The invention of dimension

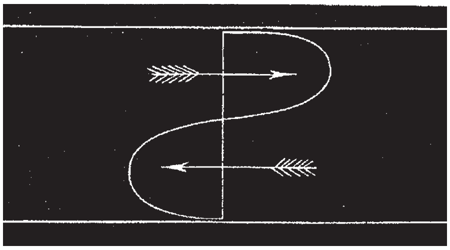
Assigning dimensions to physical quantities is not just for practicality. **Steven T. Bramwell** reflects on the deeper physical connotations of it all.

The term 'dimensions of a quantity' may conjure up a variety of images: perhaps a tool for checking equations or changing scale, a mysterious quality of nature, or a magical means of discovery. Of all the ideas of measurement theory, dimensions are perhaps the most valuable, yet equally the most elusive. What is behind their complex character?

The modern concept of dimension started in 1863 with Maxwell, who synthesized earlier formulations by Fourier, Weber and Gauss[1]. In doing so he added a nuance that we acknowledge today whenever we refer to the dimensions of, say, g ($\approx$ 9.81 m s^{-2}) as distance over time squared, rather than just the dimensional exponents (1, –2). By referring to the dimensions of a quantity, Maxwell seemed to imply that real things have natural dimensions. In the same spirit he designated units of mass, length and time as 'fundamental units'.

The consequence of Maxwell's choice was both inspiration and confusion. In the hands of virtuosos like Lord Rayleigh and Osborne Reynolds, dimensional analysis quickly became a powerful tool for discovery — identification of the Reynolds number, which describes the complexity of fluid flow, is a classic example. (The figure shows a flow pattern hand-drawn by Reynolds[2].) More generally, the method identifies relationships that are consistent with the laws of physics, involving only quantities that are relevant to a problem. Identifying these — even if only by inspired guesswork — can give huge savings of time in experiments and provide clear theoretical guidance[3,4].

After Maxwell, a sense emerged that new fundamental laws could be discovered by dimensional analysis. The product of vacuum permeability and permittivity (inversed and square-rooted) shares dimensions with speed and indeed turned out to be the speed of light. Bohr's atom was motivated by the fact that Planck's constant shares dimensions with angular



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momentum[1]. But in 1922 Bridgman insisted that dimensions are a matter of convention: human choices, like units[3]. They can't be used to discover new fundamental laws, only relations that derive from existing laws. In 1954 Maxwell's term fundamental units was replaced with 'base units'[1], just to lay down the law.

This operational conclusion about dimensions may seem, even today, just a little too bleak for physicists, who are keen to get to the truth about nature, not just to measure it. However, even if dimensions were demystified a century ago, physics was far from finished with them. The late twentieth century saw the creation of a firm theoretical basis for dimensional analysis: renormalization group methods, which now pervade many areas of theoretical physics, from particle physics to condensed matter and fluid turbulence. Starting with a microscopic model, one can 'integrate out' shorter length scales and determine how the influence of various coupling parameters evolves as one zooms out from the microscopic to the macroscopic scale. Such methods can justify the choices of which couplings and variables to choose in dimensional analysis: they are the ones that remain relevant at large scales.

But the renormalization group also enables the calculation of numbers. A good example is its ability to calculate so-called anomalous dimensions, echoing the concept of fractal geometry where, famously, a coastline's length depends 'anomalously' on the length of the measuring rod. Dimensional analysis shows[4] that anomalous dimensions are necessarily the exponents of dimensionless ratios of quantities, generated by the physics of the problem.

For example, at a critical point in a condensed-matter system (such as the gas–liquid or ferromagnetic critical point), correlations typically decay with distance to a power $d - 2 + \eta$, where d is the spatial dimensionality and η is the anomalous dimension. But here the 'distance' is in fact the ratio of physical distance l to a microscopic distance a (typically atomic size) — both length scales, and all those in between, remain relevant. One way[5] of introducing such a dimensionless ratio of dissimilar length scales is via a logarithmic integral, $\int_a^l (1/r)\mathrm{d}r = \log(l/a)$. This commonly occurs in two-dimensional systems, where the integral is related to the fundamental solution of the Laplace equation. This enables many two-dimensional systems — magnets, superfluids, crystals — to show anomalous dimensions (criticality) over a broad temperature range.

It is clear that dimensions have a life beyond the SI brochure. Over the years their stock has risen, fallen and risen again, but some of their mystery and magic has always endured. This is surely because our changing concept of dimensions reflects the evolution of physics itself — a subject that will always be concerned with the problems of how to scale, how to distinguish between relevant and irrelevant factors, and how to use mathematics to find the truth about nature. ❐

STEVEN T. BRAMWELL is at University College London, London Centre for Nanotechnology and Department of Physics and Astronomy, 17–19 Gordon Street, London WC1H 0AJ, UK.
e-mail: s.t.bramwell@ucl.ac.uk

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