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Quantum parameter estimation with general dynamics

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One of the main quests in quantum metrology, and quantum parameter estimation in general, is to find out the highest achievable precision with given resources and design schemes to attain it. In this article we present a general framework for quantum parameter estimation and provide systematic methods for computing the ultimate precision limit, which is more general and efficient than conventional methods.

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INTRODUCTION

A pivotal task in science and technology is to identify the highest achievable precision in measurement and estimation and design schemes to reach it. Quantum metrology, which exploits quantum mechanical effects such as entanglement, can achieve better precision than classical schemes and has found wide applications in quantum sensing, gravitational wave detection, quantum-enhanced reading of digital memory, quantum imaging, atomic clock synchronization, etc.;[1–11] this has gained increasing attention in recent years.[12–26]

A typical situation in quantum parameter estimation is to estimate the value of a continuous parameter x encoded in some quantum state ρ_x of the system. To estimate the value, one needs to first perform measurements on the system, which, in the general form, are described by Positive Operator Valued Measurements (POVM), $\{E_y\}$, which provides a distribution for the measurement results $p(y|x) = \mathrm{Tr}(E_y\rho_x)$. According to the Cramér–Rao bound in statistical theory,[2, 3, 27, 28] the standard deviation for any unbiased estimator of x, based on the measurement results y, is bounded below by the Fisher information: $\delta\hat{x} \geq \frac{1}{\sqrt{I(x)}}$, where $\delta\hat{x}$ is the standard deviation of the estimation of x, and $I(x)$ is the Fisher information of the measurement results, $I(x) = \sum_y p(y|x)\left(\frac{\partial \ln p(y|x)}{\partial x}\right)^2$.[29] The Fisher information can be further optimized over all POVMs, which gives

$$\delta\hat{x} \geq \frac{1}{\sqrt{\max_{E_y} I(x)}} = \frac{1}{\sqrt{J(\rho_x)}}, \quad (1)$$

where the optimized value $J(\rho_x)$ is called quantum Fisher information.[2, 3, 30, 31] If the above process is repeated n times, then the standard deviation of the estimator is bounded by $\delta\hat{x} \geq \frac{1}{\sqrt{nJ(\rho_x)}}$.

To achieve the highest precision, we can further optimize the encoding procedures $x \to \rho_x$ so that $J(\rho_x)$ is maximized. Typically the encoding is achieved by preparing the probe in some initial state ρ_0, then let it evolve under a dynamics that contains the interested parameter, $\rho_0 \xrightarrow{\phi_x} \rho_x$. Usually ϕ_x is determined by a given physical dynamics which is then fixed, while the initial state is up to our choice and can be optimized. A pivotal task in quantum metrology is to find out the optimal initial state ρ_0 and the corresponding maximum quantum Fisher information under any given evolution ϕ_x. When ϕ_x is unitary the GHZ-type of states are known to be optimal, which leads to the Heisenberg limit. However when ϕ_x is noisy, such states are in general no longer optimal. Finding the optimal probe states and the corresponding highest precision limit under general dynamics has been the main quest of the field. Recently using the purification approach much progress has been made on developing systematical methods of calculating the highest precision limit.[12, 13, 15, 18, 19] These methods, however, require smooth representations of the Kraus operators, which is not intrinsic to the dynamics.

In this article, we provide an alternative purification approach that does not require smooth representations of the Kraus operators. This framework provides systematic methods for computing the ultimate precision limit, which can be formulated as semi-definite programming and solved more efficiently than conventional methods. We also extend the Bures angle on quantum states to quantum channels, which is expected to find wide application in various fields of quantum information science.

RESULTS

Ultimate precision limit

The precision limit of measuring x from a set of quantum states ρ_x is determined by the distinguishability between ρ_x and its neighboring states ρ_{x+dx}.[30, 32] This is best seen if we expand the Bures distance between the neighboring states ρ_x and ρ_{x+dx} up to the second order of dx:[30]

$$d^2_{Bures}(\rho_x, \rho_{x+dx}) = \frac{1}{4}J(\rho_x)dx^2, \quad (2)$$

where $d_{Bures}(\rho_1, \rho_2) = \sqrt{2 - 2F(\rho_1, \rho_2)}$; here $F(\rho_1, \rho_2) = Tr\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}$ is the fidelity between two states. Thus maximizing the quantum Fisher information is equivalent as maximizing the Bures distance, which is equivalent as minimizing the fidelity between ρ_x and ρ_{x+dx}. If the evolution is given by ϕ_x, $\rho_x = \phi_x(\rho)$ and $\rho_{x+dx} = \phi_{x+dx}(\rho)$, the problem is then equivalent to finding

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out $\min_\rho F[\phi_x(\rho), \phi_{x+dx}(\rho)]$. We now develop tools to solve this problem for both unitary and open quantum dynamics.

Given two evolution ϕ_x and ϕ_{x+dx}, we define the Bures angle between them as $\Theta(\phi_x, \phi_{x+dx}) = \max_\rho \cos^{-1}[F(\phi_x(\rho), \phi_{x+dx}(\rho))]$. This generalizes the Bures angle on quantum states[33] to quantum channels. $\Theta(\phi_x, \phi_{x+dx})$ can be seen as an induced measure on quantum channel from the Bures angle on quantum states, it thus also defines a metric on quantum channels. From the definition of the Bures distance it is easy to see $\max_\rho d^2_{Bures}[\phi_x(\rho), \phi_{x+dx}(\rho)] = 2 - 2\cos\Theta(\phi_x, \phi_{x+dx})$, thus from Eq. (2) we have

$$\max_\rho J[\phi_x(\rho)] = \lim_{dx\to 0} \frac{8[1-\cos\Theta(\phi_x, \phi_{x+dx})]}{dx^2}. \tag{3}$$

The ultimate precision limit under the evolution ϕ_x is thus determined by the Bures angle between ϕ_x and the neighboring channels

$$\delta\hat{x} \geq \frac{1}{\lim_{dx\to 0} \frac{\sqrt{8[1-\cos\Theta(\phi_x,\phi_{x+dx})]}}{|dx|}\sqrt{n}}, \tag{4}$$

where n is the number of times that the procedure is repeated. If ϕ_x is continuous with respect to x, then when $dx\to 0$, $\Theta(\phi_x, \phi_{x+dx}) \to \Theta(\phi_x, \phi_x) = 0$, in this case

$$\begin{aligned}\max_\rho J[\phi_x(\rho)] &= \lim_{dx\to 0} \frac{8[1-\cos\Theta(\phi_x,\phi_{x+dx})]}{dx^2}\\ &= \lim_{dx\to 0} \frac{16\sin^2\frac{\Theta(\phi_x,\phi_{x+dx})}{2}}{dx^2}\\ &= \lim_{dx\to 0} \frac{4\Theta^2(\phi_x,\phi_{x+dx})}{dx^2},\end{aligned} \tag{5}$$

the ultimate precision limit is then given by

$$\delta\hat{x} \geq \frac{1}{\lim_{dx\to 0} 2\frac{\Theta(\phi_x,\phi_{x+dx})}{|dx|}\sqrt{n}}. \tag{6}$$

The problem is thus reduced to determine the Bures angle between quantum channels. We will first show how to compute the Bures angle between unitary channels, then generalize to noisy quantum channels.

Ultimate precision limit for unitary channels. Given two unitaries U_1 and U_2 of the same dimension, since $F(U_1\rho U_1^\dagger, U_2\rho U_2^\dagger) = F(\rho, U_1^\dagger U_2\rho U_2^\dagger U_1)$, we have $\Theta(U_1, U_2) = \Theta(I, U_1^\dagger U_2)$, i.e., the Bures angle between two unitaries can be reduced to the Bures angle between the identity and a unitary. For a $m\times m$ unitary matrix U, let $e^{-i\theta_j}$ be the eigenvalues of U, where $\theta_j \in (-\pi, \pi]$, $1 \le j \le m$, which we will call the eigen-angles of U. If $\theta_{max} = \theta_1 \ge \theta_2 \ge \cdots \ge \theta_m = \theta_{min}$ are arranged in decreasing order, then $\Theta(I, U) = \frac{\theta_{max}-\theta_{min}}{2}$ when $\theta_{max} - \theta_{min} \le \pi$,[34–39] specifically if $U = e^{-iHt}$, then $\Theta(I, U) = \frac{(\lambda_{max}-\lambda_{min})t}{2}$ if $(\lambda_{max} - \lambda_{min})t \le \pi$, where $\lambda_{max(min)}$ is the maximal (minimal) eigenvalue of H. This provides ways to compute Bures angles on unitary channels. For example, suppose the evolution takes the form $U(x) = (e^{-ixHt})^{\otimes N}$ (tensor product of e^{-ixHt} for N times, which means the same unitary evolution e^{-ixHt} acts on all N probes). Then

$$\begin{aligned}\Theta[U(x), U(x+dx)] &= \Theta[I, U^\dagger(x)U(x+dx)]\\ &= \Theta\left[I, \left(e^{-iHtdx}\right)^{\otimes N}\right].\end{aligned} \tag{7}$$

It is easy to see that the difference between the maximal eigen-angle and the minimal eigen-angle of $(e^{-iHtdx})^{\otimes N}$ is $\theta_{max} - \theta_{min} = N(\lambda_{max}|dx|t - \lambda_{min}|dx|t)$. Thus $\Theta\left(I, \left(e^{-iHtdx}\right)^{\otimes N}\right) = \frac{\theta_{max}-\theta_{min}}{2} = \frac{(N\lambda_{max}|dx| - N\lambda_{min}|dx|)t}{2}$, Eq. (6) then recovers the Heisenberg limit

$$\delta\hat{x} \geq \frac{1}{\sqrt{n}(\lambda_{max} - \lambda_{min})t}\frac{1}{N}. \tag{8}$$

This also has close connection to the quantum speed limit,[40–42] essentially the optimal probe state in this case, which is the equal superposition of the eigenvectors corresponding to λ_{max} and λ_{min}, is also the state that has the fastest speed of evolution.

Ultimate precision limit for noisy quantum channels. For a general quantum channel that maps from a m_1-dimensional to m_2-dimensional Hilbert space, the evolution can be represented by a Kraus operation $K(\rho_S) = \sum_{j=1}^d F_j\rho_S F_j^\dagger$; here the Kraus operators F_j, $1 \le j \le d$ are of the size $m_2 \times m_1$, $\sum_{j=1}^d F_j^\dagger F_j = I_{m_1}$. The channel can be equivalently represented as follows:

$$K(\rho_S) = Tr_E\left(U_{ES}(|0_E\rangle\langle 0_E| \otimes \rho_S)U_{ES}^\dagger\right), \tag{9}$$

where $|0_E\rangle$ denotes some standard state of the environment, and U_{ES} is a unitary operator acting on both system and environment, which we will call as the unitary extension of K. A general U_{ES} can be written as follows:

$$U_{ES} = (W_E \otimes I_{m_2})\underbrace{\begin{bmatrix} F_1 & * & * & \cdots & * \\ F_2 & * & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ F_d & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}}_{U}, \tag{10}$$

where only the first m_1 columns of U are fixed and $W_E \in U(p)$ ($p\times p$ unitaries) only acts on the environment and can be chosen arbitrarily; here $p \ge d$ as $p-d$ zero Kraus operators can be added.

Given a channel an ancillary system can be used to improve the precision limit, this can be described as the extended channel

$$(K \otimes I_A)(\rho_{SA}) = \sum_j (F_j \otimes I_A)\rho_{SA}(F_j \otimes I_A)^\dagger,$$

where ρ_{SA} represents a state of the original and ancillary systems. Without loss of generality, the ancillary system can be assumed to have the same dimension as the original system.

Given two quantum channels K_1 and K_2 of the same dimension, let U_{ES1} and U_{ES2} as unitary extensions of K_1 and K_2, respectively, we have[43]

$$\begin{aligned}\Theta(K_1 \otimes I_A, K_2 \otimes I_A) &= \min_{U_{ES1}, U_{ES2}} \Theta(U_{ES1}, U_{ES2})\\ &= \min_{U_{ES1}} \Theta(U_{ES1}, U_{ES2})\\ &= \min_{U_{ES2}} \Theta(U_{ES1}, U_{ES2}).\end{aligned} \tag{11}$$

This extends Uhlmann's purication theorem on mixed states[44] to noisy quantum channels. Furthermore, $\Theta(K_1 \otimes I_A, K_2 \otimes I_A)$ can be explicitly computed from the Kraus operators of K_1 and K_2 (please see supplemental material for detail): if $K_1(\rho_S) = \sum_{j=1}^d F_{1j}\rho_S F_{1j}^\dagger$, $K_2(\rho_S) = \sum_{j=1}^d F_{2j}\rho_S F_{2j}^\dagger$, then $\cos\Theta$

$(K_1 \otimes I_A, K_2 \otimes I_A) = \max_{\|W\|\le 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^\dagger)$; here $\lambda_{\min}(K_W + K_W^\dagger)$ denotes the minimum eigenvalue of $K_W + K_W^\dagger$, where $K_W = \sum_{ij} w_{ij} F_{1i}^\dagger F_{2j}$, with w_{ij} as the ij-th entry of a $d \times d$ matrix W, which satisfies $\|W\| \le 1$ ($\|\cdot\|$ denotes the operator norm, which is equal to the maximum singular value). If we substitute $K_1 = K_x$ and $K_2 = K_{x+dx}$, where $K_x(\rho_S) = \sum_{j=1}^d F_j(x)\rho_S F_j^\dagger(x)$ and $K_{x+dx}(\rho_S) = \sum_{j=1}^d F_j(x+dx)\rho_S F_j^\dagger(x+dx)$ with x being the interested parameter, then

$$\cos\Theta(K_x \otimes I_A, K_{x+dx} \otimes I_A) = \max_{\|W\|\le 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^\dagger), \quad (12)$$

where $K_W = \sum_{ij} w_{ij} F_i^\dagger(x) F_j(x+dx)$.

By substituting $\phi_x = K_x \otimes I_A$ and $\phi_{x+dx} = K_{x+dx} \otimes I_A$ in Eq. (3), we then get the maximal quantum Fisher information for the extended channel $K_x \otimes I_A$,

$$\max J = \lim_{dx \to 0} \frac{8\left[1 - \max_{\|W\|\le 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^\dagger)\right]}{dx^2}. \quad (13)$$

The maximization in Eq. (13) can be formulated as semi-definite programming: $\max_{\|W\|\le 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^\dagger) =$

$$\begin{aligned} &\textit{maximize} \quad \frac{1}{2}t \\ &s.t. \quad \begin{pmatrix} I & W^\dagger \\ W & I \end{pmatrix} \ge 0, \\ &K_W + K_W^\dagger - tI \ge 0. \end{aligned} \quad (14)$$

For example, consider two qubits with independent dephasing noises, which can be represented by four Kraus operators: $F_1(x) \otimes F_1(x)$, $F_1(x) \otimes F_2(x)$, $F_2(x) \otimes F_1(x)$, $F_2(x) \otimes F_1(x)$ with $F_1(x) = \sqrt{\frac{1+\eta}{2}} U(x)$, $F_2(x) = \sqrt{\frac{1-\eta}{2}} \sigma_3 U(x)$; here $U(x) = \exp\left(-i\frac{\sigma_3}{2}x\right)$. Figure 1 shows the maximal quantum Fisher information and the quantum Fisher information for the separable input state $|{+}{+}\rangle$, where $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$. It can be seen that the gain of entanglement is only obvious in the region of high η, i.e., low noises. It is also found that there exists a threshold for η, above the threshold the GHZ state is the optimal state that achieves the maximal quantum Fisher information, but with the decreasing of η the optimal state gradually changes from GHZ state to separable state, and this threshold increases with the number of qubits.

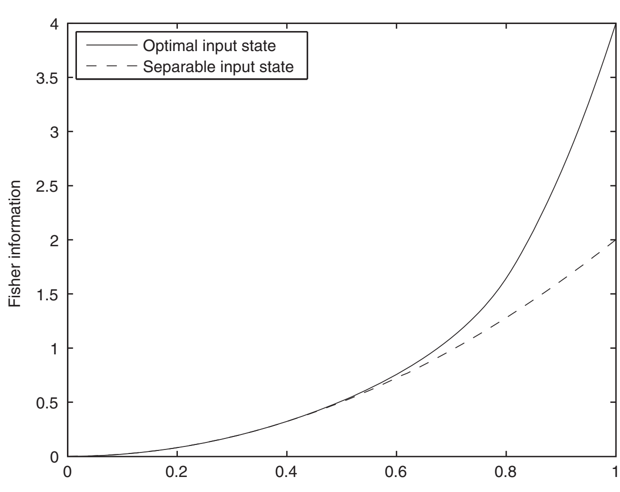

Fig. 1 Quantum Fisher information for the optimal input state and separable input state $|{+}{+}\rangle$ for two qubits with independent dephasing noises

In Fig. 2 the quantum Fisher information for the optimal state, GHZ state, and the separable state are plotted.

Parallel scheme

Previous results on the SQL (standard quantum limit)-like scaling for certain independent noise processes[12, 13, 15, 18, 45] can also be recaptured in this framework. In ref. 43 we showed that given any two channels $K_1(\rho_S) = \sum_{j=1}^d F_{1j}\rho_S F_{1j}^\dagger$, $K_2(\rho_S) = \sum_{j=1}^d F_{2j}\rho_S F_{2j}^\dagger$, we have

$$\begin{aligned} &2 - 2\cos\Theta\left(K_1^{\otimes N} \otimes I_A, K_2^{\otimes N} \otimes I_A\right) \\ &\le N\left\|2I - K_W - K_W^\dagger\right\| + N(N-1)\|I - K_W\|^2, \end{aligned} \quad (15)$$

where $K^{\otimes N}$ denote N channels in parallel as in Fig. 3, $K_W = \sum_{ij} w_{ij} F_{1i}^\dagger F_{2j}$, with w_{ij} as the ij-th entry of a $d \times d$ matrix W which satisfies $\|W\| \le 1$. This inequality is valid for any W with $\|W\| \le 1$, the smaller the right side of the inequality, the tighter the bound is. In the asymptotical limit, $N(N-1)\|I - K_W\|^2$ is the dominating term, in that case we would like to choose a W minimizing $\|I - K_W\|$ for a tighter bound. This can be formulated

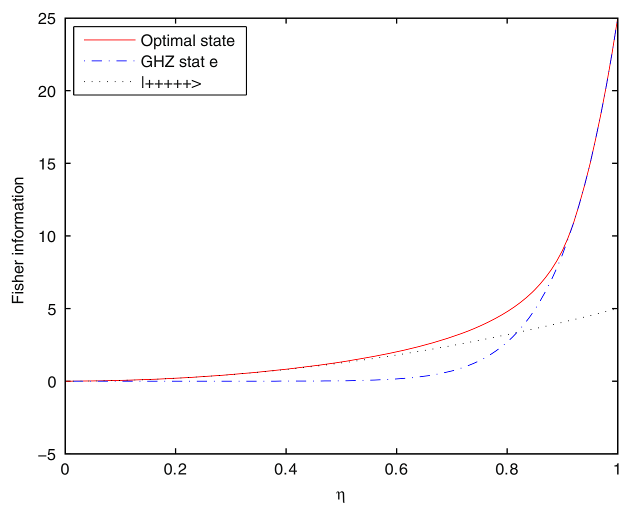

Fig. 2 Quantum Fisher information for optimal probe states, GHZ state, and separable state for five qubits under independent dephasing noises

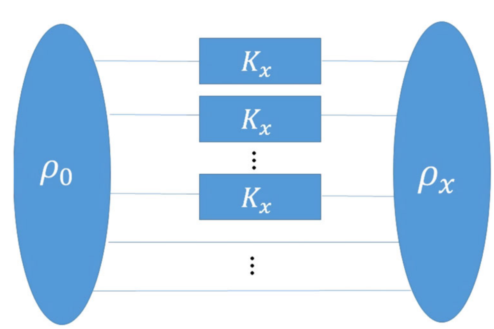

Fig. 3 N probes with independent noisy processes

as semi-definite programming with

$$\min_{\|W\|\le 1}\|I-K_W\| = \begin{array}{c}\min \quad t \\ s.t. \begin{pmatrix} I & W^\dagger \\ W & I\end{pmatrix} \ge 0, \\ \begin{pmatrix} tI & (I-K_W)^\dagger \\ I-K_W & tI\end{pmatrix} \ge 0.\end{array} \tag{16}$$

For quantum parameter estimation with the noisy channel $K_x(\rho)=\sum_{i=1}^d F_i(x)\rho F_i^\dagger(x)$, we can substitute $K_1=K_x$ and $K_2=K_{x+dx}$ into Eq. (15). If there exists a $d\times d$ matrix W with $\|W\|\le 1$ such that $\|I-K_W\|\le Ddx^2$, where $K_W=\sum_{ij} w_{ij}F_i^\dagger(x)F_j(x+dx)$, then the precision limit of $K_x^{\otimes N}$ will scale at most $\frac{1}{\sqrt{N}}$. As by substituting $K_1=K_x$ and $K_2=K_{x+dx}$ into Eq. (15),

$$\begin{aligned} &2-2\cos\Theta\left(K_x^{\otimes N}\otimes I_A, K_{x+dx}^{\otimes N}\otimes I_A\right) \\ &\le N\left\|2I-K_W-K_W^\dagger\right\| + N(N-1)\|I-K_W\|^2 \\ &\le N\left(\|I-K_W\|+\left\|I-K_W^\dagger\right\|\right)+N(N-1)D^2dx^4 \\ &\le 2DNdx^2+N(N-1)D^2dx^4. \end{aligned} \tag{17}$$

The quantum Fisher information is then bounded by

$$\max J = \lim_{dx\to 0} 8\frac{1-\cos\left[\Theta\left(K_x^{\otimes N}\otimes I_A, K_{x+dx}^{\otimes N}\otimes I_A\right)\right]}{dx^2} \le 8DN,$$

thus the precision limit has SQL scaling

$$\delta x \ge \frac{1}{\sqrt{nJ}} \ge \frac{1}{\sqrt{8nDN}}.$$

For example, consider the dephasing channel with

$$K_x(\rho_0)=U(x)\left(\frac{1+\eta}{2}\rho_0+\frac{1-\eta}{2}\sigma_3\rho_0\sigma_3\right)U^\dagger(x),$$

where $U(x)=\exp(-i\frac{\sigma_3}{2}x)$, $\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$ and $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\eta\in[0,1]$. In this case $F_1(x)=\sqrt{\frac{1+\eta}{2}}U(x)$, $F_2(x)=\sqrt{\frac{1-\eta}{2}}\sigma_3U(x)$. We choose $W=\begin{bmatrix}\cos(\xi dx) & i\sin(\xi dx)\\ i\sin(\xi dx) & \cos(\xi dx)\end{bmatrix}$ and vary ξ to minimize $\|I-K_W\|$. In this case

$$\begin{aligned} K_W &= \cos(\xi dx)F_1^\dagger(x)F_1(x+dx)+i\sin(\xi dx)F_1^\dagger(x)F_2(x+dx)+i\sin(\xi dx)F_2^\dagger(x)F_1(x+dx)+\cos(\xi dx)F_2^\dagger(x)F_2(x+dx) \\ &= \begin{pmatrix}\left[\cos(\xi dx)+i\sqrt{1-\eta^2}\sin(\xi dx)\right]e^{\frac{idx}{2}} & 0 \\ 0 & \left[\cos(\xi dx)-i\sqrt{1-\eta^2}\sin(\xi dx)\right]e^{-i\frac{dx}{2}}\end{pmatrix}, \end{aligned} \tag{18}$$

thus

$$I-K_W=\begin{pmatrix}R-iI & 0\\ 0 & R+iI\end{pmatrix}, \tag{19}$$

where $R=1-\cos(\xi dx)\cos\frac{dx}{2}+\sqrt{1-\eta^2}\sin(\xi dx)\sin\frac{dx}{2}$ and $I=\cos(\xi dx)\sin\frac{dx}{2}+\sqrt{1-\eta^2}\sin(\xi dx)\cos\frac{dx}{2}$, then $\|I-K_W\|=\sqrt{R^2+I^2}$. Expanding R and I to the second order of dx, we can get $R=\frac{1+4\xi^2+4\xi\sqrt{1-\eta^2}}{8}dx^2+O(dx^3)$ and $I=\frac{dx}{2}+\sqrt{1-\eta^2}\xi dx+O(dx^3)$. To minimize $\|I-K_W\|$ we should choose $\xi=-\frac{1}{2\sqrt{1-\eta^2}}$ when $\eta\ne 1$($\eta=1$ corresponds to the case of no dephasing error) so the first-order term in I cancels. In this case up to the second order

$$\begin{aligned}\|I-K_W\| &= |R| \\ &= \frac{\eta^2}{8(1-\eta^2)}dx^2+O(dx^3),\end{aligned} \tag{20}$$

thus $\max J=\lim_{dx\to 0} 8\frac{1-\cos\left[\Theta_{QC}\left(K_x^{\otimes N}\otimes I_A, K_{x+dx}^{\otimes N}\otimes I_A\right)\right]}{dx^2}\le\frac{\eta^2}{1-\eta^2}N$, and the precision limit $\delta x\ge\frac{1}{\sqrt{nJ}}\ge\frac{\sqrt{1-\eta^2}}{\eta\sqrt{nN}}$, which scales as $\frac{1}{\sqrt{N}}$ for any $\eta\ne 1$. This is consistent with previous studies[12, 13, 18, 19] but here with a clear procedure to obtain the value for ξ.

DISCUSSION

We discuss how our results are related to previous studies. Previous studies[12, 13] show that for an extended channel $K_x\otimes I_A$ the maximal quantum Fisher information is given by

$$\max J = 4\min_{\{\hat{F}_j(x)\}}\left\|\sum_{j=1}^d \dot{\hat{F}}_j^\dagger(x)\dot{\hat{F}}_j(x)\right\| \tag{21}$$

where the minimization is over all smooth representations of equivalent Kraus operators of the channel K_x. Note that this can be equivalently written as

$$\begin{aligned}\max J &= 4\min_{\{\hat{F}_j(x)\}}\left\|\sum_{j=1}^d \dot{\hat{F}}_j^\dagger(x)\dot{\hat{F}}_j(x)\right\| \\ &= 4\min_{\{\hat{F}_j(x)\}}\left\|\sum_{j=1}^d\lim_{dx\to 0}\frac{\left(\hat{F}_j^\dagger(x+dx)-\hat{F}_j^\dagger(x)\right)}{dx}\frac{\left(\hat{F}_j(x+dx)-\hat{F}_j(x)\right)}{dx}\right\| \\ &= 4\min_{\{\hat{F}_j(x)\}}\left\|\frac{2I-\sum_{j=1}^d\left(\hat{F}_j^\dagger(x)\hat{F}_j(x+dx)+\hat{F}_j^\dagger(x+dx)\hat{F}_j(x)\right)}{dx^2}\right\| \\ &= 4\frac{2-\max_{\{\hat{F}_j(x)\}}\lambda_{\min}\left[\sum_{j=1}^d\left(\hat{F}_j^\dagger(x)\hat{F}_j(x+dx)+\hat{F}_j^\dagger(x+dx)\hat{F}_j(x)\right)\right]}{dx^2},\end{aligned} \tag{22}$$

where the optimization is over all smooth representations of equivalent Kraus operators. In previous studies the equivalent Kraus operators are represented by $\hat{F}_j(x)=\sum_{i=1}^d\omega_{ji}(x)F_i(x)$ and $\hat{F}_j(x+dx)=\sum_{i=1}^d\omega_{ji}(x+dx)F_i(x+dx)$, where $\omega_{ji}(x)$ is ji-entry of $W_E(x)\in U(d)$, and $W_E(x)$ is required to be smooth with respect to x. It is easy to see that in this case Eq. (22) is a special case of Eq. (13) with W restricted to the form $W_E^\dagger(x)W_E(x+dx)$. This restriction arises as the operator W_E in Eq. (10), which originally can be arbitrary chosen, was assumed to depend on x smoothly in previous studies.[12, 13] Such restriction is not intrinsic to the dynamics. Furthermore without the restriction the set $\{W|\|W\|\le 1\}$ is a convex set, which allows a direct formulation as the semi-definite programming. While with the restriction $W=W_E^\dagger(x)W_E(x+dx)$ needs to be unitary which does not form a convex set, to circumvent this difficulty previous study needs to resort to the Lie algebra of the unitaries and formulated the semi-definite programming on the tangent space instead.[15] That, however, comes with a cost on the computational complexity. The complexity of semi-definite programming is determined by the number of variables (A) and the size of the constraining matrices (B) as $O(A^2B^2)$,[46] while the number of variables in the semi-definite programming here is in the same order as previous studies (both in the order of d^2), the size of the constraining matrices differ: the constraining matrices here have the total size of $2d+m_1$, while previous formulation needs a size of m_1+dm_2.[15] The difference can be significant when the system gets large (note that for generic channels d is in the order of m_1m_2). For example, for N-qubit system, $m_1=m_2=2^N$, the difference quickly becomes large with the increase of N. Also since any choice of allowed W leads to a lower bound on the precision limit, expanding the set of allowed W from the unitaries to $\{W| \|W\| \leq 1\}$ also provides more room for obtaining useful lower bounds.

CONCLUSION

In conclusion, we presented a general framework for quantum metrology that provides systematical ways to obtain the ultimate precision limit. This framework relates the ultimate precision limit directly to the geometrical properties of the underlying dynamics, which eases the analysis on utilizing quantum control methods to alter the underlying dynamics for better precision limit.[47, 48] The tools developed here, such as the generalized Bures angle on quantum channels that can be efficiently computed using semi-definite programming, are expected to find wide applications in various fields of quantum information science.

METHODS

For more details on the derivation of the formulas for the ultimate precision limit, please see the Supplemental Information.

COMPETING INTERESTS

The authors declare no competing interests.

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