Some Concepts of Graph Theory

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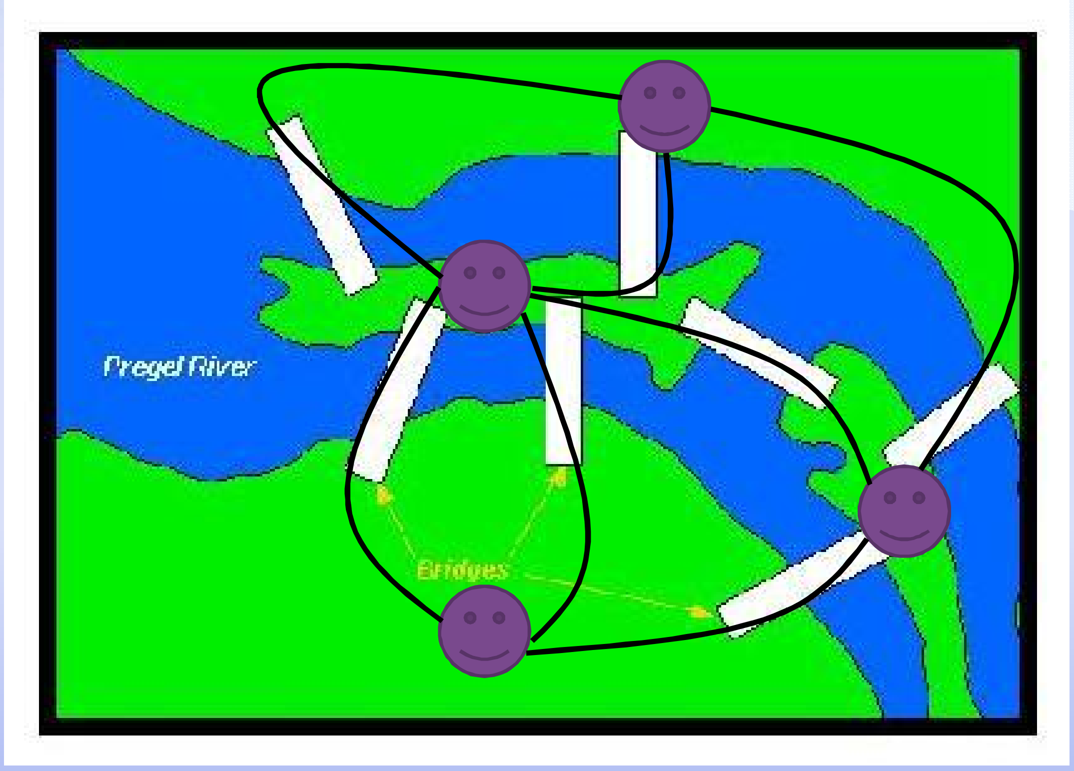
The story begins…

- königsberg bridge on the Pregel River

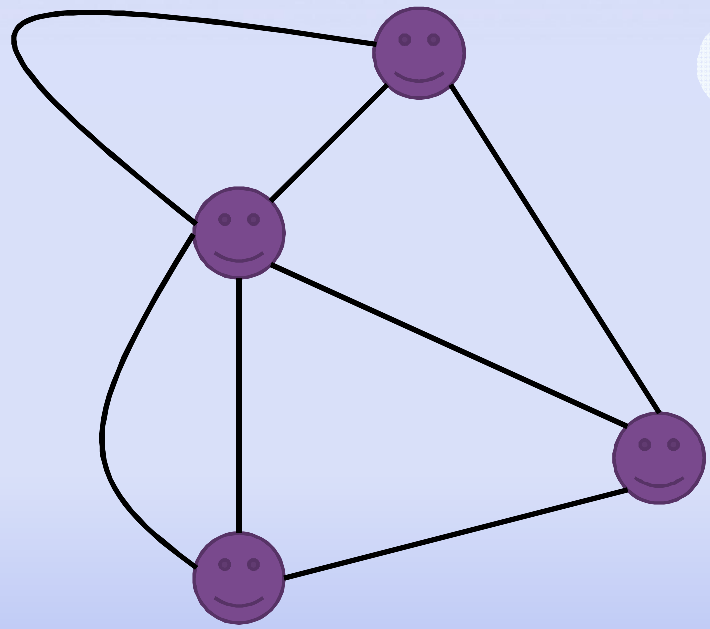


The königsberg bridge problem

Find a traversal through the cities that would cross each bridge once and only once.

Formalized version:

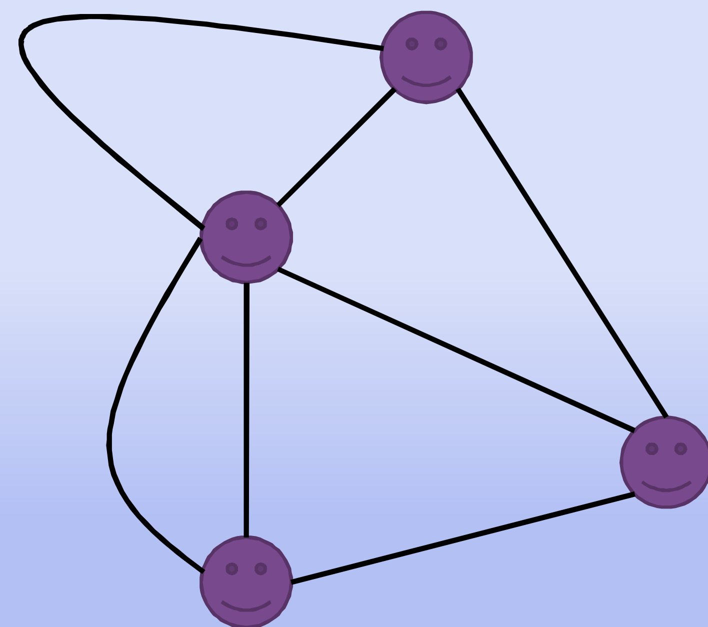
Find an Euler walk in the shown graph.



A graph

- $G = (V, E)$
  - V: Set of vertices $\{v1, v2, \ldots, v_m\}$
  - E: Set of edges $\{e1, e2, \ldots, e_n\}$
    - $E \subseteq V \times V$

- Order of the graph: m
- Size of the graph: n



Vertex incidence, self-loops and degree

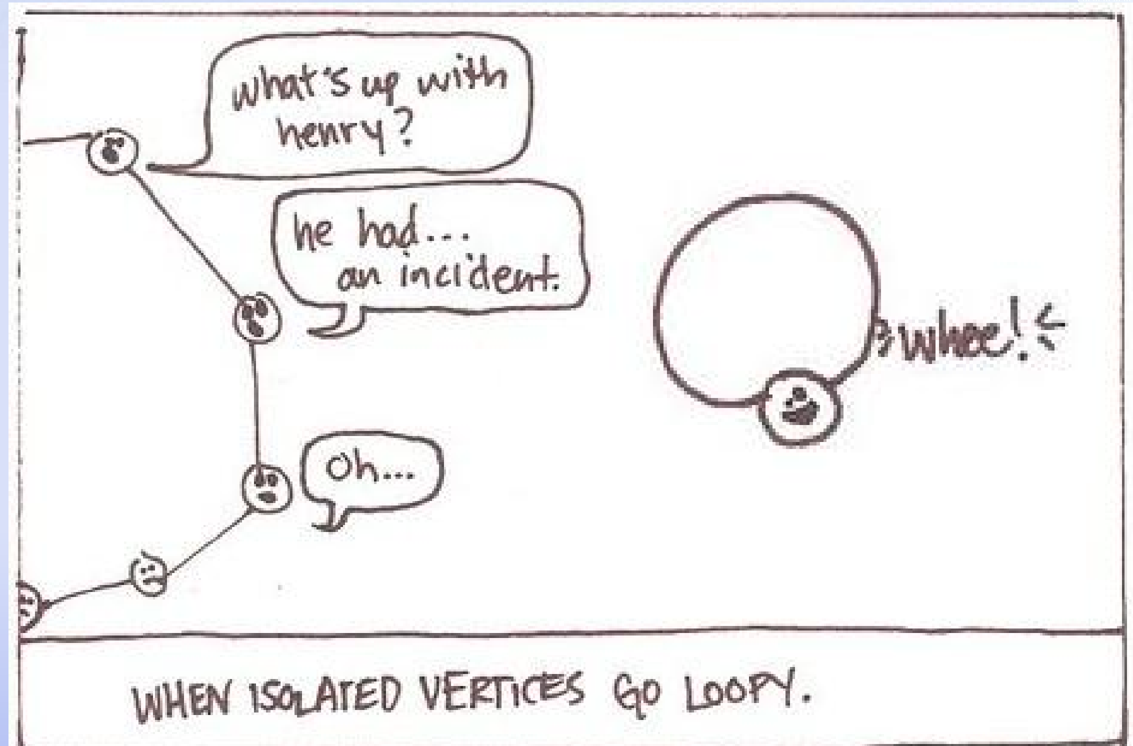
Given, $G = (V, E)$

$I(v) = \{(i, j) \subseteq V \times v \mid (i, j) \in E\}$

$d(v) = |I(v)|$

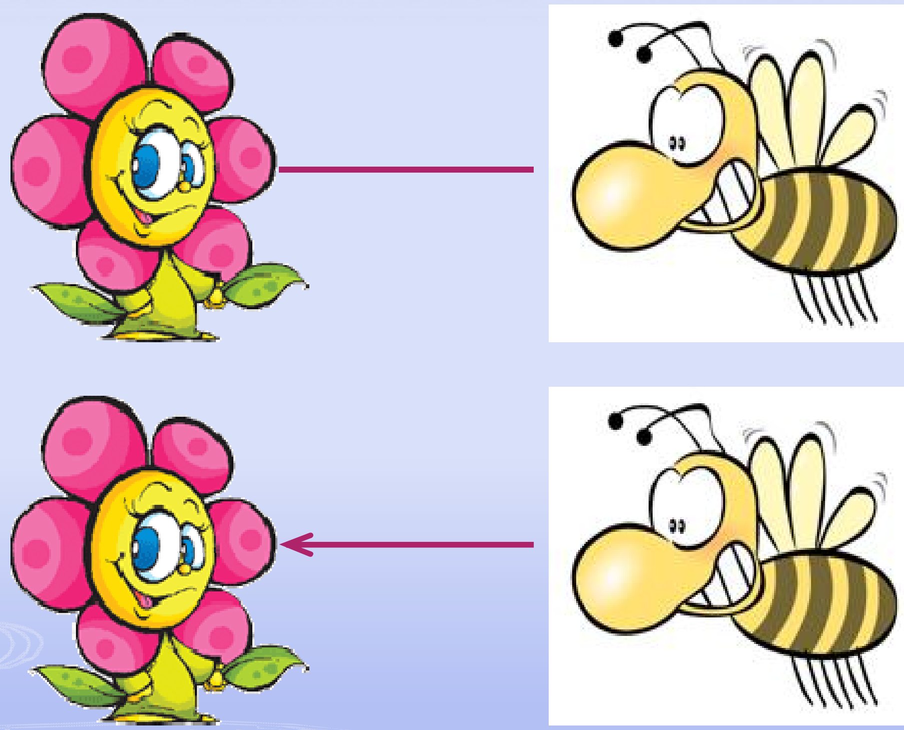
$d(G) = (1/|V|).\sum_{v \in V} d(v)$

$\sum_{v \in V} d(v)$ is twice the size of the graph i.e. $2|E|$



Directed graph

- Distinguishing vertex pairs



In-degree and out-degree

- In-degree: $d^-(v)$
- Out-degree: $d^+(v)$

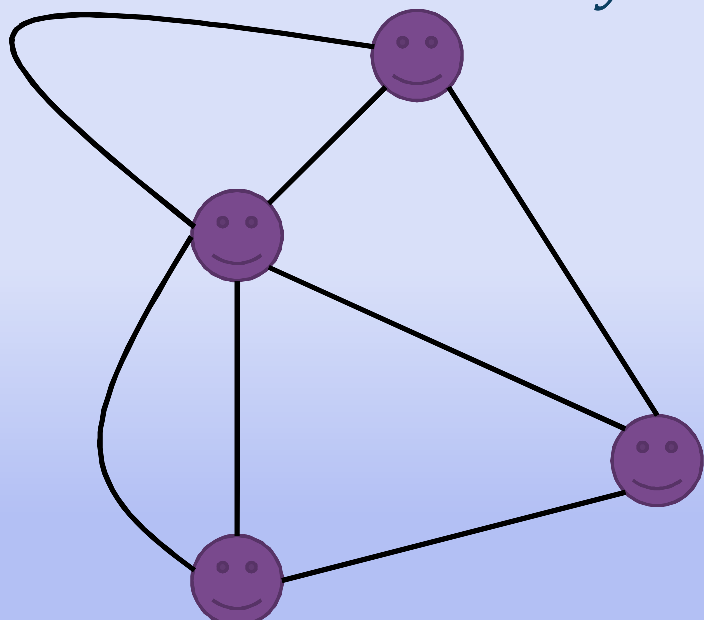
$$\sum_{v \in V} d^-(v) = \sum_{v \in V} d^+(v) = |E|$$

Significant theorems

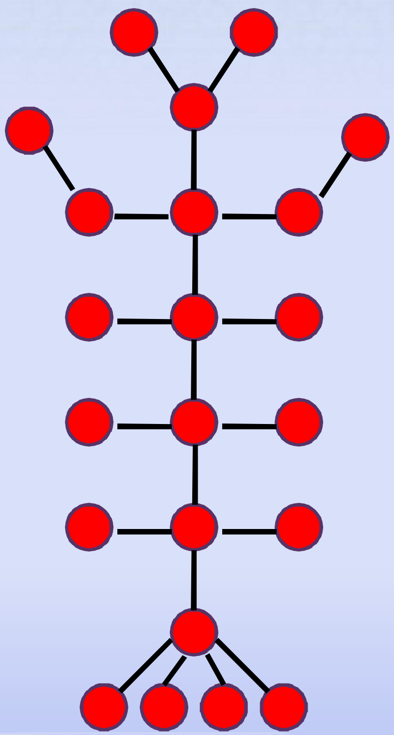
- The number of vertices of odd degree in a graph is always even.

Walks, paths and cycles

- A *walk* is a sequence of edges in a graph
  - A *walk* in which no edge is repeated is a *trail*
  - A *trail* in which no vertex is repeated is a *path*
  - A *path* which have same start and end vertices is a *cycle*



Tree




Sometimes I wonder about graph theorists...
Mommy, what's a lobster?
Well honey, a lobster is a tree...
quoi?
2007
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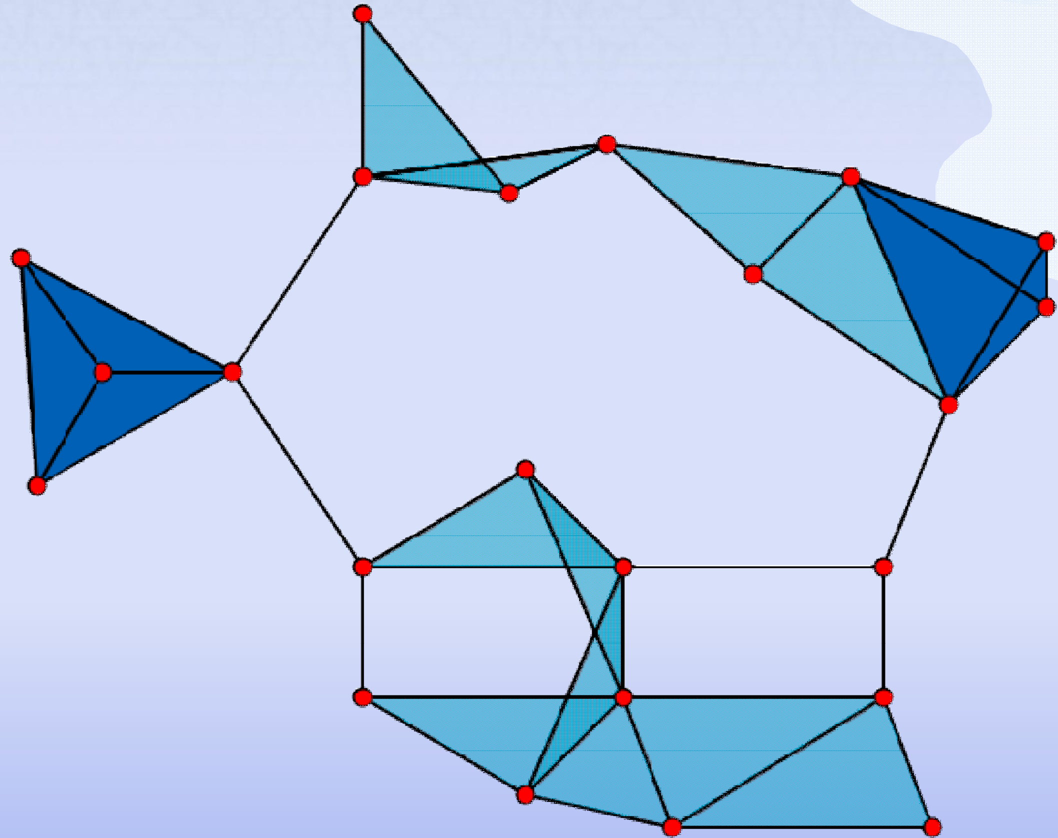
Properties of a tree

- A tree is a connected graph with no cycles
- There is one and only one path between every pair of vertices
- The degree of a tree with n vertices is $2(n-1)/n$

Cliques: Complete subgraphs

- 1-vertex cliques (vertices) – 23,
- 2-vertex cliques (edges) – 42,
- 3-vertex cliques (light blue triangles) – 19 (11 are maximal),
- 4-vertex cliques (dark blue trianges) – 2 (both are maximum and maximal).

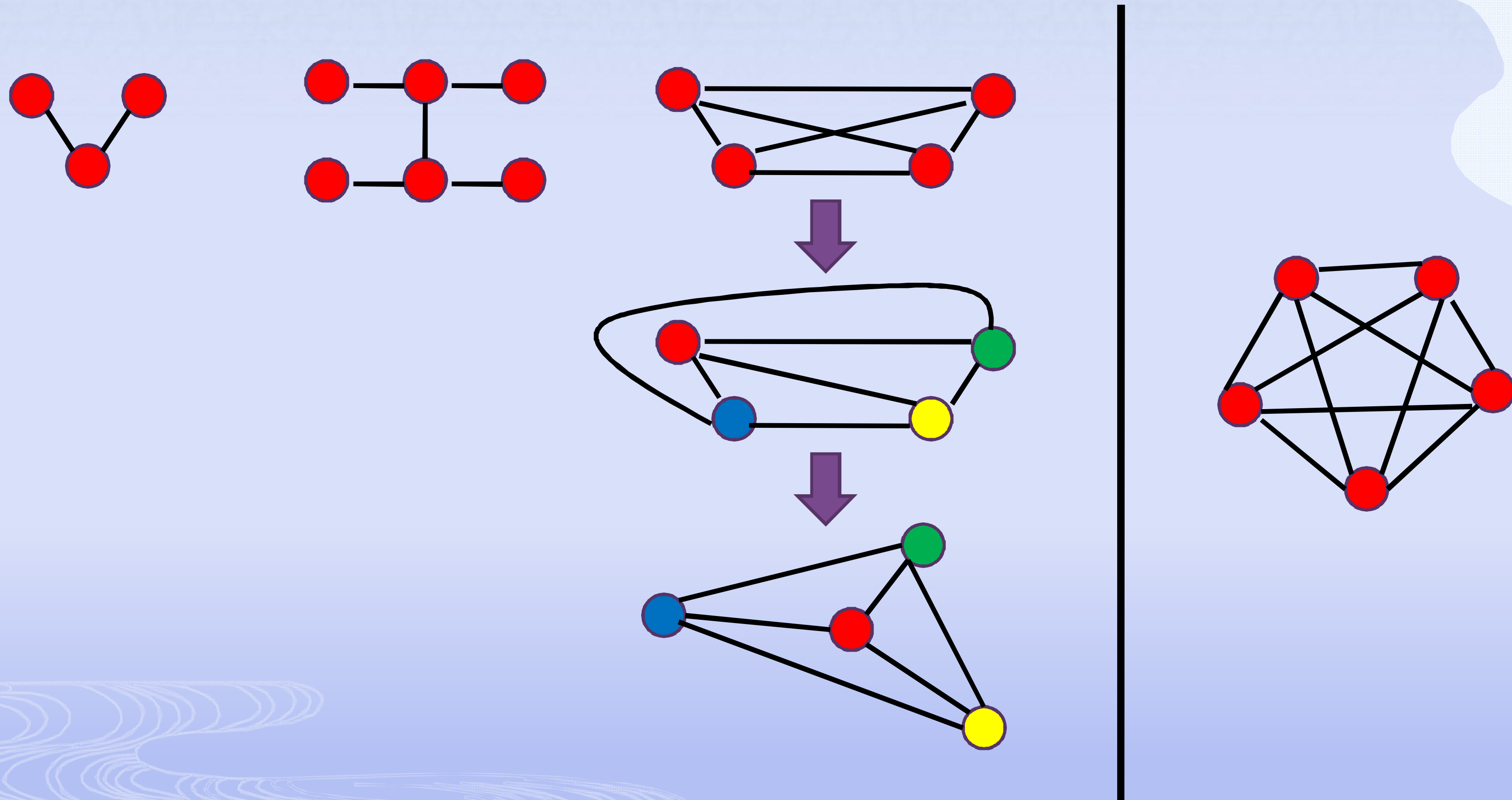
Clique number of the graph: 4



Bipartite graph and biclique

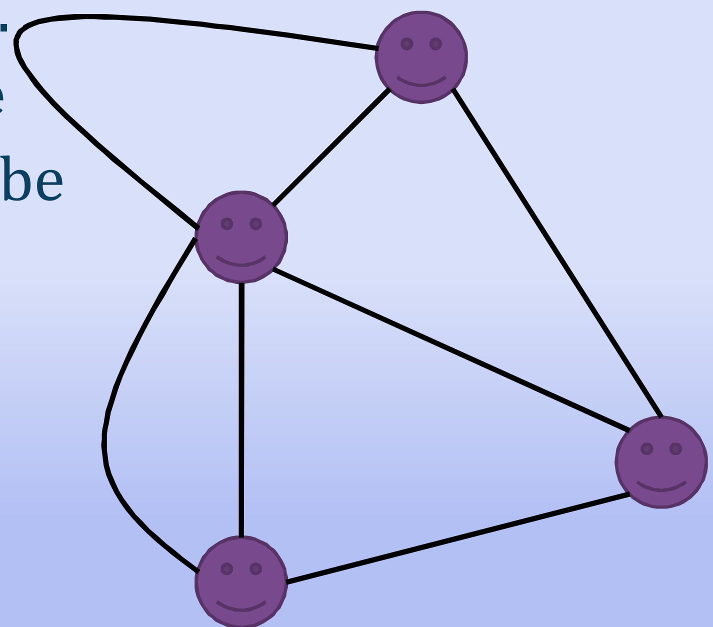
$G = (V1, V2, E)$, where $E \subseteq V1 \times V2$

Planar and non-planar graphs



The story ends…

- An Euler walk is a walk that traverses every edges of a graph once
  - **Criterion:** During any walk in the graph, the number of times one enters a non-terminal vertex equals the number of times one leaves it.
  - Thus, the degree values of the non-terminal vertices should be even.
  - But, this graph has degree values 3, 3, 5, 5.
  - So, no solution exists.



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Thank you