

## ARTICLE OPEN

## Detecting topological phases of microwave photons in a circuit quantum electrodynamics lattice

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Topology is an important degree of freedom in characterising electronic systems. Recently, it also brings new theoretical frontiers and many potential applications in photonics. However, the verification of the topological nature is highly nontrivial in photonic systems, as there is no direct analogue of quantised Hall conductance for bosonic photons. Here we propose a scheme of investigating topological photonics in superconducting quantum circuits by a simple parametric coupling method, the flexibility of which can lead to the effective *in situ* tunable artificial gauge field for photons on a square lattice. We further study the detection of the topological phases of the photons. Our idea uses the exotic properties of the edge state modes, which result in novel steady states of the lattice under the driving-dissipation competition. Through the pumping and the photon-number measurements of merely few sites, not only the spatial and the spectral characters but also the momentums and even the integer topological quantum numbers with arbitrary values of the edge state modes can be directly probed, which reveal unambiguously the topological nature of photons on the lattice.

npj Quantum Information (2016) 2, 16015; doi:10.1038/npjqi.2016.15; published online 7 June 2016

## INTRODUCTION

Charged particles in two dimensions exhibit integer quantum Hall effect when exposed to a perpendicular magnetic field,<sup>1</sup> characterised by the quantised transverse conductances in transport experiments. This novel effect can be explained by the integer topological Chern numbers describing the global behaviour of the energy bands.<sup>2,3</sup> Such topological insulating integer quantum Hall effect phase is robust against disorder and defects, because the band topology remains invariant as long as the band gaps are preserved. Therefore, in the connection between the topologically nontrivial material and the trivial vacuum, there exist unavoidably the edge state modes (ESMs) spatially confining at the boundary and spectrally traversing the band gaps.<sup>4</sup> The presence of these gapless ESMs thus serves as an unambiguous signature of the topological nontriviality of the bulk band structure.

Recently, the concept of topology has been extended to circuit quantum electrodynamics (QED) lattice,<sup>5,6</sup> where the electrons are replaced by microwave photons hopping between superconducting transmissionline resonators (TLRs).<sup>7,8</sup> Although the idea of topological photonics was first developed in photonic crystals,<sup>9–13</sup> circuit QED enjoys the time-resolved engineering of a large-scale lattice at the single-site level.<sup>14,15</sup> The demonstrated strong coupling between superconducting qubits and TLRs<sup>7,8</sup> further allows the effective photon–photon interaction,<sup>16–19</sup> which can hardly be achieved in other physical systems, indicating prospective future of investigating strongly correlated photonic liquids.<sup>20,21</sup> As photons are charge neutral, there have been several proposals of synthesising artificial magnetic fields on a TLR lattice, with predicted strengths much stronger than those in conventional electronic materials.<sup>22–25</sup> Nevertheless, the synthetic

Abelian gauge field has not been implemented so far despite the extensive theoretical studies, partially because of the complicated circuit elements required in these schemes. In addition, the detection of the integer topological invariants has also been addressed in recent research,<sup>26–28</sup> which is nontrivial in the sense that the Hall conductance measurement cannot be transferred to circuit QED because of the absence of fermionic statistics.

Here we propose a theoretical scheme of implementing topological photonics in a two-dimensional circuit QED lattice. The distinct merit of our proposal is that we couple the TLRs by parametric frequency conversion (PFC) method, which is simple in experimental setup and feasible with state-of-the-art technology.<sup>29–32</sup> The lattice in our scheme is formed by TLRs connected to the ground through the superconducting interference devices (SQUIDs),<sup>29,30,33,34</sup> where the tunable photon hopping with nontrivial phases between TLRs can be induced through the dynamic modulation of the SQUIDs, allowing the arbitrary synthesis of time- and site-resolved gauge fields on the square lattice.<sup>11,33</sup> Moreover, with the driving-dissipation mechanism being used, various quantities of the ESMs can be measured by the pumping and the steady-state photon-number (SSPN) detection of only few sites on the lattice.<sup>32</sup> In particular, the integer topological winding numbers of the ESMs with arbitrary values can be directly probed through the realisation of the adiabatic pumping process.<sup>4,27</sup> Such a measurement is equivalent to the measurement of the Chern numbers of the bulk bands and thus clearly examine the topological nontriviality of the photons. Furthermore, our detailed discussions show unambiguously that our proposal is very robust against various potential imperfection sources in experimental realisations because of the topological nature of the ESMs, pinpointing the feasibility with current level of

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Received 21 July 2015; revised 24 February 2016; accepted 2 April 2016

technology. Being flexible for the extension to more complicated lattice configurations and the incorporation of effective photon correlation, our scheme serves therefore as a promising and versatile platform for the future investigation of various photonic quantum Hall effects.

## RESULTS

### The lattice

We start with a square lattice consisting of TLRs with four different lengths placed in an interlaced form, as shown in Figure 1a. At their ends, the TLRs are commonly grounded by SQUIDs with effective inductances much smaller than those of the TLRs<sup>33,34</sup> (Supplementary Information).<sup>35</sup> Because of their very small inductances, the grounding SQUIDs impose low-voltage shortcuts at the ends of the TLRs. Therefore, the lowest eigenmodes of the lattice can be approximated by the  $\lambda/2$  modes of the TLRs with their ends being the nodes, and the whole lattice can be described by

$$\mathcal{H}_S = \sum_{\mathbf{r}} \omega_{\mathbf{r}} a_{\mathbf{r}}^{\dagger} a_{\mathbf{r}}, \quad (1)$$

with  $a_{\mathbf{r}}^{\dagger}/a_{\mathbf{r}}$  being the creation/annihilation operators of the  $\mathbf{r}$ th photonic mode and  $\omega_{\mathbf{r}}$  being the eigenfrequency. We further specify the eigenfrequencies of the four kinds of TLRs as yellow— $\omega_0$ , blue— $\omega_0+\Delta$ , green— $\omega_0+3\Delta$  and red— $\omega_0+4\Delta$ , respectively, with  $\omega_0/2\pi \in [10, 20]$  GHz and  $\Delta/2\pi \in [1, 3/2]$  GHz.

Such a configuration is for the following application of the dynamic modulation method and can be achieved through the length selection of the TLRs in the millimetre range<sup>30–32</sup> (Supplementary Information).

We then consider how to implement on the TLR lattice the effective tight-binding Hamiltonian

$$\mathcal{H}_T = \mathcal{T} \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} a_{\mathbf{r}}^{\dagger} a_{\mathbf{r}'} e^{-i\theta_{\mathbf{r}\mathbf{r}'}} + \text{h.c.}, \quad (2)$$

in the rotating frame of  $\mathcal{H}_S$ . Here  $\mathcal{T}$  is the uniform hopping amplitude, and

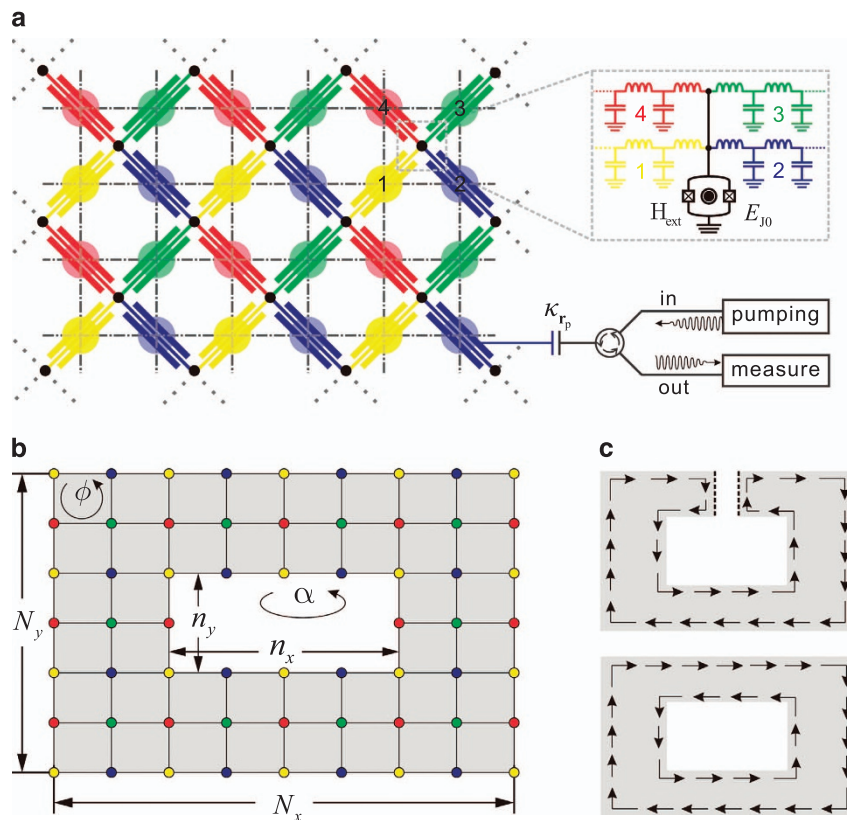
$$\theta_{\mathbf{r}\mathbf{r}'} = \int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{A}(\mathbf{x}) \cdot d\mathbf{x}, \quad (3)$$

is the  $\mathbf{r} \rightarrow \mathbf{r}'$  hopping phase manifesting the presence of a vector potential  $\mathbf{A}(\mathbf{x})$  through Peierls substitution.<sup>3</sup> For each plaquette of the lattice, the summation of the hopping phases around its loop has the physical meaning

$$\theta_{\text{plaquette}} = \oint \mathbf{A}(\mathbf{x}) \cdot d\mathbf{x} = \iint \mathbf{B}(\mathbf{x}) \cdot d\mathbf{S}, \quad (4)$$

i.e., the synthetic local magnetic field for the microwave photons.

However, it is nontrivial to have complex hopping constants between TLRs, because the physical coupling between two TLRs takes real coupling constants, regardless of whether it is capacitive<sup>14,15</sup> or inductive.<sup>31,32</sup> We then consider the dynamic modulation method studied in recent experiments.<sup>30–32</sup>



**Figure 1.** (a) Sketch of the square TLR lattice with the four colours (yellow, blue, green and red) denoting the different lengths of the TLRs and the black dots representing the grounding SQUIDs (see the lumped circuit magnified representation on the right side). Each TLR has the role of a photonic site (the large coloured rounds), and the effective hopping between them (the dotted-dash lines) can be induced through the dynamic modulation of the SQUIDs. The pumping and the consequent steady-state measurement can be performed through the external coil connected to the pumping site(s) (lower right). (b) Configuration of the proposed lattice. The coloured rounds and the solid lines label the TLRs with corresponding lengths and the photon hopping branches, respectively. (c) Spatial geometry of the lattice. The lattice shown in **b** can be obtained from the gluing of a simply connected plane by the two dashed sides (the upper panel). Through this process, the opposite chiralities of the inner and the outer ESMs (the arrows) are formed (the lower panel).

The grounding SQUIDs can be modelled as flux-tunable inductances, and it is now experimentally possible to modulate the SQUIDs

by a.c. magnetic flux oscillating at very high frequencies (the experiment-achieved range is typically 8–10 GHz,<sup>29,35</sup> which is much higher than the following proposed 1~6 GHz). Such a.c. modulation introduces a small a.c. coupling

$$\mathcal{H}_{AC} = \sum_{(r,r')} \mathcal{T}_{rr'}^{ac}(t) (a_r + a_r^\dagger) (a_{r'} + a_{r'}^\dagger), \quad (5)$$

in addition to the d.c. contribution of the SQUIDs, which is irrelevant because the TLRs are largely detuned (Supplementary Information). We then assume that the a.c. modulation of the grounding SQUIDs contains three tones with frequencies being  $\Delta$ ,  $2\Delta$  and  $4\Delta$ . By bridging the frequency differences between the TLRs, the  $2\Delta/4\Delta$  tones induce the vertical blue  $\Leftrightarrow$  green/red  $\Leftrightarrow$  yellow parametric hoppings, and the  $\Delta$  tone establishes the horizontal yellow  $\Leftrightarrow$  blue and green  $\Leftrightarrow$  red parametric hoppings (Supplementary Information). When experiencing the PFC process between the TLR sites, the microwave photons adopt the phases of the a.c. modulating pulses, leading to the effective controllable complex hopping constants in the rotating frame of  $\mathcal{H}_S$ .<sup>32,33</sup> Through the application of the developed three-tone PFC pulses to each of the grounding SQUIDs, every vertical hopping branch and every pair of horizontal hopping branches can be independently controlled by a modulating tone threaded in one of the grounding SQUIDs (Supplementary Information), implying that the artificial magnetic field for microwave photons with Landau gauge

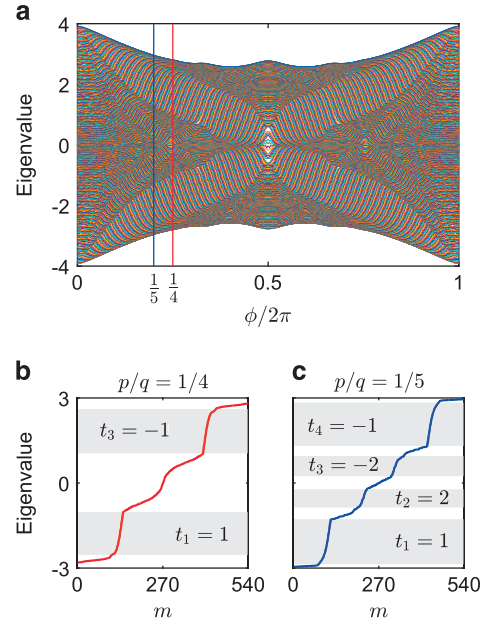
$$\mathbf{A} = [0, A_y(x), 0], \quad \mathbf{B} = B\mathbf{e}_z = \left[0, 0, \frac{\partial}{\partial x} A_y(x)\right], \quad (6)$$

can be created with *in situ* tunability. A further estimation demonstrates that the uniform hopping strength can be synthesised in the range  $|\mathcal{T}/2\pi| \in [5, 15]$  MHz (refs 30,31; Supplementary Information).

For the investigation convenience of the ESM physics, in what follows we endow a nontrivial ring geometry to the TLR lattice, i.e., an  $N_x \times N_y$  square lattice with an  $n_x \times n_y$  vacancy at its middle, as shown in Figure 1b. Through the careful setting of the hopping phases, we penetrate a uniform effective magnetic flux  $\phi$  in each plaquette of the lattice and an extra  $a$  at the central vacancy. In Figure 2a, the energy spectrum of a finite lattice is calculated with  $N_x = N_y = 24$  and  $n_x = n_y = 6$ . In the rational situation  $\phi/2\pi = p/q$  with  $p, q$  being co-prime integers, the unit cell of the lattice is enlarged by  $q$  times, leading to  $q$  nearly flat magnetic bands and the fractal Hofstadter butterfly spectrum<sup>36</sup> (Figure 2a). These  $q$  magnetic bands have nontrivial topological band structures, and between the  $q$  bands there exist ESMs traversing the  $q-1$  band gaps.<sup>3,4,13</sup> The lattice spectrums of the situations of interest  $p/q = 1/4$  and  $p/q = 1/5$  are shown in Figure 2b,c, respectively, where the flatness of the band steps and the stiffness of the connections between the steps imply the degeneracy of the Landau levels and the spectral location of the ESMs.

#### Probing the ESMs: the spatial and spectral information

Compared with fermionic electronic systems, the photonic nature of circuit QED allows multiple occupation of a particular mode at the same time and the non-equilibrium driving-dissipation competition. Here we propose the following scheme of probing the topological nature of the ESMs. With the detailed modelling being discussed in Materials and Methods, we emphasise that the physics behind is that the exotic properties of the ESMs result in the novel steady states of the lattice, and the information of the ESMs can be extracted from the SSPNs of only few sites on the lattice versus the pumping frequency and the pumping sites.



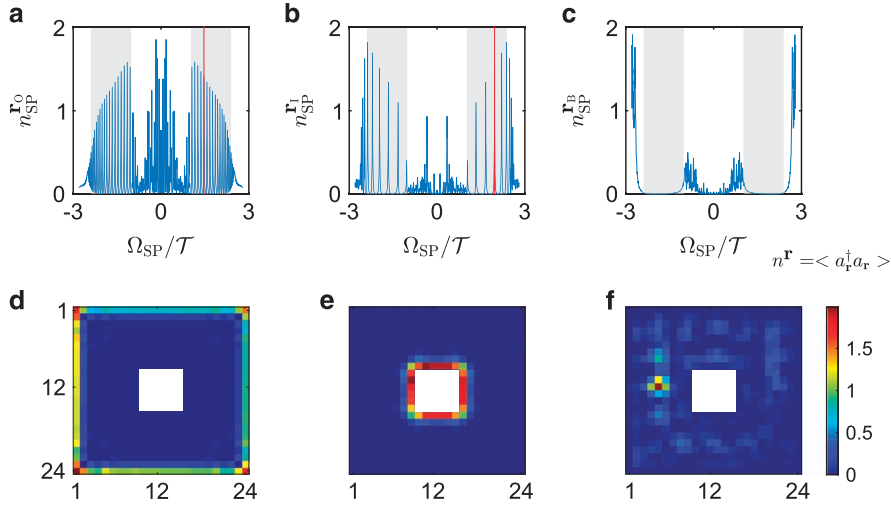
**Figure 2.** (a) Hofstadter butterfly spectrum of the proposed lattice with  $N_x \times N_y = 24 \times 24$ ,  $n_x \times n_y = 6 \times 6$  and  $a/2\pi = 0$ , where energy is in units of  $\mathcal{T}$ . The situations of the rational effective magnetic fields  $\phi/2\pi = 1/4$  and  $\phi/2\pi = 1/5$  are denoted by the red and blue lines, respectively. Their eigenenergies are shown in **b** and **c** with  $m$  being the index labelling the 540 eigenvalues from smallest to largest. The band gaps are highlighted with their topological winding numbers marked.

First, let us consider the single-site driving of a particular site  $\mathbf{r}_p$  described by

$$\mathcal{H}_{SP} = \mathcal{P}_S a_{\mathbf{r}_p} e^{i\Omega_{SP}t} + \text{h.c.}, \quad (7)$$

with  $\mathcal{P}_S$  being the pumping strength and  $\Omega_{SP}$  being the detuning in the rotating frame of  $S$ . The SSPN on the pumping site  $n_{SP}^{\mathbf{r}_p} = \langle a_{\mathbf{r}_p}^\dagger a_{\mathbf{r}_p} \rangle$  in the situation  $p/q = 1/4$  and  $a = 0$  is numerically simulated based on Equation (16) and plotted in Figure 3. In what follows, we show that the spatial and spectral information of the ESMs can be distilled by measuring the dependence of the single-site SSPN  $\langle a_{\mathbf{r}_p}^\dagger a_{\mathbf{r}_p} \rangle$  on  $\Omega_{SP}$  and  $\mathbf{r}_p$ .

If we choose  $\mathbf{r}_p = \mathbf{r}_O = (1, 24)$  as an outer edge site (OES), significant  $n_{SP}^{\mathbf{r}_p}$  can be detected when  $\Omega_{SP}$  falls in the 1st and 3rd gaps, indicated by the highlighted spectrum comb in Figure 3a (for the even  $q = 4$ , the 2nd gap is closed as a Dirac point form). This can be attributed to the excitation of the outer ESMs. However, if  $\Omega_{SP}$  is chosen deeply in the magnetic bands,  $n_{SP}^{\mathbf{r}_p}$  has bare value because in this situation  $\mathcal{H}_{SP}$  can only excite bulk state modes (BSMs), which spread over the whole lattice—i.e., the weight of  $\mathbf{r}_p$  in the mode function becomes diluted. The situation of pumping an inner edge site (IES)  $\mathbf{r}_p = \mathbf{r}_I = (9, 13)$  is similar, where the comb-like spectrum of  $n_{SP}^{\mathbf{r}_p}$  centralised in the band gaps can also be found in Figure 3b. Meanwhile, there are still several interesting differences. As the number of the IESs is smaller than that of the OESs, Figure 3b contains fewer peaks than Figure 3a.<sup>3</sup> In addition, different pumping strengths have been used in the numerical simulation of Figure 3a,b such that the obtained  $n_{SP}^{\mathbf{r}_p}$  and  $n_{SP}^{\mathbf{r}_p}$  are in the same region. This choice can also be traced back to the small number of the IESs, which results in the concentration of the mode functions in the inner edge. Another observation is the opposite trends of  $n_{SP}^{\mathbf{r}_p}$  and  $n_{SP}^{\mathbf{r}_p}$  versus  $\Omega_{SP}$ : In the 1st gap of Figure 3a,b, the peaks of  $n_{SP}^{\mathbf{r}_p}/n_{SP}^{\mathbf{r}_p}$  increase/decrease with increasing  $\Omega_{SP}$ . In contrast, when we set  $\mathbf{r}_p = \mathbf{r}_B = (5, 13)$  as a bulk site (BS), the lattice will have detectable  $n_{SP}^{\mathbf{r}_p}$  if  $\Omega_{SP}$  falls in the magnetic bands.



**Figure 3.** Steady state of the proposed TLR lattice with  $\phi/2\pi = 1/4$ ,  $\alpha = 0$  and  $T/2\pi = 10\text{MHz}$ . The SSPN on the pumping site  $n_{\text{SP}}^{\mathbf{r}_p}$  versus  $\Omega_{\text{SP}}$  are displayed in (a–c) with  $\mathbf{r}_p = [(1, 24), (9, 13), (5, 13)]$  and  $\mathcal{P}_S/T [0.5, 0.25, 0.23]$ , respectively. The representative SSPN distributions on the whole lattice are presented in (d–f), with the pumping frequencies  $\Omega_{\text{SP}}/T [1.47, 1.97, 2.69]$  marked by the corresponding red lines in (a–c).

When we choose  $\Omega_{\text{SP}}$  in the band gaps, the lattice cannot be excited because no BSM spectrally populates in the band gaps and no ESM spatially populates in the bulk of the lattice (notice the marked window at the 1st and 3rd gaps in Figure 3c).

The above illustration can be experimentally detected by the proposed measurement scheme sketched in Figure 1a: A particular pumping site  $\mathbf{r}_p$  is capacitively connected to an external coil with input/output ports for pumping/measurement. The steady state of the lattice can be prepared by injecting microwave pulses through the input port for a sufficiently long time. During the steady-state period, energy will leak out of the  $\mathbf{r}_p$ th TLR from the coupling capacitance, which is proportional  $\omega_{\mathbf{r}_p} \langle a_{\mathbf{r}_p}^\dagger a_{\mathbf{r}_p} \rangle$  with the proportional constant being determined by the coupling capacitance. The target observable  $\langle a_{\mathbf{r}_p}^\dagger a_{\mathbf{r}_p} \rangle$  can therefore be measured by simply integrating the energy flowing to the output port in a given time duration. Actually, this measurement scheme has already been used in a recent experiment in which both the amplitude and the phase of a coherent state of a TLR were measured.<sup>32</sup> Here we emphasise that what we want to measure is the expectation value  $\langle a_{\mathbf{r}_p}^\dagger a_{\mathbf{r}_p} \rangle$ , whereas the detailed probability of the multi-mode coherent steady-state projected to the Fock basis is nevertheless not needed. It is this weak requirement that greatly simplifies our measurement.

We further calculate for each pumping situation a typical steady-state photon distribution and display them in Figure 3d–f, respectively. In Figure 3d,e, the steady states correspond to the excitation of an inner or outer ESM. The confinement and uniformity of the steady states clearly reflects the ESM mode functions localised and uniformly distributed on the edge, whereas the extended spatial distribution of the BS pumping steady state in Figure 3f illustrates intuitively the difference between the BSMs and the ESMs.

#### Probing the ESMs: measuring the momentum

Although the single-site pumping provides a route of discovering the spatial and spectral properties of the ESMs, the more interesting physics comes from the multisite inhomogeneous pumping, which proves to be an efficient method of measuring the momentums of the ESMs. We pump  $m$  consecutive OESs as

$$\mathcal{H}_{\text{MP}} = \mathcal{P}_M \sum_{j=1}^m \left[ a_{\mathbf{r}_j} e^{i(\Omega_{\text{MP}} t - j k_p)} + \text{h.c.} \right], \quad (8)$$

and investigate the summed SSPN  $n_{\text{MP}} = \sum_{j=1}^m \langle a_{\mathbf{r}_j}^\dagger a_{\mathbf{r}_j} \rangle$  on the  $m$  pumping sites. Here  $\mathcal{P}_M$  is the homogeneous pumping strength,  $k_p$  is the phase gradient of the pumping between neighbouring sites,  $\Omega_{\text{MP}}$  is the frequency detuning in the rotating frame and  $\mathbf{r}_j$  for  $j = 1, 2 \dots m$  denotes the  $j$ th of the  $m$  pumping sites.

Suppose  $\Omega_{\text{MP}}$  matches the eigenfrequency of a particular outer ESM, there arises an interesting question: how does  $n_{\text{MP}}$  depend on  $k_p$ ? For a photon in that ESM, we can imagine its propagation around the edge with its ESM momentum  $k_0$  (this can be verified by the discussion of the coherent dynamics in Discussion). Therefore, the reduced ESM mode function on the  $m$  pumping sites can be represented by a vector

$$\mathbf{k}_0 = \left( 1, e^{i k_0}, e^{2i k_0} \dots e^{i(m-1)k_0} \right) \quad (9)$$

where the  $e^{i k}$  factor denotes the phase delay between two consecutive sites and the equal-weighting character of  $\mathbf{k}_0$  reflects the uniform spatial distribution of the ESM on the confined edge (see Figure 3d,e). It is this form of  $\mathbf{k}_0$  that inspires the inhomogeneous multisite pumping  $\text{MP}$ , which can be represented by another vector

$$\mathbf{k}_p = \left( 1, e^{i k_p}, e^{2i k_p} \dots e^{i(m-1)k_p} \right). \quad (10)$$

On the basis of the above physical picture, we can conjecture that the maximum of  $n_{\text{MP}}$  will emerge at the point

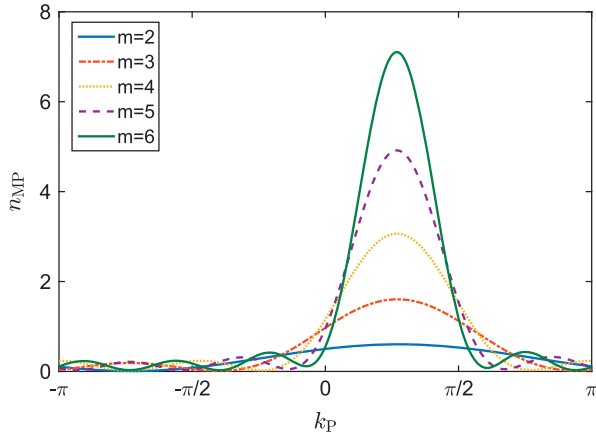
$$k_p = k_0, \quad (11)$$

where the excitations from the  $m$  pumping sites constructively interfere with each other. The dependence of  $n_{\text{MP}}$  on  $k_p$  and  $m$  is plotted in Figure 4, where the positions of the peaks infer the value of  $k_0$ . In addition, the full width of half maximum (FWHM) of the peaks decreases with the increase of  $m$ . This can be understood by considering the two extreme cases. If  $m = 1$ , there is certainly no peak because the steady state is independent of  $k_p$ . Meanwhile, when all the OESs participate in the inhomogeneous pumping and the lattice size grows up, the inner product  $|\mathbf{k}_p^\dagger \mathbf{k}_0|^2$  describing the interference between the pumping sites becomes nonzero if  $k_p = k_0$ . In this situation, the peaks in Figure 4 approach a  $\delta$ -like function. As implied in Figure 4, for a moderate  $m = 5$  the FWHM is already sharp enough to discriminate the ESM momentums with a satisfactory resolution.



### Probing the ESMs: the integer topological invariants

We further consider the measurement of the integer topological quantum numbers of the system. The topological property of an electronic Bloch band is captured by the quantised Hall conductance, which turns out to be its Chern number.<sup>3</sup> This



**Figure 4.**  $n_{\text{MP}}$  versus  $k_p$  and  $m$  with  $\mathcal{P}_M/\mathcal{T} = 0.1$  and  $\Omega_{\text{MP}}/\mathcal{T} = 1.67$ . The pumping sites start from the OES site  $(\lfloor 7 - m/2 \rfloor, 24)$  and end at  $(\lfloor 6 + m/2 \rfloor, 24)$ . The other parameters are the same as those in Figure 3.

transport measurement is nevertheless inaccessible in circuit QED systems because of the absence of Fermi statistics. Meanwhile, the presence of the ESMs provides an alternative way of probing the topological invariants according to the bulk-edge correspondence.<sup>4</sup> In the rational situation  $\phi/2\pi = p/q$ , the eigenenergies of the ESMs can be represented by the zero points of the Bloch functions winding around the  $q - 1$  holes of a complex energy surface, which correspond to the  $q - 1$  gaps of the lattice.<sup>3</sup> The topological quantum numbers of the ESMs are given by the Diophantine equation

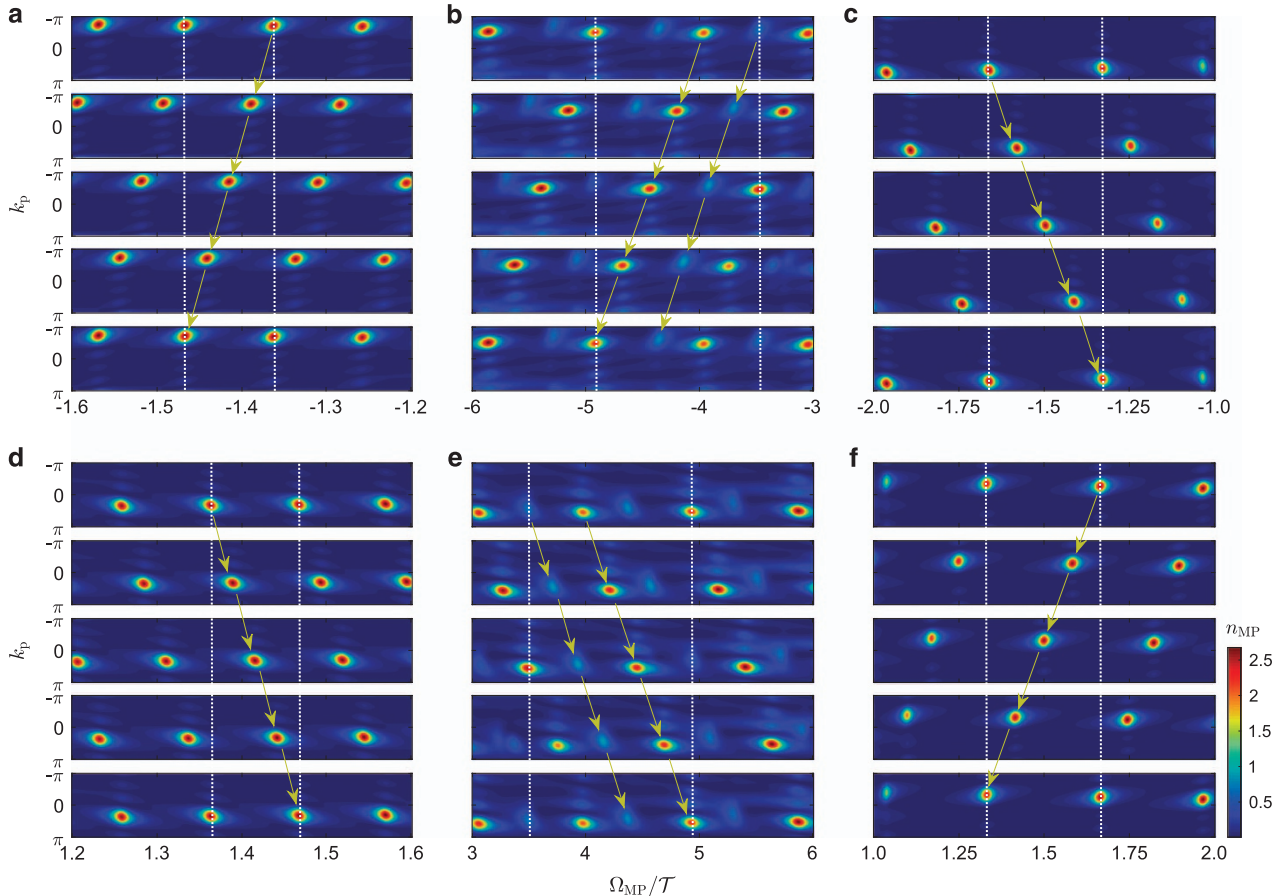
$$h = s_h q + t_h p, |t_h| \leq q/2, \quad (12)$$

where  $h$  is the gap index and  $t_h, s_h$  are integers. In particular,  $t_h$  is the topological winding number of the  $h$ th gap, which is related to the Chern number  $\mathcal{C}_h$  of the  $h$ th band as

$$\mathcal{C}_h = t_h - t_{h-1}. \quad (13)$$

Measuring the winding numbers of the ESMs is thus equivalent to measuring the Chern numbers of the magnetic bands. For  $q = 4$ , we have  $\mathcal{C}_1 = \mathcal{C}_3 = 1$  and  $t_1 = -t_3 = 1$ , whereas for  $q = 5$  we have  $\mathcal{C}_3 = -4$  and  $\mathcal{C}_j = 1$  for the other four bands, and  $t_1 = -t_4 = 1$ ,  $t_2 = -t_3 = 2$  (see Figure 2b,c).

As the spatial configuration of the proposed lattice is equivalent to the Laughlin cylinder<sup>3,4</sup> (i.e., it can be regarded as the rolling of a two-dimensional simply connected plane and the consequent threading with an effective magnetic flux  $\alpha$ ; see Figure 1c), the adiabatic pumping of the ESMs can be realised through the control of  $\alpha$  in the central vacancy. Once  $\alpha$  increases



**Figure 5.** Adiabatic pumping of the proposed lattice represented by  $n_{\text{MP}}$  versus  $\Omega_{\text{MP}}$  and  $k_p$ . Here we set  $\phi/2\pi = 1/4$  for **a, c, d** and **f**, and  $\phi/2\pi = 1/5$  for **b** and **e**. The panels from top to bottom in each of the subfigures correspond to  $a/2\pi = 0, 1/4, 1/2, 3/4$  and  $1$ , respectively. For **a, b, d** and **e**, the OESs  $(4, 1) - (8, 1)$  are pumped, whereas for **c** and **f** the IESs  $(9, 9) - (13, 9)$  are pumped. Notice the ranges of the pumping frequency are chosen in a mirror form for the upper and lower subfigures. The other parameters are the same as those in Figures 3 and 4.

monotonically from 0 to  $2\pi$ , an integer number of ESMs will be transferred with the spectrum of the lattice returned to its original form. Such an integer is exactly the winding number of the ESMs. In Figure 5a,d, the dependence of  $n_{MP}$  on  $k_p$  and  $\alpha$  is numerically calculated for  $p/q = 1/4$ . Guided by the dashed white lines and the solid yellow arrows, the ESM peaks in the 1st and 3rd gaps move by one with the opposite moving directions, in agreement with the relation  $t_1 = -t_3 = 1$ . This scheme is in principle general to measure integer topological invariants with any value: The peaks of the ESMs in the  $h$ th gap move by  $|t_h|$  during the variation of  $\alpha$ , with the moving direction indicating the sign of  $t_h$ , and the Chern numbers  $C_h$  can be calculated from Equation (13) after all  $t_h$  are obtained. The situation  $\phi/2\pi = 1/5$  is also shown in Figure 5b,e, where the movements in the 2nd and 3rd gaps cross two peaks, indicating  $t_2 = -t_3 = 2$ . In addition, the opposite moving directions can also be observed in the pumping of the inner ESMs and the outer ESMs in the same gap (Figure 5c,f).

## DISCUSSION

### Robustness against imperfection factors

The imperfection in realistic experiments accompanies inescapably with the ideal scheme proposed above, including the diagonal and off-diagonal disorders  $\omega_r \rightarrow \omega_r + \delta\omega_r$ ,  $\mathcal{T} \rightarrow \mathcal{T} + \delta\mathcal{T}_{rr}$  added into  $\mathcal{H}_T$  by the fabrication errors of the circuit and the low-frequency  $1/f$  background noises,<sup>37</sup> and the non-nearest-neighbour hoppings induced the residual long-range coupling between TLRs, which are not presented in the proposed  $\mathcal{H}_T$ . Understanding these effects is thus crucial for our scheme. As briefly summarised below and as studied in detail in Supplementary Information, these imperfection factors lead to unwanted terms, which are all much smaller than the band gaps of the lattice ( $\sim \mathcal{T}$ ). Moreover, some of the resulted effects can be further suppressed by the slight refinement of the proposed scheme. Such small fluctuations cannot destroy the topological properties of the ESMs, because they are not strong enough to close and reopen the band gaps. Therefore, the presence of these imperfections can only renormalise negligibly the predicted results. From this point of view, our scheme enjoys the topological protection against the imperfection factors, which pinpoints its feasibility based on the current level of technology.

In particular, the fabrication error including the deviations of the realised circuit parameters from the ideal settings (e.g., the lengths and the unit capacitances or inductances of the TLRs) leads to the disorder of the eigenmodes' frequencies  $\delta\omega_r$ , whereas the long-range coupling induced by the finite inductances of the grounding SQUIDs results in the next-nearest-neighbour hoppings through the current-division mechanism.<sup>38,39</sup> As evaluated in Supplementary Information, these effects are both at the level of  $[10^{-2}, 10^{-1}]\mathcal{T}$ . In addition, these errors can be further corrected by simple revision of the proposed PFC scheme, including the refined choice of the TLRs' lengths and the corresponding renormalisation of the modulating frequencies of the grounding SQUIDs (Supplementary Information).

In actual experimental circuits, the  $1/f$  noise at low frequencies far exceeds the thermodynamic noise.<sup>37</sup> The  $1/f$  noise in superconducting quantum circuits can generally be traced back to the fluctuations of three degrees of freedom, namely the charge, the flux and the critical current. Because of its low-frequency property, we can treat the  $1/f$  noises as quasi-static—i.e., the noises do not vary during an experimental run, but vary between different runs. First, the proposed circuit is insensitive to the charge noise, as it consists of only TLRs, which are linear elements, and grounding SQUIDs, which have very small charging energies and very large effective Josephson coupling energies. Such insensitivity roots in the same origin of the charge insensitivity of transmon qubits<sup>40</sup> and the flux insensitivity of the

low decoherence flux qubit.<sup>41</sup> Second, the flux  $1/f$  noises that penetrated in the loops of the grounding SQUIDs shift the d.c. bias points of the grounding SQUIDs in a quasi-static way. The consequent effect is then the fluctuations

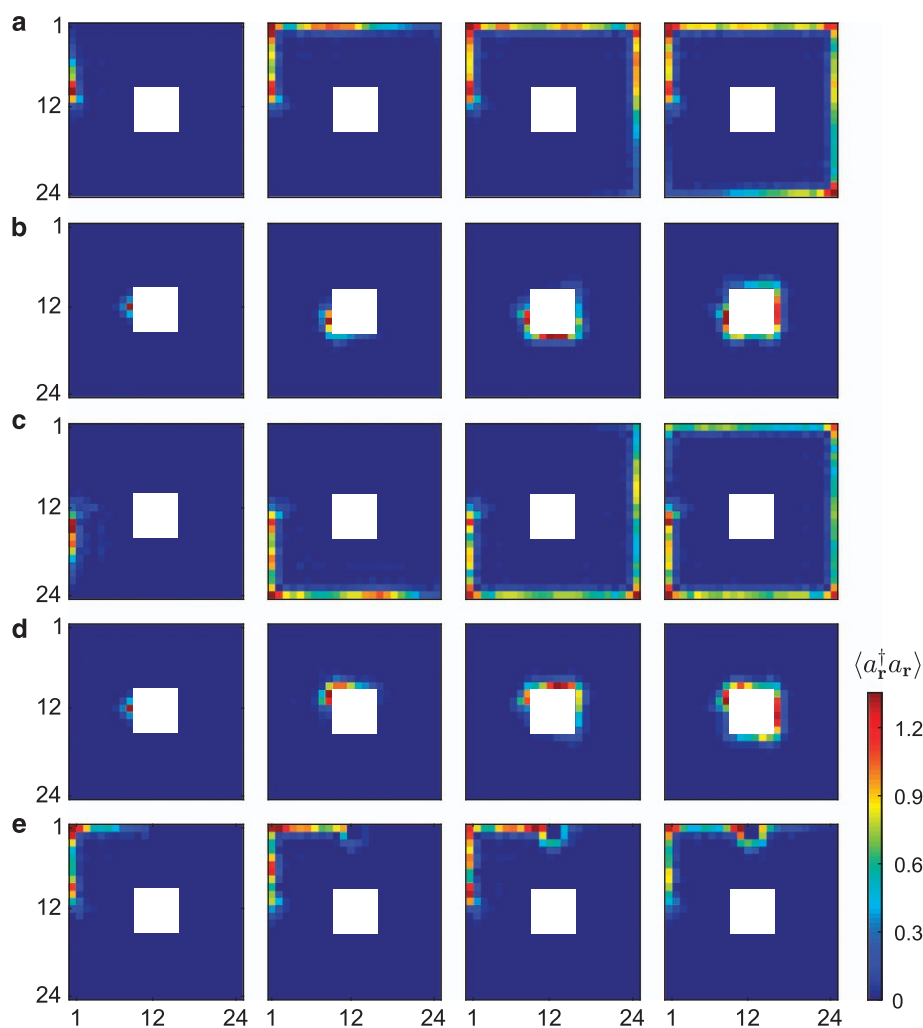
$$\delta\omega_r < 10^{-3}\mathcal{T}, \delta\mathcal{T}_{rr} < 10^{-4}\mathcal{T}, \quad (14)$$

where the detailed evaluation can be found in Supplementary Information. These flux-noise-induced diagonal and off-diagonal fluctuations are both much smaller than the band gaps and the spectral spacing between the ESM peaks ( $\sim 10^{-1}\mathcal{T}$ , see Figure 5). Such small fluctuations can thus neither destroy the topological properties of the ESM nor mix the resolution of the ESMs in the SSPN measurement. Therefore, we come to the conclusion that our scheme can survive in the presence of the  $1/f$  flux noise. The influence of the critical current noise is similarly analysed, with results indicating that the induced effects are even smaller than those of the flux  $1/f$  noises and can then be safely neglected (Supplementary Information).<sup>42</sup>

### Coherent chiral photon flow dynamics

A natural further step beyond the previous SSPN investigation is the study of coherent dynamics of the lattice, which is becoming experimentally possible because of the recent extension of the coherence times of superconducting circuits.<sup>7,8</sup> Such an investigation can offer an intuitive insight into the chiral property of the ESMs. We assume that the lattice is initially prepared in its ground state and then a driving  $s_p$  is added, with  $\mathbf{r}_p$  being an edge state and  $\Omega_{s_p}$  being the eigenfrequency of an ESM. The time evolution of the lattice is calculated with screenshots of the photon flow dynamics shown in Figure 6. The chirals of the ESMs result in the unidirectional photon flows around the edge with the directions determined by  $\Omega_{s_p}$  and  $\mathbf{r}_p$ . Chosen  $\mathbf{r}_p = (1, 13)$  as an OES, the photon flow is clockwise/counterclockwise if  $\Omega_{s_p}$  falls in the 1st/3rd gap (Figure 6a,c), indicating the opposite chirals of the ESMs in different gaps. The energy separation of ESMs with different chirals can be understood by regarding the unidirectional flow as a rotating spin.<sup>12</sup> Placed in an artificial magnetic field  $\mathbf{B}$ , such spin has split energies with one spinning direction lower and the opposite direction higher. Moreover, it is observed from Figure 6a,b (see also Figure 6c,d) that the chiral of the inner ESMs is opposite to that of the outer ESMs in the same gap. This oppositeness can be explained by the spatial configuration of the lattice. As shown in the upper panel of Figure 1c, by 'tearing' the lattice apart we can get a simply connected plane where there is only one edge existing. Now we consider the inverse: i.e., we 'glue' the two sides marked by dashed lines together, and obtain the ring geometry shown in Figure 1a and the lower panel of Figure 1c. During this gluing, the ESM flow (marked by the arrows) cancels itself on the glued sides, leaving two closed circulations with opposite directions. This 'tearing-and-gluing' process can also be tested in our lattice configuration because the horizontal hopping branches can be adiabatically tuned on and off. Another interesting observation is the different velocities of the photon flows between Figure 6a,c. Notice that the pumping frequencies in these two subfigures are not mirror to each other; such a difference in flowing velocity reflects the momentum difference between the corresponding ESMs.

The chiral flow in the presence of disorder and defect is also calculated. Here we assume that  $\delta\omega_r$  and  $\mathcal{T}_{rr}$  are normally distributed with  $\sigma(\delta\omega_r) = \sigma(\mathcal{T}_{rr}) = 0.05\mathcal{T}$  much larger than those estimated in Supplementary Information. In addition, a  $2 \times 2$  hindrance is placed on the upper outer edge with  $\delta\omega_{(12-13,23-24)}/\mathcal{T} = 30$ . As displayed in Figure 6e, the survival of the chiral photon flow under disorder and its circumvention around the hindrance clearly verify the topological robustness of the ESMs.



**Figure 6.** Chiral flow dynamics in the presence of single-site pumping. We set  $\mathbf{r}=(1, 13)$ ,  $\mathcal{P}_S/T=2$  for **(a,c,e)** and  $\mathbf{r}=(9, 13)$ ,  $\mathcal{P}_S/T=1.5$  for **(b and d)**. The synthesised magnetic field is set as  $\phi/2\pi=1/4$  and  $a/2\pi=0$ . In addition,  $\Omega_{SP}/T$  is chosen as  $[-1.76, -1.97, 1.47, 1.97, -1.75]$  for **a–e**, respectively. The times of the panels from left to right are arithmetic progressions with the first terms  $T_1/2\pi=[6, 1, 6, 1, 15]T^{-1}$  and the common differences  $\Delta T/2\pi=[13, 5, 13, 5, 2.5]T^{-1}$  for **(a–e)**, respectively. In particular, in the calculation of **(e)**, the effect of lattice disorder and defect is incorporated (see the main text). The other parameters are set the same as those used in Figure 3.

We remark that the proposed unidirectional photon flow can also be detected through the photon-number measurement of only few sites neighbouring to the pumping sites or the defect sites. We first establish the PFC process and pump the lattice with  $\mathcal{H}_{SP}$  for a duration less than the time scale during which the photon flow circulates around the whole edge loop and then remove them. The energy leaked out from the edge sites neighbouring to the pumping sites can be observed, which is proportional to the photon numbers stored in those TLRs. From Figure 6 we expect that the injected photons tend to flow towards the sites on a particular direction with phase delay, whereas it leaves the sites on the opposite direction negligibly excited.<sup>24,33</sup> The comparison of the measured photon numbers on the opposite directions thus reveals the chiral property of the ESMs. The circumvention of the photon flow around the hindrance can be detected in a similar manner by measuring the sites neighbouring to the hindrance sites.

#### Extension in the future

The flexibility of the proposed PFC method is not limited by the square lattice configuration in this paper. For instance, a brick wall lattice (i.e., a stretched honeycomb lattice) can be obtained in a

straightforward manner by closing some of the vertical hopping branches in the original square lattice. This generalisation may pave an alternative way to the study of photonic graphene.<sup>43</sup> In addition, although the cross talk between the diagonal next-nearest-neighbour TLRs is suppressed in our scheme because of frequency mismatch, it can indeed be opened by adding another tone to the modulating pulses of the SQUIDs. This may offer potential facilities in the future study of anomalous quantum Hall effect in the checkerboard lattice configuration.<sup>25</sup>

In recent research, lattice configurations supporting a dispersionless flat band have been investigated extensively, including the Lieb and the Kagomé lattices.<sup>25</sup> These band structures provide an idea platform of achieving strongly correlated phases as the kinetic energy is quenched.<sup>44,45</sup> These lattice configurations can also be synthesised by the variation of the proposed square lattice. While the photonic topological insulator considered in this paper can be understood in the single-particle picture, the introduction of interaction significantly complicates the problem and may lead to much richer but less explored physics. On the other hand, with the demonstrated strong coupling between TLRs and superconducting qubits<sup>7,8</sup> (and also atomic system, see ref. 46), the Bose–Hubbard<sup>16,18</sup> and Jaynes–Cummings–Hubbard nonlinearities<sup>17,19</sup> can be



incorporated into the proposed lattice. A further research direction should therefore be the implementation of photonic fractional Chern insulators and the understanding of strong correlation in the proposed architecture and its potential hybrid-system generalisations, which may utilise the advantages of different physical systems.<sup>46,47</sup>

## CONCLUSION

In conclusion, we have proposed a method of implementing topological photonics in a circuit QED lattice. The effective magnetic field for microwave photons can be synthesised through the proposed parametric approach, and the topological properties of the ESMs can be extracted from the steady states of the lattice under pumping. Moreover, being flexible to incorporate effective photon-photon interaction, our proposal may offer a new route towards the investigation of nonequilibrium photonic quantum Hall fluids in on-chip superconducting quantum circuits. Taking the advantage of simplicity in setup and topological robustness against potential imperfections, the realisation of this scheme is envisaged in the future experiments.

## MATERIALS AND METHODS

### Steady state of the lattice

We explicitly consider the pumping of the lattice described by  $\mathbf{P}^\dagger \mathbf{a} e^{-i\Omega t} + \text{h.c.}$  in the rotating frame of  $\mathcal{H}_S$ . Here  $\mathbf{P}$  and  $\mathbf{a}$  are the vectors composed of the pumping strengths and the annihilation operators of the lattice sites, respectively, and  $\Omega$  is the monochromatic detuning playing the role of Fermi surface. In the presence of dissipation, the evolution of the lattice is described by the master equation

$$\frac{d\rho}{dt} = -i[\mathbf{a}^\dagger(\mathcal{B} - \Omega\mathcal{I})\mathbf{a} + \mathbf{P}^\dagger\mathbf{a} + \mathbf{a}^\dagger\mathbf{P}, \rho] + \frac{1}{2}\sum_r \kappa_r (2a_r \rho a_r^\dagger - a_r^\dagger a_r \rho - \rho a_r^\dagger a_r), \quad (15)$$

where  $\rho$  is the density matrix of the lattice,  $\kappa_r$  is the decay rate of the  $r$ th TLR and the matrix  $\mathcal{B}$  is defined by  $\mathbf{a}^\dagger \mathbf{a} = \mathcal{H}_T$ . As a linear system (i.e., there is no photon-photon interaction), the lattice can be described in the picture of multi-mode coherent state, and its steady state can thus be determined by

$$i \frac{d\langle \mathbf{a} \rangle}{dt} = \left[ \mathcal{B} - \Omega\mathcal{I} - \frac{1}{2}i\mathcal{K} \right] \langle \mathbf{a} \rangle + \mathbf{P} = 0, \quad (16)$$

where  $\mathcal{K}$  is the diagonal matrix of the TLRs' decay rates. From Equation (16) our idea emerges that  $\Omega$  can be used to select the mode we are interested in, and the information of that mode can be extracted from the dependence of  $\langle \mathbf{a} \rangle$  on  $\mathbf{P}$ . In addition, as the pumping sites are coupled to external coil, they suffer more severe decoherence than the other conventional sites. Therefore, we set the decay rates of the pumping sites as uniformly  $\kappa_r/2\pi = 2$  MHz and those of the conventional sites as uniformly  $\kappa_r/2\pi = 100$  kHz throughout the numerical simulation of this paper—i.e., there is a 20-times difference between them.<sup>30–32</sup>

## ACKNOWLEDGEMENTS

We thank Z. D. Wang (HKU), M. Gong (CUHK), Z. Q. Yin (THU) and L. Y. Sun (THU) for helpful discussions. This work was supported in part by the National Fundamental Research Program of China (Grants No. 2012CB922103 and No. 2013CB921804), the National Science Foundation of China (Grants No. 11374117, No. 11574353 and No. 11375067) and the PCSIRT (Grant No. IRT1243).

## CONTRIBUTIONS

Y.H. and Z.Y.X. proposed the idea. Y.P.W. carried out all calculations under the guidance of Y.H. Z.Y.X., W.L.Y. and Y.W. participated in the discussions. Y.H., Y.P.W. and Z.Y.X. contributed to the interpretation of the work and the writing of the manuscript.

## COMPETING INTERESTS

The authors declare no conflict of interest.

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Supplementary Information accompanies the paper on the *npj Quantum Information* website (<http://www.nature.com/npjqi>)