i^{(A)}\}$  and  $\{N_i^{(B)}\}$ , that give descriptions of measurement devices. Note that an EW can be generally factorized into local measurements.<sup>9</sup> We can restrict the consideration to normalized EW without loss of generality, i.e.,  $\text{tr}W = 1$ , on a Hilbert space  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$  where  $d_i = \dim \mathcal{H}_i$  and  $D = d_1 d_2$ .

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### Mirrored structural physical approximations

It turns out that our approach is directly related to the structural physical approximation (SPA)<sup>14,15</sup> of EWs. Recall, that for an EW  $W$  its SPA can be written as a non-negative observable

$$P = (1-p)W + p \frac{l_1 \otimes l_2}{D}, \quad (11)$$

where  $p \in (0, 1)$  is a minimal number such that that  $P \geq 0$ . Let us call a positive operator  $P$  a positive SPA (p-SPA) in the sense that an EW  $W$  is admixed with a positive fraction  $1-p > 0$ . In a similar vein, we introduce the negative SPA (n-SPA) with a negative fraction of an EW  $W$  as follows

$$Q = (1-q)W + q \frac{l_1 \otimes l_2}{D} \quad (12)$$

with maximal  $q > 1$  such that  $Q \geq 0$ . Clearly, both  $P$  and  $Q$  are density operators in  $\mathcal{H}_1 \otimes \mathcal{H}_2$  belonging to the boundary of space of states. We call a pair  $(P, Q)$  mirrored SPAs to an EW  $W$ .

**Proposition.** Let  $(W^{(-)}, W^{(+)})$  be mirrored EWs. Then, p-SPA to  $W^{(+)}$  coincides with n-SPA to  $W^{(-)}$ .

**Proof.** Let us begin with p-SPA to  $W^{(+)}$  and n-SPA to  $W^{(-)}$ ,

$$\tilde{W} = (1-p_{\pm})W^{(\pm)} + p_{\pm} \frac{l_1 \otimes l_2}{D},$$

with  $p_+ < 1 < p_-$  and  $D = d_1 d_2$ . Then, it holds that for any  $\sigma \in S_{\text{sep}}$

$$p_+/D \leq \text{tr}[\tilde{W}\sigma] \leq p_-/D \quad (13)$$

which shows  $L(\tilde{W}) = p_+/D$  and  $U(\tilde{W}) = p_-/D$ .  $\square$

The result in Eq. (13) can be compared to Eq. (3) by replacing an observable  $A$  with the resulting non-negative operator  $\tilde{W}$  after SPA. Note also that the universal relation (10) reads

$$(p_- - 1)W^{(-)} + (1 - p_+)W^{(+)} = (p_- - p_+) \frac{l_1 \otimes l_2}{D}.$$

Note also that mirrored EWs can be constructed by replacing  $A$  in Eqs. (6) and (7) with the operator  $\tilde{W}$  after p- (n-) SPA of  $W^{(+)}$  ( $W^{(-)}$ ).

Then, if  $\tilde{W}$  belongs to the boundary of  $S(\mathcal{H})$ , i.e.  $\lambda_{\min}(\tilde{W}) = 0$ , then  $W^{(-)}$  detects an entanglement of  $\rho = n(l_1 \otimes l_2 - \Pi)$ , where  $\Pi$  is a projector onto a support of  $\tilde{W}$  and  $n$  is a normalization factor. Indeed, one has  $\text{tr}(W^{(-)}\rho) = -L(\tilde{W})n/n < 0$ . On the other hand, for  $\text{tr}\tilde{W}^2 > U(\tilde{W})$ , we have  $\text{tr}(W^{(-)}\tilde{W}) < 0$ , and hence  $W^{(-)}$  detects entanglement of  $\tilde{W}$ .

The framework may be equivalently applied to completely positive (CP) maps. To simplify the discussion let us restrict to trace preserving maps, that is, a CP map  $\tilde{\Phi}$  corresponds to a quantum channel. Recall that a linear map  $\Phi$  is positive if  $\text{tr}(P\Phi(Q)) \geq 0$  for any pair of rank-1 projectors  $P$  and  $Q$ . A bipartite state  $\rho$  is separable if  $[\text{id} \otimes \Phi](\rho) \geq 0$  for all positive maps  $\Phi$  (where "id" stands for the identity map, that is,  $\text{id}(X) = X$ ). Now, using Choi–Jamiołkowski isomorphism, a map  $\tilde{\Phi}$  is CP if the following bipartite operator  $\tilde{W} = \sum_{i,j} |i\rangle\langle j| \otimes \tilde{\Phi}(|i\rangle\langle j|)$  is positive. Noting that

$$\text{tr}(\tilde{W} P^T \otimes Q) = \text{tr}[P\tilde{\Phi}(Q)],$$

let us introduce the following bounds

$$L(\tilde{\Phi}) = \min \text{tr}[P\tilde{\Phi}(Q)], \quad U(\tilde{\Phi}) = \max \text{tr}[P\tilde{\Phi}(Q)],$$

where minimization (maximization) runs over rank-1 projectors  $P$  and  $Q$ . Then, we construct positive maps as follows

$$\Phi^{(-)} = \frac{1}{m_-} [d_2 U(\tilde{\Phi}) \Phi_* - \tilde{\Phi}],$$

$$\Phi^{(+)} = \frac{1}{m_+} [\tilde{\Phi} - d_2 L(\tilde{\Phi}) \Phi_*],$$

with the depolarization channel  $\Phi_*(\rho) = l_2 \text{tr} \rho / d_2$ ,  $m_- = d_2 U(\tilde{\Phi}) - 1$ , and  $m_+ = 1 - d_2 L(\tilde{\Phi})$ . For any separable state  $\sigma \in$

$S_{\text{sep}}$ , one has

$$L(\tilde{\Phi})\sigma_1 \otimes l_2 \leq (\text{id} \otimes \tilde{\Phi})(\sigma) \leq U(\tilde{\Phi})\sigma_1 \otimes l_2,$$

where  $\sigma_1 = \text{tr}_2 \sigma$  denotes a reduced state. Similarly

$$L(\tilde{\Phi})l_1 \otimes \sigma_2 \leq (\tilde{\Phi} \otimes \text{id})(\sigma) \leq U(\tilde{\Phi})l_1 \otimes \sigma_2,$$

with  $\sigma_2 = \text{tr}_1 \sigma$ .

An example of such construction has been considered in ref. 16. We recall that a set of product vectors  $|\psi_j\rangle = |\alpha_j\rangle \otimes |\beta_j\rangle$  ( $j = 1, \dots, K$ ) defines an unextendible product basis (UPB) in  $\mathcal{H}_1 \otimes \mathcal{H}_2$  if there is no product vector orthogonal to all  $|\psi_j\rangle$ .<sup>17</sup> Let us choose the projector onto UPB,  $\Pi_{\text{UPB}} = |\psi_1\rangle\langle\psi_1| + \dots + |\psi_K\rangle\langle\psi_K|$ . Then, an EW in the following

$$W = (\Pi_{\text{UPB}} - L(\Pi_{\text{UPB}})l_1 \otimes l_2) / [K - L(\Pi_{\text{UPB}})D] \quad (14)$$

shares precisely the same structure with  $W^{(+)}$ . The witness detects a bound entangled state  $\rho = (l_1 \otimes l_2 - \Pi_{\text{UPB}}) / (D - K)$ .

The presented framework can be immediately generalized for the multipartite case. Clearly, multipartite case is much more subtle since there are several levels of separability.<sup>3,4</sup> The simplest notion is the full separability corresponding to pure product vectors  $|\Psi\rangle = |\psi_1\rangle \otimes \dots \otimes |\psi_N\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$ . Now for any  $\tilde{W} \geq 0$  in  $\mathcal{H}$  one defines  $U(\tilde{W}) = \max_{|\Psi\rangle} \langle\Psi|\tilde{W}|\Psi\rangle$  and  $L(\tilde{W}) = \min_{|\Psi\rangle} \langle\Psi|\tilde{W}|\Psi\rangle$ , where it suffices to consider the optimization over product states. Finally, one defines a pair of EWs via (6) and (7), generalized to multipartite systems (see also ref. 18). For example, the EW in Eq. (14) can be immediately generalized as follows.<sup>13,16</sup>

Let  $|\psi_j\rangle \in \mathcal{H}$  ( $j = 1, \dots, K$ ) denote UPB and  $\Pi_{\text{UPB}} = |\psi_1\rangle\langle\psi_1| + \dots + |\psi_K\rangle\langle\psi_K|$ , from which one can construct a witness  $W^{(+)} = (\Pi_{\text{UPB}} - L(\Pi_{\text{UPB}})l) / [K - L(\Pi_{\text{UPB}})D]$ , with  $l = l_1 \otimes \dots \otimes l_N$  and  $D = d_1 \dots d_N$ .

### DISCUSSION

In this section, we illustrate the systematic construction of mirrored EWs with examples and then give a summary and outlook.

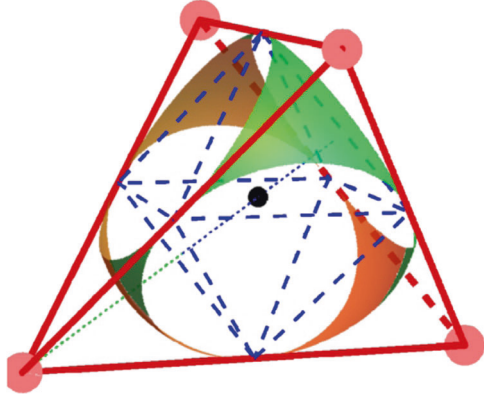
**Example 1** (Two-qubit EWs). Consider the following EWs for two-qubit states

$$W^{(+)} = \frac{1}{8} \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ -4 & 0 & 0 & 1 \end{pmatrix}, \quad W^{(-)} = \frac{1}{8} \begin{pmatrix} 3 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 3 \end{pmatrix},$$

which are written as block matrices in the computational basis for qubit states, e.g.  $\text{Diag}[1, 1, 1, 1] = \sum_{i,j=0,1} |i\rangle\langle i| \otimes |j\rangle\langle j|$ . The EW  $W^{(+)}$  can detect entangled states in the one-parameter state family (isotropic states)  $\rho_a = (1-a)|\phi^+\rangle\langle\phi^+| + a l_2 \otimes l_2 / 4$  with the Bell state  $|\phi^+\rangle = (|00\rangle + |11\rangle) / \sqrt{2}$  for  $a < 3/5$  (its known that for  $a < 2/3$  the states are entangled). The mirrored one,  $W^{(-)}$ , detects entangled states in the class of (isotropic) states  $\rho_\beta = (1-\beta)|\phi^-\rangle\langle\phi^-| + \beta l \otimes l / 4$  with the Bell state  $|\phi^-\rangle = (|00\rangle - |11\rangle) / \sqrt{2}$  for  $\beta < 1/3$ . Note also that there is no EW that can detect entangled states  $\rho_a$  and  $\rho_\beta$  at the same time. This is visualized in Fig. 2. The (green) hyperplane corresponds to the witness  $W^{(+)}$  and the (orange) hyperplane to the witness  $W^{(-)}$ . Both witnesses cover the full state space in this case. The upper and lower bounds are computed to be  $U(W^{(\pm)}) = 1/2$  with  $p_+ = 3/5$  and  $p_- = 7/5$  which give rise to the single observable as follows,

$$\tilde{W} = \frac{1}{10} \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}, \quad (15)$$

for which the separability window is given by  $[L(\tilde{W}), U(\tilde{W})] = [3/20, 7/20]$ .



**Fig. 2 Visualization of the “two-qubit EW”.** This magic simplex<sup>35–37</sup> represents the Hilbert space of all bipartite qubit states which partial traces correspond to the maximally mixed states, so-called locally maximally mixed states. They can be written in the form  $\rho = \frac{1}{4}(I \otimes I + c_1 \sigma_1 \otimes \sigma_1)$  with  $c_i \in \mathbb{R}$ . All points  $\{c_1, c_2, c_3\}$  inside the (red) tetrahedron represent states (satisfy the positivity condition). All states inside the (blue) double pyramid are separable states. The (pink) dots correspond to the four Bell states and the dotted line represents one isotropic state  $\rho_\alpha$  (SEP (blue):  $\alpha \geq \frac{2}{3}$ ; ENT (green)  $\alpha < \frac{2}{3}$ ). The (light and dark) green surface represents  $\text{tr}[W^{(+)}\rho] = 0$  and the (light and dark) orange surface  $\text{tr}[W^{(-)}\rho] = 0$ . In this case, the single observable that contains mirrored EWs  $W^{(\pm)}$  covers the full space of all locally maximally mixed states. However, note that these witnesses are not optimal.

**Example 2** (Two-qutrits EWs). The Choi EW  $W^{(+)}$  obtained from the Choi map in  $d=3$  ref.<sup>19</sup> is given by

$$W^{(+)} = \frac{1}{6} \left( \sum_{i=0}^2 [2|ii\rangle\langle ii| + |i, i-1\rangle\langle i, i-1|] - 3P_+ \right),$$

where  $P_+ = |\phi^+\rangle\langle\phi^+|$  with the Bell state  $|\phi^+\rangle = (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$ . The Choi EW is known to be nondecomposable and can detect bound entangled states. One can find  $U(W^{(+)}) = 2/9$  with  $p_+ = 3/5$  and  $p_- = 7/5$  so that the other one  $W^{(-)}$  is given by

$$W^{(-)} = \frac{2}{9} I \otimes I - \tilde{W}. \quad (16)$$

Interestingly, while  $W^{(+)}$  has a single negative eigenvalue  $-1/6$ , its SPA-mirrored EW  $W^{(-)}$  has two negative eigenvalues both  $-1/9$ . Note that both witnesses  $W^{(\pm)}$  have the same upper bound  $U(W^{(\pm)}) = 2/9$ . Finally, the separability window  $[L(\tilde{W}), U(\tilde{W})]$  of the observable reads is  $[3/45, 7/45]$ . The p-SPA of the Choi EW is given by

$$\tilde{W} = \frac{2}{5} W^{(+)} + \frac{3}{45} I \otimes I \quad (17)$$

which corresponds to a separable state as shown in ref.<sup>20</sup>. Interestingly,  $W^{(+)}$  is nondecomposable, whereas  $W^{(-)}$  is decomposable.

**Example 3** (No nontrivial partner). Here we show that not every EW  $W^{(+)}$  has a nontrivial partner  $W^{(-)}$ . Consider  $W^{(+)} = |\psi\rangle\langle\psi|^T$ , where

$$|\psi\rangle = \sum_i s_i |e_i\rangle \otimes |f_i\rangle, \quad (18)$$

denotes the Schmidt decomposition of  $|\psi\rangle$ . We assume  $s_1 \geq s_2 \geq \dots \geq s_d$ . One finds  $U(W^{(+)}) = \lambda_{\max} = s_1^2$ , and hence

$$W^{(-)} = \frac{1}{Ds_1^2 - 1} (s_1^2 I_1 \otimes I_2 - |\psi\rangle\langle\psi|^T), \quad (19)$$

is a positive operator and hence cannot be used as a witness.

**Example 4** (Genuine multipartite EWs). In ref.<sup>21</sup> a general formalism to detect various types of multipartite entanglement

was introduced which works for any number of particles and dimensions. It turned out to be efficient to guarantee the security in quantum secret sharing protocols<sup>22</sup> or to prove experimentally the genuine multipartite entanglement of four physical photons entangled in their orbital angular momentum degrees of freedom<sup>23</sup> or of two photons entangled in polarization and orbital angular momentum degrees of freedom.<sup>24</sup> Let us here consider three qubits. For instance, the function

$$Q_{\text{GHZ}}(\rho) := 2(\sqrt{\rho_{22}\rho_{77}} + \sqrt{\rho_{33}\rho_{66}} + \sqrt{\rho_{44}\rho_{55}} - |\rho_{18}|)$$

(with  $\rho_{ij}$  being the elements of the density matrix  $\rho$ ) is  $\geq 0$  for all fully separable as well as for all biseparable states. Thus a violation of  $0 \leq Q_{\text{GHZ}}(\rho)$  proves that the state  $\rho$  is genuinely multipartite entangled. Moreover, the function is maximized for the Greenberger–Horne–Zeilinger state, e.g.  $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ . The factor 2 is chosen to set  $Q_{\text{GHZ}}(|\text{GHZ}\rangle\langle\text{GHZ}|) = \pm 1$ . Now let us apply our procedure to obtain a new bound. Different to what we considered so far is that this inequality correspond to a nonlinear witness, but of course this witness can be linearized by using (i)  $\sqrt{\rho_{ii}\rho_{jj}} \leq \frac{\rho_{ii} + \rho_{jj}}{2}$  and (ii) assuming  $\rho_{18}$  to be purely real or imaginary. This gives  $Q_{\text{GHZ}}^{\text{lin}} = 1 - \rho_{11} - \rho_{88} + 2\text{Re}\{\rho_{18}\}$  and obviously still the same maxima  $\pm 1$  for  $|\text{GHZ}\rangle$ . But considering the optimization over all pure fully separable states  $\sigma_{\text{sep}}$  results in

$$\begin{aligned} 0 \leq Q_{\text{GHZ}}(\sigma_{\text{sep}}) &\leq \frac{1}{2}, \\ 0 \leq Q_{\text{GHZ}}^{\text{lin}}(\sigma_{\text{sep}}) &\leq 1. \end{aligned} \quad (20)$$

Consequently, the upper bound (mirrored witness) loses its predictive power.

Another physically distinct genuine multipartite entangled state is the  $W$ -state or Dicke state, e.g. in the computational basis given by  $|\text{Dicke}\rangle = \frac{1}{\sqrt{3}}\{|001\rangle + |010\rangle + |100\rangle\}$  and the function<sup>21</sup>

$$Q_{\text{Dicke}}(\rho) = 2 \left( \frac{\rho_{22} + \rho_{33} + \rho_{55}}{2} + \sqrt{\rho_{11}\rho_{44}} + \sqrt{\rho_{11}\rho_{66}} \right. \\ \left. + \sqrt{\rho_{11}\rho_{77}} - |\rho_{23}| - |\rho_{25}| - |\rho_{35}| \right),$$

which gives  $-1$  for the Dicke state  $|\text{Dicke}\rangle$  in the computational basis. However, note that though  $|\text{GHZ}\rangle$  is also a genuine multipartite entangled state, but the minimum over all local unitaries leads only to  $\min Q_{\text{Dicke}}(|\text{GHZ}\rangle) = -\frac{3}{4}$ . Such as an optimization over all local unitaries of the GHZ-witness function  $Q_{\text{GHZ}}(|\text{Dicke}\rangle) = -0.628$  which does not equal the maximal value  $-1$ .

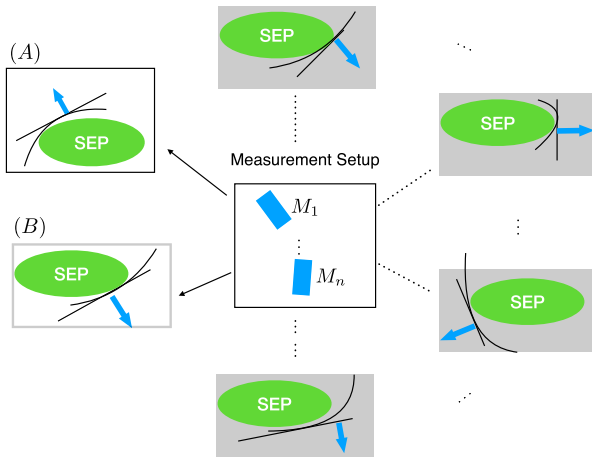
Now optimizing over all pure separable states of  $Q_{\text{Dicke}}$  leads to

$$0 \leq Q_{\text{Dicke}}(\sigma_{\text{sep}}) \leq 1. \quad (21)$$

Indeed the upper bound is again a nontrivial one since  $\max Q_{\text{Dicke}}(\rho_{\text{ent}}) = 1.5$  (actually the  $\rho_{\text{ent}}$  equals  $|\text{GHZ}\rangle$ ). Thus this expresses a certain duality between the two possible genuinely multipartite entangled states. Of course, one can also consider the optimization over all fully separable and biseparable states, then, however, both criteria  $Q_{\text{GHZ/Dicke}}$  provide no longer nontrivial upper bounds, they are optimal in this sense.

## Summary and outlook

We have presented the framework of mirrored EWs. It is shown that a single observable corresponds to a pair of EWs, called mirrored EWs, in the sense that entanglement detection with mirrored EWs can be performed by realizing a single observable in practice. Then, a single observable has both upper and lower bounds for separable states, where mirrored EWs give the characterization of the bounds, respectively. Thus, entangled states can be detected by violations of either of the bounds. It is shown that mirrored EWs can be constructed from bounds of an observable for separable states, as well as via structural physical approximations. Equivalently, mirrored EWs can be rephrased that



**Fig. 3** An EW, Eq. (1), corresponds to an observable that can be estimated in a measurement setup that contains a set of POVM elements. The box (A) shows a schematic structure of the entanglement detection with an EW and its nonlinearization. The present work shows that box (B) exists: the measurement setup prepared for the EW in box (A) can be used to construct the other EW in box (B), i.e., its mirrored one. This EW detects other types of entangled states utilizing the same information from the experiment (center), i.e. no other experimental setup is needed. It remains open to disclose the gray boxes, i.e., the possibility of constructing more than two EWs from a single measurement setup (see also Discussion).

an EW generally has a nontrivial upper bound for separable states, such that the upper bound is characterized by its pair, i.e., this shows that an EW can generate its paired one. Examples are presented to illustrate how the framework of mirrored EWs works in detail to detect different types of entangled states.

Finally, nonlinearities are introduced to improve mirrored EWs, i.e., mirrored nonlinear EWs. It is shown that both subtraction and addition of nonlinear polynomials can improve mirrored EWs: the subtraction improves standard EWs that detect entangled states from violations of the lower bound, and the addition does the other detection scheme with a nontrivial upper bound. We emphasize that our findings show, via the effort of mirrored EWs, the recorded experimental knowledge that has been used to detect entanglement from Eq. (1) can be utilized once again to see the violation of an upper bound in Eq. (27), without acquiring additional experiments. A larger set of entangled states than the one from the very definition of a witness can be detected.

The structure of SPA behind mirrored EWs can be generalized by replacing the identity operator with some other separable operators. That is, the operator  $I_1 \otimes I_2/D$  in a positive and negative SPAs in Eqs. (11) and (12) can be replaced with a full-rank, non-negative and unit-trace operator  $X \in S_{\text{sep}}(\mathcal{H})$ . Note that for a full-rank, non-negative and unit-trace operator  $X$ , there always exist such  $p_X$  and  $q_X$ .<sup>25</sup> In future investigations, it would be interesting to extend mirrored EWs by generalizing SPAs and to explore further structures of entangled states (cf. Fig. 3).

EWs are a useful tool not only to investigate theoretical and experimental aspects of entangled states but also to certify properties having a convex structure, such as fidelities of states or quantum operations; see e.g. refs.<sup>26,27</sup> Our results, that the EWs can be indeed mirrored, pave a new avenue in the certification of entanglement and other quantum properties. In future investigations, it would be interesting to find the theoretical characterizations of mirrored EWs in the view of (non-)decomposability, extremality of EWs, etc.<sup>4,8</sup> Although EWs apply single-copy-level measurements that are feasible with current technologies, it is worth to extend the framework of mirrored EWs to multi-copy scenarios, e.g., with more assumptions on state preparation<sup>28</sup> and

with more advanced technologies; see e.g. refs.<sup>29–31</sup> From a practical point of view it is also worth to consider the statistical errors in an implementation of nonlinear and mirrored EWs, e.g., the implementation should be constructed such that the ranges of statistical errors of the paired EWs do not overlap each other within the separability window.

## METHODS

EWs can be found as linear hyperplanes that distinguish entangled states from the set of separable states on the space of Hermitian operators. Since the set of separable states is convex, one may improve EWs by introducing nonlinearities such that the convex set is better characterized and a larger set of entangled states can be detected. For instance, it has been found that uncertainty relations in terms of variances offer such a possibility.<sup>32,33</sup>

A systematic method of constructing nonlinear EWs has been introduced.<sup>34</sup> Suppose that some entangled state  $\rho$  is detected by a positive map  $\Lambda$ , that is,  $[\text{id} \otimes \Lambda](\rho)$  possesses an eigenvector  $|\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$  corresponding to a negative eigenvalue. Then the following formula

$$W = (\text{id} \otimes \Lambda^*)(|\phi\rangle\langle\phi|) \quad (22)$$

where  $\Lambda^*$  denotes a dual map, defines an EW detecting  $\rho$ , i.e.  $\text{tr}[W\rho] < 0$ . A nonlinear functional is then constructed by subtracting polynomials from an EW. To be specific, let us consider subtracting a single nonlinear polynomial as follows,<sup>34</sup>

$$\mathcal{F}^{(1)}(\rho) = \text{tr}[W\rho] - \frac{1}{s^2(\chi)} \text{tr}[\tilde{X}\rho] \text{tr}[\tilde{X}^\dagger \rho], \quad (23)$$

where one can choose  $\tilde{X} = (\text{id} \otimes \Lambda^*)(|\phi\rangle\langle\chi|)$  for some state  $|\chi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$  and  $s(\chi)$  is the largest Schmidt coefficient of  $|\chi\rangle$ . The parameters are chosen such that  $\mathcal{F}^{(1)}[\sigma] \geq 0$  for all  $\sigma \in S_{\text{sep}}$ . It is clear that a witness  $W$  does not detect a larger set of entangled states than its nonlinear functional  $\mathcal{F}$  since  $\mathcal{F}[\rho] \leq \text{tr}[W\rho]$ .

In what follows, we revisit mirrored EWs in Example 1 to demonstrate the improvements and illustrate construction of a nonlinear functional from an observable  $A$  that links mirrored EWs. With respect to an observable  $A$ , the improvement can be made by both adding and subtracting polynomials.

**Example 5** (Nonlinear mirrored EWs). We recall that entangled states  $|\phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$  are detected by mirrored EWs  $W^{(\pm)}$  in Example 1,  $\text{tr}[W^{(\pm)}|\phi^\pm\rangle\langle\phi^\pm|] = -1/4 \mp 1/8$ .

From the EWs, positive maps  $\Lambda_{(\pm)}^*$  are denoted as follows; see also Eq. (22)

$$W^{(\pm)} = [\text{id} \otimes \Lambda_{(\pm)}^*](|\phi^\pm\rangle\langle\phi^\pm|).$$

According to Eq. (23), a nonlinear polynomial can be constructed as  $\tilde{X}^{(\pm)} = (\text{id} \otimes \Lambda_{(\pm)}^*)(|\phi^\pm\rangle\langle\phi^\mp|)$ , where we have chosen  $|\chi\rangle = |\phi^\mp\rangle$ ,

$$\tilde{X}^{(+)} = \frac{1}{8} \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 4 & 0 & 0 & -1 \end{pmatrix}, \quad \tilde{X}^{(-)} = \frac{1}{8} \begin{pmatrix} 3 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & -3 \end{pmatrix}.$$

Since  $|\chi\rangle = |\phi^\mp\rangle$ , we have  $s^2(\chi) = 1/2$ . Then, a pair of nonlinear EWs are obtained from mirrored EWs,

$$\mathcal{F}_{(\pm)}^{(1)}(\rho) = \text{tr}[W^{(\pm)}\rho] - 2\text{tr}[\tilde{X}^{(\pm)}\rho] \text{tr}[\tilde{X}^{(\pm)\dagger}\rho], \quad (24)$$

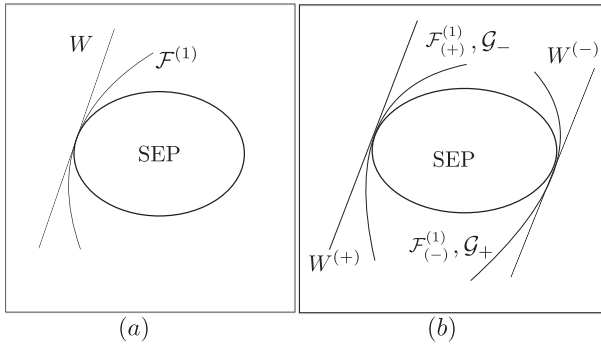
which is non-negative for all separable states.<sup>34</sup>

For instance, we consider a state  $|\mu\rangle = a|00\rangle + b|11\rangle$  with  $a, b \in \mathbb{R}$  and  $a^2 + b^2 = 1$ :

$$\text{tr}[W^{(+)}|\mu\rangle\langle\mu|] = \frac{1}{8}(a^2 + b^2 - 8ab), \quad \text{and} \\ \text{tr}[W^{(-)}|\mu\rangle\langle\mu|] = \frac{1}{8}(3a^2 + 3b^2 + 8ab).$$

The state  $|\mu\rangle$  with  $ab = 1/8$  is entangled but not detected by  $W^{(+)}$  since  $\text{tr}[W^{(+)}|\mu\rangle\langle\mu|] = 0$ . It can be detected by the nonlinear witness in Eq. (24): we have that  $\mathcal{F}_{(+)}^{(1)}(|\mu\rangle\langle\mu|) = -1/64 < 0$ . For  $ab = -3/8$ , we have  $\text{tr}[W^{(-)}|\mu\rangle\langle\mu|] = 0$  but  $\mathcal{F}_{(-)}^{(1)}(|\mu\rangle\langle\mu|) = -9/8 < 0$ . Thus, the improvement by nonlinear polynomials in Eq. (24) is shown.

We recall that the mirrored witnesses  $W^{(\pm)}$  characterize the lower and upper bounds of an observable  $A$  for separable states; see Eqs. (3), (6) and (7). It is straightforward to find a nonlinear functional with the observable  $A$



**Fig. 4** A schematic view of nonlinear and mirrored entanglement witnesses. **a** A nonlinear EW  $\mathcal{F}^{(1)}$  can be obtained by subtracting a polynomial from a witness  $W$ , so that a larger set of entangled states can be detected<sup>34</sup>; see Eq. (24). **b** Mirrored EWs  $W^{(-)}$  and  $W^{(+)}$  that characterize upper and lower bounds of an observable for separable states, respectively, can also be improved, which are denoted by  $\mathcal{F}_{(\pm)}^{(1)}$ . For the observable, nonlinear EWs are introduced by both subtracting or adding a polynomial; see  $\mathcal{G}_{\pm}$  in Eq. (25).

that links mirrored EWs  $W^{(\pm)}$  from Eq. (24): we write by  $\mathcal{G}$  a nonlinear functional from an observable  $A$  via nonlinear EWs in Eq. (24)

$$\mathcal{G}_{\pm}[\rho] = \text{tr}[A\rho] \pm 2n_{\pm} \text{tr}[\tilde{\chi}^{(\mp)} \rho] \text{tr}[\tilde{\chi}^{(\mp)} \rho] \quad (25)$$

where  $n_{\pm}$  are from Eq. (8). Note that both addition and subtraction of nonlinear polynomials are essential to improve an observable  $A$ . It holds that for  $\sigma \in S_{\text{sep}}$

$$\begin{aligned} \mathcal{F}_{(+)}^{(1)}[\sigma] \geq 0 &\iff L(A) \leq \mathcal{G}_{-}[\sigma] \\ \mathcal{F}_{(-)}^{(1)}[\sigma] \geq 0 &\iff \mathcal{G}_{+}[\sigma] \leq U(A). \end{aligned} \quad (26)$$

This shows how to improve an observable  $A$  with nonlinear polynomials (cf. Fig. 4). For the lower bound  $L(A)$  for separable states, which is characterized by an EW  $W^{(-)}$ , the improvement is made by subtracting a nonlinear polynomial, i.e.,  $\mathcal{G}_{-}$  in Eq. (25). Note that this is similar to the way that a standard EW is improved by nonlinear polynomials; see Eq. (24). For the upper bound  $U(A)$ , however, the improvement is obtained by adding a nonlinear polynomial, i.e.,  $\mathcal{G}_{+}$  in Eq. (25).

The improvement of an observable  $A$  can be rephrased on the level of standard EWs. The separability window in Eq. (3) can be rewritten for all  $\sigma \in S_{\text{sep}}$  by

$$0 \leq \text{tr}[W\sigma] \leq U(W). \quad (27)$$

Note that from Eqs. (6) and (9), there exists a witness  $W$  that characterizes the upper bound, i.e., for any state  $\rho$  on  $\mathcal{H}_1 \otimes \mathcal{H}_2$ , it holds that

$$\text{tr}[W\rho] \leq U(W) \iff 0 \leq \text{tr}[W'\rho].$$

Let us recall that the detection method by a lower bound  $\text{tr}[W\sigma] \geq 0$  for  $\sigma \in S_{\text{sep}}$  is improved by subtracting nonlinear polynomials. The improvement in Eq. (26) means the following statements are equivalent:

- the detection method by an upper bound  $\text{tr}[W\sigma] \leq U(W)$  for  $\sigma \in S_{\text{sep}}$  is improved by adding nonlinear polynomials, and
- the detection method by a lower bound  $\text{tr}[W'\sigma] \geq 0$  for  $\sigma \in S_{\text{sep}}$  is improved by subtracting nonlinear polynomials.

Thus, it is shown that nonlinear EWs can be generally constructed by both subtracting and adding nonlinear polynomials.

## DATA AVAILABILITY

Datasets were generated or analyzed during the current study.

## CODE AVAILABILITY

The results were computed by pencil or the use of commercial computer software as Mathematica.

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## AUTHOR CONTRIBUTIONS

All authors have equally contributed to the main result, the examples and the writing.

## COMPETING INTERESTS

The authors declare no competing interests.

## ADDITIONAL INFORMATION

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