NOTES AND COMMENTS

A NOTE ON THE INITIAL PROGRESS OF NEW GENES

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Received 27.iv.63

Bodmer and Parsons (1960) have investigated the initial progress of new genes with various genetic systems. They inquire under what conditions can a new gene, arising in a population by mutation or migration, increase in frequency and become established in that population. Among other things they find that, for random mating, in the situation in which initially there is only one gene present at a given locus in the population, and a new allele of that gene is introduced, the new gene will become established in the population if and only if the viability of the heterozygote which it forms with the original wild allele exceeds the viability of the original wild homozygote. Furthermore, again with random mating, they obtain conditions under which a third allele, C say, of two allelic genes A and B which are already at stable equilibrium in the population, can increase in frequency. These conditions might be expressed roughly as follows: the newly arising heterozygotes AC and BC must between them possess a certain selective advantage over the already established genotypes AA, AB and BB. Thus, for example, suppose that AC is at a selective disadvantage with respect to all three of the already existing genotypes, then C can still establish itself in the population provided that the disadvantage of AC is offset by a sufficiently large selective advantage of the other heterozygote BC over AA, AB and BB.

The purpose of this note is to show that these results can be derived from general properties of random mating systems, in a form somewhat different to that given by Bodmer and Parsons, and to suggest that this form is perhaps of some interest from the point of view of elucidating general principles.

It is convenient to work in terms of a system of n alleles, because one can then appeal to the notation and results of Owen (1954) and Mandel (1959). Thus we consider alleles $A_1, A_2, \dots, A_n$ which we suppose are initially in stable equilibrium in the population, and that subsequently a new allele A_{n+1} is introduced. Let $a_{ij} = a_{ji} \geq 0$ ($i, j = 1, 2, \dots, n$) denote the relative viability of the genotype $A_i A_j$; these viability parameters form a symmetric, non-negative matrix (a_{ij}) , whose determinant will be denoted by Δ . Let $a_{n+1, j}$ ($j = 1, 2, \dots, n+1$) denote the relative viabilities of the newly arising genotypes $A_{n+1} A_j$ when A_{n+1} is introduced into the population. Further, let p_i ($i = 1, 2, \dots, n+1$) be the frequency of allele A_i in a given generation of interest, with $0 \leq p_i \leq 1$ for each i , and p'_i ($i = 1, 2, \dots, n+1$) that in the following generation, with $0 \leq p'_i \leq 1$ for each i . Then, with random mating, the two sets of gene frequencies p_i and p'_i are connected by the relations

$$V p'_i = p_i \sum_{j=1}^{n+1} a_{ij} p_j \quad (i = 1, 2, \dots, n+1) \quad (1)$$

where

$$V = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij} p_i p_j$$

Now let

$$V' = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij} p_i' p_j'$$

Then it has been shown by a number of writers (Mulholland and Smith, 1959; Scheuer and Mandel, 1959; Atkinson, Watterson and Moran, 1960; and Kingman, 1961) that, under the above conditions, $V' \geq V$ with equality if and only if the population is in equilibrium.

Suppose then that we consider the population in the state $p_i = P_i + x_i$ ($i = 1, 2, \dots, n+1$), where the P_i ($i = 1, 2, \dots, n$) are the equilibrium gene frequencies for the system of n alleles, $P_{n+1} = 0$, and the x_i ($i = 1, 2, \dots, n$), $x_{n+1} > 0$ denote small displacements from the equilibrium state for the n alleles, involving the introduction of a small proportion of the new gene A_{n+1} .

Then we have

$$\sum_{i=1}^{n+1} x_i = 0.$$

Obviously, in view of the above remarks, the new gene will persist in the population if and only if

$$\sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij} (P_i + x_i) (P_j + x_j) > \sum_{i=1}^n \sum_{j=1}^n a_{ij} P_i P_j \quad (3)$$

and, neglecting second-order terms in the x_i , this condition reduces to

$$\sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij} x_i P_j > 0$$

which is

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i P_i + x_{n+1} \sum_{j=1}^n a_{n+1, j} P_j > 0. \quad (4)$$

If we write now

$$V_e = \sum_{i=1}^n \sum_{j=1}^n a_{ij} P_i P_j$$

then clearly

$$V_e = \sum_{j=1}^n a_{ij} P_j, \text{ all } i = 1, 2, \dots, n$$

and hence

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i P_j = V_e \sum_{i=1}^n x_i = -x_{n+1} V_e.$$

Thus the necessary and sufficient condition for the new gene A_{n+1} to persist in the population can be written

$$x_{n+1} \left(\sum_{j=1}^n a_{n+1, j} P_j - V_e \right) > 0 \quad (5)$$

or, since $x_{n+1} > 0$,

$$\sum_{j=1}^n a_{n+1, j} P_j > V_e \quad (6)$$

or

$$\sum_{j=1}^n (a_{n+1, j} - a_{ij}) P_j > 0, \text{ for any } i = 1, 2, \dots, n. \quad (7)$$

Bodmer and Parsons (1960) consider the cases $n = 1$ and $n = 2$ respectively. For their model with $n = 1$, $a_{ij} = b$ and $a_{n+1, j} = h$ so that the condition $h > b$ follows immediately from (7). For $n = 2$ and with the viability matrix as given by Bodmer and Parsons (1960, p. 285) we have from (7) taking $i = 1$

$$\frac{(g-a)(h-b) + (f-h)(h-a)}{(h-a) + (h-b)} > 0$$

or, taking $i = 2$

$$\frac{(g-h)(h-b) + (f-b)(h-a)}{(h-a) + (h-b)} > 0.$$

In either of these inequalities the denominator is positive, $h > a$, b being the stability conditions for the two allele system (Fisher, 1922), while the numerator in each case is $G + F + C$, so that the condition (2.3) of Bodmer and Parsons (1960, p. 286) follows at once.

In attempting to formulate some general principles, one might note that the quantity V_e is a weighted average of the viability parameters for the already established genotypes, the weights being the zygotic genotype frequencies at equilibrium; thus the condition for the new gene to increase, in the form (6), may be expressed as follows: the weighted average of the viabilities of the newly formed heterozygotes (the weights being the equilibrium gene frequencies P_j) must exceed the weighted average of the viabilities of the already existing genotypes (with weights $P_i P_j$). This has already been noted, for the case $n = 2$, by Haldane (1957). That is to say, the newly formed heterozygotes must have some selective advantage over the old genotypes. Moreover, it can be shown (Mandel, 1959) that $V_e > a_{ii}$ for all $i = 1, 2, \dots, n$ and thus it follows from (6) that a necessary (but not sufficient) condition for the new gene to increase is that the viability of at least one of the newly formed heterozygotes must be greater than that of each of the already existing homozygotes. This appears to be as far as one can go towards the formulation of general principles, since sufficient conditions will be based on values of the P_i , and these in turn depend upon the relative magnitudes of the a_{ij} ($i, j = 1, 2, \dots, n$).

One might note here, incidentally, that the statement of Bodmer and Parsons (1960, p. 286) "... if $h > f, g$ we must have both f and $g > a, b$ for C to increase ..." appears to be incorrect, as can be seen by means of a variety of examples; for instance, if we take $a = 0.02, b = 0.60, f = 0.80, g = 0.58$ and $h = 1.00$ for which $h > f, g$ and $g < b$ and $G + F + C = 0.028 > 0$ so that C will in fact increase.

Finally, one might ask the following question: under what conditions can the frequency of a newly arising gene increase in the population when the already existing alleles are not necessarily in a state of stable equilibrium, but are themselves undergoing changes in gene frequency? Under these circumstances equation (4) takes the form

$$V \sum_{i=1}^n \frac{x_i p_i'}{p_i} + x_{n+1} \sum_{j=1}^n a_{n+1, j} p_j > 0 \quad (8)$$

and this reduces to

$$\sum_{i=1}^n x_i \sum_{j=1}^n (a_{ij} - a_{n+1, j}) p_j > 0. \quad (9)$$

Clearly the left hand members of these conditions are now no longer independent of the displacements x_i , and the question of whether the new allele will increase now depends upon the nature of the disturbance which is created among the old alleles by the introduction of the new gene. For instance, if the new gene is introduced into the population at the expense of allele A_1 only, while the frequencies of $A_2, A_3, \dots, A_n$ remain unchanged, then condition (9) reduces to

$$\sum_{j=1}^n (a_{1j} - a_{n+1,j}) p_j < 0$$

and this condition may clearly be interpreted in a very simple fashion.

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CYTOLOGICAL STUDIES IN THE GRAMINEÆ (II)

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This second list of chromosome numbers continues that published earlier by Singh and Godward (1960), which included data utilised by Clayton (1962) in support of the separation of the genera *Tetrapogon* and *Chloris*. It is accordingly numbered table 2 to follow table 1 of the previous paper.