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Objective evaluation of refractive data and astigmatism: quantification and analysis

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Abstract

The aim of this study was to present methods to improve the analysis of refractive data. A comparison of methods is used to analyse refractive powers using individual powers and aggregate data. Equations are also developed for the representation of the average power of a lens or refractive data as a univariate measure, which includes spherical, coma, and/or other aberrations. The equations provide a precise representation of refractive power, which is useful for comparing individual and aggregate data. Average lens power in the principal meridian can be adequately computed as can the average lens power through orthogonal and oblique meridians, providing a good univariate representation of astigmatism and refractive power. Although these formulae are perhaps not as easy to use as, for example, the spherical equivalent, they are more precise and superior in principle involving fewer approximations and are not subject to systematic bias. These effects are of significance when dealing with high-powered lenses such as intraocular lenses or the cornea. They need to be taken into account particularly for calculations of intraocular lens power, toric intraocular lenses, and cornea refractive surgery, especially if outcomes are to be improved. Such issues are of particular importance when dealing with aggregate data and determining statistical significance of treatment effects.

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Introduction

There are many clinical situations where a measure of refractive outcome is needed to determine the effectiveness of a given treatment or intervention. A univariate measure is usually sought, particularly when comparing population samples or testing for an association between a characteristic feature of a population and a treatment effect. It is important, therefore, that such a measure is not subject to bias or systematic error, as this may lead to erroneous assumptions regarding a treatment effect. Similarly, in the analysis of refractive data, there is a natural tendency to want to view refractive powers as a univariate measure, that is, as a single number. The easiest and most convenient form is a scalar number. This underpins the popularity of using the 'spherical equivalent' often referred to as the 'mean spherical equivalent' (MSE) or more correctly 'the nearest equivalent sphere' (NES)[1] as such a measure. If, however, a univariate measure is to be used, one needs to have confidence that any approximations that are made do not discard useful information and more importantly, that the measure does not introduce a systematic bias or a non-constant error. For example, although the spherical equivalent is popular, it unfortunately loses information and introduces a non-constant error, which may affect the statistical analysis and interpretation of results.[2,3] In addition, treating the components of refractive powers as independent values, leads to similar issues. For example, treating the 'cylinder' independently of the spherical component whether as a vector or scalar measure may result in misleading results.[4–6] To understand these issues, it is necessary to review the arithmetic of treating paraxial refractive powers, how the spherical equivalent is derived, and whether there are more robust methods of analysis. Furthermore, optical aberrations are important components of refractive powers and it is therefore necessary to consider and include, where possible, measures of these aberrations in the analysis of refractive data.

Adding refractive data

Refractive data are conventionally expressed as sphere/cylinder$_x$ axis, from which it is apparent that there are three components that are co-dependent.[4–11] Despite this, there is a tendency to treat each component independently. For example, consider the following two refractive powers, $+2/+2_{x90}$ and $+1/+1_{180}$. If they were to be added together what would be the result? There are three possibilities depending on whether each component is treated independently or dependently. If they are treated independently as scalar values this leads to the following situation,

$$\begin{vmatrix} Sphere \\ 2 \\ 1 \\ \overline{3} \end{vmatrix} + \begin{vmatrix} Cylinder \\ 2 \\ 1 \\ \overline{3} \end{vmatrix} +$$

or $(+2/+2_{90})+(+1/+1_{180}) \neq +3/+3$, which is incorrect. If they are treated independently as vectors, this leads to

$$\begin{vmatrix} Sphere \\ 2 \\ 1 \\ \overline{3} \end{vmatrix} + \begin{vmatrix} Cylinder \\ 2_{90} \\ 1_{180} \\ \overline{1_{90}}^{1} \end{vmatrix} +$$

or $(+2/+2_{90})+(+1/+1_{180}) \neq +3/+1_{90}$, which again is incorrect. Furthermore, treating the sphere and cylinder independently leads to a particular problem under transposition. If, however, they are treated dependently, then

$$\begin{vmatrix} Sphere & Cylinder \\ 2^1 & 2_{90} \\ 1 & 1_{180} \\ \overline{4} & \overline{1_{90}}^{1} \end{vmatrix} +$$

or $(+2/+2_{90})+(+1/+1_{180}) = +4/+1_{90}$, which is the correct (paraxial) result. In addition, as evident in Table 1, if treated dependently the same result is obtained under transposition as would be expected, whereas different results are obtained if the data are treated independently.

It is evident that the space of cylinders is not closed under addition or subtraction, that is, a cylinder plus a cylinder does not necessarily equal a cylinder. Clearly, in any analysis of changes in refractive powers, the components need to be treated as co-variates.[4–11]

Calculating refractive change

Changes or summations of refractive powers are self-evident when there is no change or rotation of axes. More commonly, however, there is an axis rotation. It is therefore, necessary to be able to refer to a standard system. This method was principally developed by Long.[12] Consider for example, a postoperative refractive result of $+1/+2_{x75}$. If one had an intended or target refraction of $0/+0.5_{x150}$ then what is the difference between the intended and actual outcomes? This can be achieved using Long's method as follows:

$$S/C_A = \begin{matrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{matrix}$$

where, in the 2×2 matrix, the cell in the first row and first column is denoted by f_{11} and the cell in the first row and second column is denoted by f_{12} and so on. Long[12] showed that refractive data can be transformed into four independent components given by, $f_{11} = S + C\sin^2 A$, $f_{12} = -C\sin A\cos A$, $f_{21} = -C\sin A\cos A$, and $f_{22} = S + C\cos^2 A$. Therefore,

$$S/C_A = \begin{matrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{matrix} = \begin{matrix} S + C\sin^2 A & -C\sin A\cos A \\ -C\sin A\cos A & S + C\cos^2 A \end{matrix}.$$

For a thin lens, $f_{12} = f_{21}$ so that,

$$S/C_A = [f_{11} \quad f_{12} \quad f_{22}] = [S + C\sin^2 A \quad -C\sin A\cos A \quad S + C\cos^2 A].$$

Returning to the example, the postoperative refractive error of $+1/+2_{x75}$ and target of $0/+0.5_{x150}$ can then be transformed to,

$$\begin{aligned} +1/+2_{75} &= [1 + 2\sin^2{}_{75} - 2\sin_{75}\cos_{75} \quad 1 + 2\cos^2{}_{75}] \\ &= 2.87 \quad -0.50 \quad 1.13 \end{aligned}$$

$$\begin{aligned} 0/+0.5_{150} &= [0 + 0.5\sin^2{}_{150} - 0.5\sin_{150}\cos_{150} \quad 0 + 0.5\cos^2{}_{150}] \\ &= 0.13 \quad 0.22 \quad 0.38. \end{aligned}$$

Thus, to calculate the difference, we have

$$\begin{aligned} &[f_{11}\ f_{12}\ f_{22}]_{\text{Postop}} - [f_{11}\ f_{12}\ f_{22}]_{\text{Target}} \\ &= [0.13 - 2.87 = -2.74] \quad [0.22 - (-0.50) = 0.72] \\ &\quad [0.38 - 1.13 = -0.76] = [-2.74 \quad 0.72 \quad -0.76]. \end{aligned}$$

This difference is then back transposed into S/C_A using Keating's or other methods,[4–6,9–10,13] yielding $-2.97/+2.45_{162}$. The usefulness of this method is that, it can be applied to large data sets to provide means, standard deviations, confidence intervals, and statistical tests. For example, if the following are pre- and postoperative refractive results from five subjects (Table 2). Using Long's formalism[7] and the methods of Harris,[4–6,14,15] and Kaye and Harris[10] allows one to calculate the total, mean, SD, upper (UCI) and lower (LCI) confidence intervals, and SEM. In this particular example, the mean difference of $-0.44/+1.59_{146}$ (SD $4.91/1.61_{115}$) between the pre- and postoperative data is significant $P=0.04$ (see method of Harris for statistical analysis[14]).

Plotting refractive powers

Refractive data plots are preferably shown on 3D axes. Harris developed an excellent method for representing data on standard axes.[16] Although there are other methods,[17] one of the attractions of this method, is that the length of a power vector of 1D is that of a cylinder of 1D. With this in mind it allows one to visualise and interpret the distance of a point from the origin. The Euclidian distance $\|a\|$ between a point and the origin on a 2D plot, such as Y and X, is, $\|a\|=\sqrt{x^2+y^2}$. For a 3D plot, the distance of a point from the origin is $\|a\|=\sqrt{x^2+y^2+z^2}$. The distance of a point in $f_{11}\quad f_{12}\quad f_{22}$ from the origin is then, $\|a\|=\sqrt{f_{11}{}^2+f_{12}{}^2+f_{22}{}^2}$. This is fine for a single point, however, because the axis often changes between different refractive powers, the plotting of each point then requires a rotation of axes. Harris showed that it is possible to provide a solution to this by multiplying f_{12} by $\sqrt{2}$.[16] This then gives, $h_1=f_{11}$, $h_2=\sqrt{2}\,f_{12}$, $h_3=f_{22}$. The distance of a refractive power point from the origin (Figure 1) is then $\|h\|=\sqrt{f_{11}{}^2+2f_{12}{}^2+f_{22}{}^2}$, which if substituted by the terms of S/C_A becomes $\|h\|=\sqrt{S^2+(S+C)^2}$ and it is evident that there is no need to rotate the axes.[16] If one considers an example of say a preoperative refractive power of $0/+1_{90}\cdot[1\quad 0\quad 0]$, a postoperative of $0/+1_{180}\,[0\quad 0\quad 1]$, the difference being $-1/+2_{180}\,[-1\quad 0\quad 1]$, then a plot of these points, as shown in Figure 2, would be $h_{\mathrm{Preop}}=[1\quad 0\quad 0]$, $h_{\mathrm{Difference}}=[-1\quad 0\quad 1]$, and $h_3=[0\quad 0\quad 1]$. The length of each of these powers from the origin is, $\|h\|_{\mathrm{Preop}}=\sqrt{1^2+0^2+0^2}=1\mathrm{D}$, $\|h\|_{\mathrm{Postop}}=\sqrt{0^2+0^2+1^2}=1\mathrm{D}$, $\|h\|_{\mathrm{Difference}}=\sqrt{-1^2+0^2+1^2}=\sqrt{2}\approx 1.41\mathrm{D}$.

Table 1 Adding refractive powers

	Sphere (dioptres)	*Cylinder (dioptres)*	*Axis (degrees)*	*NES (dioptres)*
	+3	+3	90	+4.50
	−3	−4	90	−5.00
Sum	0	−1	90	−0.50
Transposition to a +ve cylinder				
	+3	+3	90	+4.50
	−7	+4	180	−5.00
Dependent sum	−1	+1	180	−0.50
Independent sum (vector)	−4	+1	180	−3.50
Independent sum (scalar)	−4	+7		−0.50

Abbreviation: NES, nearest equivalent sphere.[1]

Astigmatism

In the analysis of refractive power, a commonly used term is astigmatism. Astigmatism refers to the absence of stigmatism (a point focus). Aberrations therefore are types of astigmatism. There are three main types of non-chromatic aberrations: spherical, off axis (coma), and oblique. Most higher orders of aberration are combinations of these main aberrations. Aberrations are important in the analysis of refractive power and any univariate measure needs to take them into account. If, however, one considers only paraxial powers, then astigmatism is limited to that found within a cylinder. A cylinder, however, is not the same as astigmatism. Indeed, if one accepts the concept of there being a

Table 2 Preoperative, postoperative, and the difference between pre- and postoperative refractive data

Subject	S/C_A	f_{11}	f_{12}	f_{22}	S/C_A	f_{11}	f_{12}	f_{22}	S/C_A	f_{11}	f_{12}	f_{22}
1	$-2/1_{13}$	−1.95	−0.22	−1.05	$7.00/-4.00_{11}$	6.86	0.73	3.14	$2/3_{10}0$	4.91	0.51	2.09
2	$0.5/3_{47}$	2.10	−1.50	1.90	$-2.00/-3.49_{34}$	−3.12	1.63	−4.38	$-1/-1.5_{5}$	−1.01	0.13	−2.49
3	$-2/5_{90}$	3.00	0.00	−2.00	$-0.01/4.02_{179}$	−0.01	0.09	4.01	$3/-1_{5}$	2.99	0.09	2.01
4	$1/3_{67}$	3.54	−1.08	1.46	$-8.83/3.65_{164}$	−8.53	0.99	−5.47	$-5/1_{5}$	−4.99	−0.09	−4.01
5	$-4/-1_{165}$	−4.07	−0.25	−4.93	$5.00/1.00_{165}$	5.07	0.25	5.93	$1/0_{175}$	1.00	0.00	1.00
Mean	$-1.15/1.90_{70}$	0.53	−0.61	−0.93	$-0.44/1.59_{146}$	0.05	0.74	0.65	$-0.30/0.90_{98}$	0.58	0.13	−0.28
SD	$2.36/1.41_{123}$	3.35	0.64	2.78	$4.91/1.61_{115}$	6.23	0.62	5.20	$2.75/1.12_{102}$	3.82	0.23	2.80
Total	$-5.74/9.48_{70}$	2.63	−3.04	−4.63	$-2.22/7.95_{146}$	0.27	3.69	3.23	$-1.48/4.48_{98}$	2.90	0.64	−1.40
UCI	$4.36/2.89_{103}$	7.10	0.65	4.52	$9.48/4.14_{125}$	12.26	1.94	10.84	$5.09/3.09_{101}$	8.06	0.58	5.20
LCI	$-8.09/3.76_{48}$	−6.04	−1.87	−6.37	$-12.24/2.78_{10}$	−12.16	−0.47	−9.54	$-6.99/1.32_{15}$	−6.90	−0.32	−5.76
SEM	$0.24/-0.09_{33}$	0.22	0.04	0.18	$0.42/-0.10_{25}$	0.40	0.04	0.34	$0.25/-0.07_{12}$	0.25	0.01	0.18

Data are presented as dioptres (apart from axis in degrees) in spherocylinder notation (S/C_A), sphere (S), cylinder (C), axis (A), and in the corresponding Long's formalism f_{11}, f_{12}, and f_{22}. Mean, SD, upper (UCI) and lower (LCI) 95% confidence intervals, and standard error of the mean (SEM).

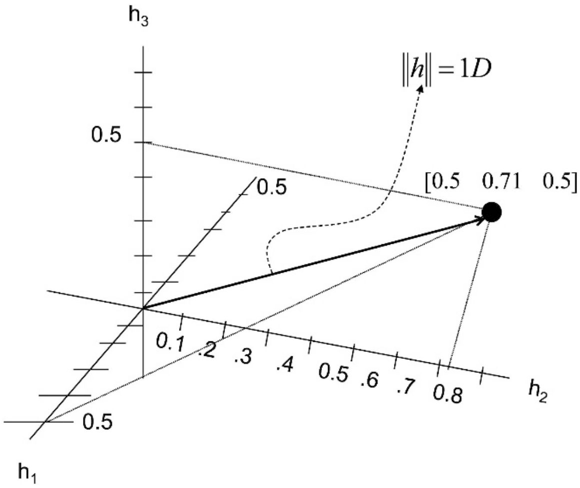

Figure 1 Representation of the components of refractive powers. Data is transformed from S/C_A notation into Long's formalism f_{11}, f_{12}, f_{22} and the power vectors of Harris, where $h_1 = f_{11}$, $h_2 = \sqrt{2}\, f_{12}$, $h_3 = f_{22}$. Units are in dioptres. One point is plotted and shown to have an Euclidian length of 1D.

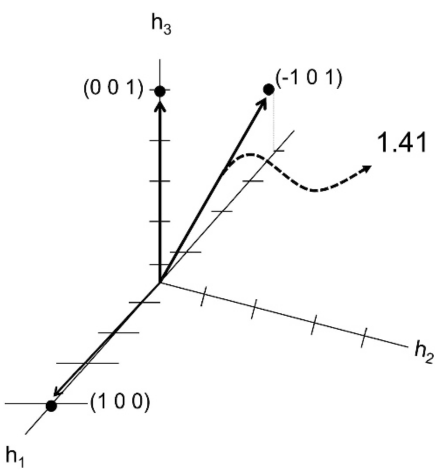

Figure 2 Change in refractive power. Representation (dioptres) of preoperative (1D), postoperative (1D), and the respective difference (surgically induced refractive effect) of 1.41D.

spherical equivalent, then it follows that a cylinder has a spherical component. Accepting for the moment the errors introduced by the NES, it is then possible to subtract the spherical component represented by the NES away from a cylinder, leaving the astigmatic component.[1] If one considers the refractive powers in Figure 3, it is evident that subtraction of the NES, results in a Jackson crossed cylinder (JCC). Consider a refractive power of S/C_A. It follows that $S/C_A = \left[S + \frac{C}{2}\right] + \left[S/C_A - \left(S + \frac{C}{2}\right)\right]$, which simplifies to $S/C_A = \left[S + \frac{C}{2}\right] + \left[-\frac{C}{2}/C_A\right]$, that is, $S/C_A = NES + JCC$. Thus if one applies this to a cylinder, we have, $C_A = \frac{C}{2} - \frac{C}{2}/C_A$ or a cylinder of $+1_{90}$ yields a NES of $+0.5$ and a JCC of $-0.5/+1_{90}$. What is important, however, is that the space of spheres and that of JCCs are closed under the arithmetic operations of addition and subtraction. The separation of the paraxial stigmatic and astigmatic components is also evident in the method of Thibos *et al*, in which the spherical component (M) is $M = S + C/2$ and the astigmatic components are J_0 and J_{45}.[17] Although this is a reasonable determination of paraxial astigmatism, a better approximation, which is not subject to a systematic error introduced by the spherical equivalent, is to use the equations below for average lens power denoted by $\langle F \rangle$ as described below equation 7.[2] This then gives for a cylinder $0/C_A = \langle C \rangle + \langle -C \rangle / C_A$ with $\langle -C \rangle / C_A$ the astigmatic component.

The paraxial approximations: errors

It is essential to appreciate the errors introduced by the paraxial approximations. Consider a section through the principal meridian of a lens cylinder (Figure 4) of sag height S, half cord length Y and radius R. Orthogonal and non-orthogonal oblique sections at angles θ and γ away from the principal meridian, result in elliptical sections with major radii (semi-diameters) $R \sec \theta$ and $R \sec \gamma$, respectively. The focal length (f) and back vertex power (F) of a lens of refractive index n in a meridian can be determined as follows. A ray parallel to the axis of the lens subtends at z, an angle of incidence α, with the normal to the lens surface (Figure 5). The angle of refraction β, that is, $\sin^{-1}\left[\frac{\sin\alpha}{n}\right]$, subtends an angle $(\alpha - \beta)$ with the axis of the lens. The power of a lens for a section through the principal meridian (F_P), where γ and θ are both zero, is

$$F(\alpha) = F_P = \frac{n}{R} \frac{\tan(\alpha - \beta)}{\sin\alpha - \tan(\alpha - \beta)\{\cos\alpha - 1\}} \qquad [1]$$

and

$$\tan\alpha_{\mathrm{Max}} = \frac{Y}{\sqrt{R^2 - Y^2}}.^{2,3}$$

For the treatment of all sections, principal, orthogonal and non-orthogonal oblique, the reader is referred to references therein.[2,3] For the power of a lens in the principal meridian, (θ and γ both zero), and for very small angles of incidence, that is, paraxial rays or rays close to the optic axis, $\tan\alpha \approx \alpha$, $\sin\alpha \approx 0$, $\cos\alpha \approx 1$, F can be approximated by $F \approx \frac{2Sn}{S^2 + Y^2} \frac{\alpha - \beta}{\alpha}$. In this paraxial case, $\beta \sim \alpha n_1/n$ and this reduces to the first approximation (AP1) or (F_{AP1}),

$$F_{\mathrm{AP1}} = \frac{2S(n - n_1)}{S^2 + Y^2}. \qquad [2]$$

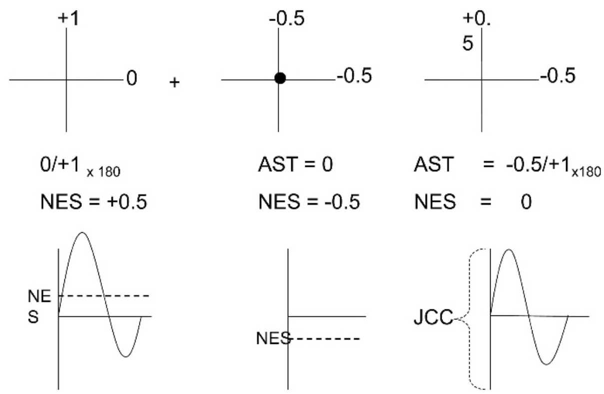

Figure 3 Paraxial astigmatism. Subtraction of the spherical component represented by the nearest equivalent sphere (NES) away from a cylinder, leaving the astigmatic component, the Jackson cross cylinder (JCC). Data in first row as a power cross. Units in dioptres.

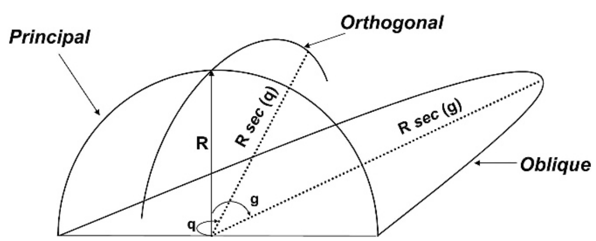

Figure 4 Light ray refracted through a section of a thin lens. Radius (R), angles of incidence (α), emergence or refraction (β), and back vertex focal length (f).

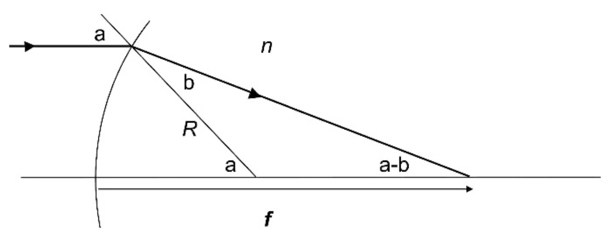

Figure 5 Section through the meridian of a spherical lens. Radius (R), sag height (S), and half cord length (Y). As the lens power increases, R reduces and for an equivalent cord length or segment size, S increases relative to Y. $R = \frac{Y^2+S^2}{2S}$.

Furthermore, if the S^2 term is ignored (which has been the current clinical practice), then a further or second (AP2) approximation F_{AP2} is introduced, given by

$$F_{AP2} = \frac{2S(n-n_1)}{Y^2}. \qquad [3]$$

Thus, two approximations have been introduced and it is equation 3, F_{AP2}, that is used to derive the formula for the spherical equivalent of a lens, as will be shown.[2] Note, the first approximation, F_{AP1} is useful for providing a reasonable approximation for power in non-principal meridians without being subject to the systematic bias that occurs with the second approximation (F_{AP2}).[2,3] The potential error of the F_{AP2} approximation was highlighted by Pascal in his paper on the power of cylinders in oblique meridians.[18]

Orthogonal meridians

Consider the second approximation $F_{AP2}(\theta)$. For a cylindrical lens and for sections off the principal meridian, the second approximation $F_{AP2}(\theta)$ then becomes,

$$F_{AP2}(\theta) = \frac{2S(n_2-1)\cos^2\theta}{Y^2}. \qquad [4]$$

This equation is often written approximated as

$$F_{AP2}(\theta) = \frac{(n_2-1)\cos^2\theta}{R}. \qquad [5]$$

Once we accept F_{AP2} as an approximation for the power of a lens at angle θ, the mean or average power of a lens can be determined by integratingF_{AP2} over θ. Using the definition of the average of a function, the orthogonal meridional average of F_{AP2} that is, $\langle F_{AP2}\rangle$ is given by,

$$\langle F_{A2}\rangle = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{2S(n_2-1)}{Y^2}\cos^2\theta \, d\theta = \frac{S(n_2-1)}{Y^2} = \frac{1}{2}\frac{(n_2-1)}{R}. \qquad [6]^{2,3}$$

Thus, the average approximated power of a cylindrical lens is 'half' of the lens power in its principal meridian (10). This then is the justification of taking the spherical equivalent of a spherocylinder to be 'sphere' 0.5C, or half of the cylinder power combined with the spherical part of the refractive power. Equation [12] for average lens power has a systematic bias brought about by the exclusion of the square of the sag or S^2 term (Figure 6). In essence, although the contribution of the S^2 term is small, its effect is dependent on the lens parameters and power of the lens and importantly, is therefore, not constant (Figure 6). A particular consequence of this is that for aggregate data an unaccountable error is introduced into the summation or mean of a sample of lens powers.

An improved paraxial approximation not subject to a systematic error

If aberrations are not of concern, say for small (paraxial) lens segments, then it is worth considering at this point the average power based upon the first paraxial approximation F_{AP1}. The radius of orthogonal sections away from the principal meridian is $R = \frac{2S}{S^2+Y^2\sec^2(\theta)}$, and

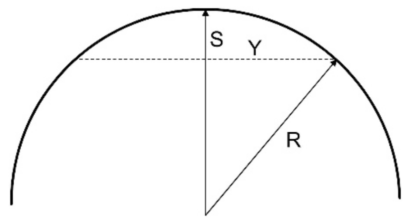

Figure 6 Sections through the meridians of a lens cylinder. Principal meridian of radius (R) and orthogonal and non-orthogonal oblique meridians of radii ($R\sec\theta$) and ($R\sec\gamma$) respectively.

which if substituted gives $F_{AP1}=\frac{2S(n_2-1)}{S^2+Y^2\sec^2\theta}$.[2] Using precisely the same assumptions that are made using F_{AP2} to derive the spherical equivalent that is, excluding the effect of spherical aberration for small sections and oblique non-orthogonal meridional power, a formula for the average power based on F_{AP1} that is $\langle F_{A1}\rangle$ can be derived as follows:

$$\langle F_{A1}\rangle=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{2S(n_2-1)}{S^2+Y^2\sec^2\theta}\mathrm{d}\theta=\frac{2(n_2-1)}{S}\left(1-\frac{Y}{\sqrt{S^2+Y^2}}\right). \qquad [7]^2$$

For a lens of refractive index n_2 in air. For a hemicylinder, the cord passes through the centre of the cylinder and Y and S are then equal to the radius R. The average power then reduces to $\langle F_{A1}\rangle=\frac{2(n_2-1)}{R}\left[1-\frac{1}{\sqrt{2}}\right]\approx 0.59\frac{n_2-1}{R}$. That is, the average power of a hemicylinder is 0.59 (a proportionality factor $P_F=0.59$) of its power in the principal meridian. Depending on the size of the lens segment (between a cord length of zero to that equal to the diameter) P_F will vary between 0.5 and 0.59, that is $0.5<P_F\leq 0.59$. The spherical equivalent of NES assumes a cord length of zero hence the use of 0.5 as the proportionality factor. In addition, it should be noted that invariance under transposition assumes that P_F is 0.5 (exclusion of S^2 and use of the second paraxial approximation F_{AP2}). If we are given a lens of power F, then NES, the nearest equivalent spherical power is just $\frac{1}{2}F$, and the difference between $\langle F_{A1}\rangle$ and NES, ΔF is,

$$\Delta F=\frac{2(n_2-1)}{S}\left[1-\frac{Y}{\sqrt{S^2+Y^2}}-\frac{S^2}{2(S^2+Y^2)}\right]. \qquad [8]^2$$

It is clear that as the power of the lens increases so does the error introduced by the spherical equivalent.[14]

One of the important differences between the F_{AP1} and F_{AP2} approximations is that the F_{AP1} approximation retains the correct relationship between two lens powers. In particular, use of F_{AP1} approximation makes it relatively easy to calculate the effective power of two orthogonal cylinders at an angle θ. For example, the effective power of two orthogonal equal power cylinders at an angle θ is

$$F(\theta)+F(\theta+90)^0=\frac{2S(n_2-1)\left[2Y^2+S^2\sin(2\theta)\right]}{Y^4+Y^2S^2+\frac{1}{4}\sin^2(2\theta)} \qquad [9]$$

which importantly, is dependent on θ. Essentially two orthogonal cylinders, cannot be represented by a spherocylinder combination and vice versa. Similarly, the power of a spherocylinder with a spherical component S and a cylinder with cylindrical power $-C=2S$ is not zero. Thus strictly speaking, a JCC, as its name defines, can only be constructed from two crossed cylinders, that is, two orthogonal cylinders of equal but opposite power and not by a spherocylinder combination.[2] This is an important result and although the magnitude of the error is small for small powers, it is significant for aggregate or high powers. In particular it becomes clear that the magnitude of the error is amplified as the samples or data sets increase in size. A consequence of this, would be the incorrect assumption that the mean keratometric power reflects the average of two keratometric orthogonal cylinders, $k1$ and $k2$. The cornea is not composed of two orthogonal cylinders so that the error introduced by taking the mean of $k1$ and $k2$ is very small. The relevance of this is apparent in the calculation of intraocular lens (IOL) power, particularly toric IOLs. For example, the methods used for the calculation of IOL power, treat the cornea as being composed of two orthogonal (perpendicular) keratometric powers, $k1$ and $k2$. When $k1$ and $k2$ are different as is the general case, the arithmetic mean of $k1$ and $k2$ is used as a measure for the calculation of IOL power. If it is decided to use a toric IOL, then the power of the IOL based upon $k2$ ($k2>k1$) is used for the sphere (IOL_{k2}) and the difference powers between IOL_{k2} and IOL$k1$ (the latter based on $k1$) is used for the cylinder component of the IOL. It is clear, however, from equations 8 and 9 that the average of $k1$ and $k2$ is not a sphere, and that the cornea is not composed of two perpendicular cylinders of powers $k1$ and $k2$. The cornea has varying powers and as such, it would be preferable to take several (not orthogonal) keratometric measurements to determine an approximation of the spherical IOL component with the cylinder IOL power (if a toric lens is intended) then based upon the difference between the highest (IOL_{k2}) and lowest (IOL_{k1})—the latter being in current practice. In addition, because of spherical aberrations for a high-powered lens such as the cornea are significant, it would be preferable to include an approximation according to the profile of the cornea—see reference 3 for non-spherical shapes and equation 11 below for a spherical shape.

Aberrations

Treatment of power in clinical optics has largely been limited by paraxial approximations.[19] The limitations of these approximations, is evident by the recognition of the importance of spherical aberration in the design of lenses, particularly intraocular lenses.[20,21] The gulf between standard refractive data based say on retinoscopy and that measured with systems, is largely due to the use of paraxial approximations for the former and the extension to geometric optics for the latter. Even for the micro-lens arrays used in aberrometers or wave front systems, it is perhaps unfortunate that approximations are also made for the individual micro-lens. Light rays refracted at the surface of a spherical lens do not come into focus at the same point, with peripheral rays being more refracted than paraxial rays (Figure 7). Rays not parallel to the axis of the lens produce coma and rays off the axis produce oblique astigmatism. Although spherical aberration is minimal for lens segments with small sag heights, it increases for more powerful lenses such as the cornea and also varies according to the refractive index of the lens. The change in power of a lens from paraxial to marginal rays is shown in Figures 7 and 8. There is, however, no need to exclude peripheral rays in the analysis of refractive powers. The contribution of spherical aberration to the power of the lens can be taken into account by calculating the average power of the lens in that meridian, that is, keeping θ and γ constant and letting α increase from zero to $\alpha_{\max}$.[2,3] Other aberrations (coma or oblique) can similarly be determined by varying θ and γ.[2,3] It has been shown that the average focal length $\langle f_{\mathrm{P}} \rangle$ and power $\langle F_{\mathrm{P}} \rangle$ of a lens in its principal meridian, which includes spherical aberration,[2,3] is

$$\langle F_{\mathrm{P}} \rangle = \frac{n}{R}\left(\frac{\alpha_{\mathrm{Max}}(n^2-1)}{\alpha_{\mathrm{Max}}(n^2-1)+\sin\alpha_{\mathrm{Max}}+nE\left(\sin\alpha_{\mathrm{Max}},\frac{1}{n}\right)}\right). \qquad [10]$$

For small segments such as the width of the pupil (5 mm), $\alpha_{\mathrm{Max}} \sim \frac{Y}{R}$ and equation [10] can be simplified to $\langle F \rangle \cong \left(\frac{6Rn^2(n-1)}{6R^2n^2-Y^2}\right)$ which is useful for calculating the average power of the cornea or an intraocular lens.[2,3] For example, for a pupil size of 5 mm and (and refractive index of 1.37), the average anterior surface power (for a spherical shape) $\langle K_{\mathrm{Anterior}} \rangle$ of the cornea is

$$\langle K_{\mathrm{Anterior}} \rangle \cong \left(\frac{6Rn^2(n-1)}{6R^2n^2-Y^2}\right) = \frac{4.17R}{11.26R^2-0.0025^2}. \qquad [11]^2$$

Therefore, if for example the measured anterior surface corneal radii are 7.8 mm and 8 mm, based upon a spherical profile for each section (which may not be the case, see reference 3), gives average anterior surface powers of $\langle K_{\mathrm{Anterior}} \rangle = 47.87$ and $\langle K_{\mathrm{Anterior}} \rangle = 46.65$ for k1 and k2, respectively.[2]

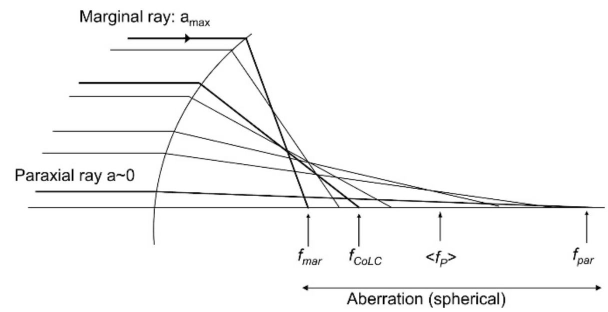

Figure 7 Light rays refracted through a section of a thin lens. Longitudinal spherical aberration. Paraxial ($\alpha \sim 0$) and marginal ($\alpha_{\max}$) light rays. Focal length ranges between f_{Paraxial} and f_{Marginal}, with average focal length $\langle f \rangle$. Focal length of circle of least confusion (f_{CoLC}), the position of which coincides with the densest collection of rays.

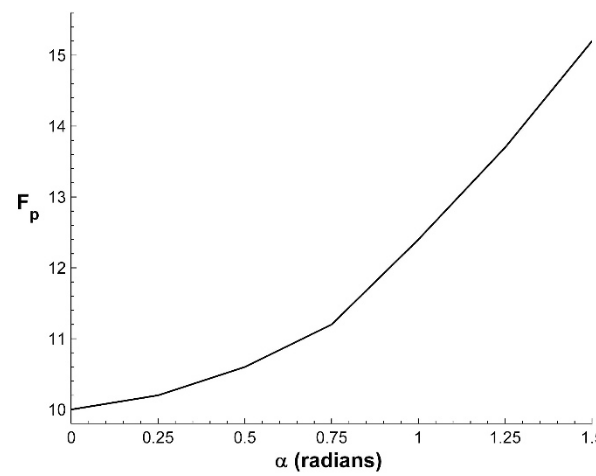

Figure 8 Power of a lens. Power of a lens (F_{P}) according to the angle of incidence (α) for $+10$ dioptre lens ($\alpha \sim 0$). As the angle of incidence increases so does the power of the lens (see Figure 6). Angle of incidence in radians.

In clinical practice, the paraxial power of a lens is usually given as $F \cong \frac{(n-1)}{R}$. Thus, for a standard trial lens of refractive index 1.5 and width of 20 mm, $\langle F \rangle \cong \left(\frac{F}{1-\frac{0.01^2F^2}{3.38}}\right)$. Hence, for a lens of stated power of 10D, and for a small section of the lens, including spherical aberration, gives an average power of 10.03D. For section through a spherical lens, a good univariate representation of average lens power, that includes aberrations that arise through orthogonal and oblique meridians, can be similarly approximated by $\langle L \rangle \cong \left(\frac{1.13R}{4.10R^2-0.0001}\right)$,[2] or for example, $\langle L \rangle \cong \left(\frac{0.56F}{1.025-0.0001F^2}\right)$ for a lens from a standard trial lens set of segment of width 20 mm and $n = 1.5$.[2]

In addition to that, provided the contour of a surface is known, similar equations can be developed for the average refractive power of any continuous surface. For example, a parabolic or other shape used to simulate the corneal profile.[2,3]

Discussion

This article reviews some of the issues that arise in the analysis of refractive data. In particular it is suggested how such data could best be analysed and represented. Methods are presented for the representation of the average power of a lens or refractive data as a univariate measure of refractive power, which includes spherical, coma and/or other aberrations. As opposed to the systematic bias brought about by use of the spherical equivalent, these methods have no such bias. Treatment effects or associations between biological variables and refractive error can thus be made more precisely. This may have significant effects for aggregate data, high powers, and analyses or representations of lens power, which depend on the constancy of this approximation. Average lens power in the principal meridian can thus be adequately computed as can the average lens power through orthogonal and oblique meridians providing a good univariate representation of lens power. Although these formulae are perhaps not as easy to use as, for example, the spherical equivalent, they are more precise and superior in principle involving fewer approximations and are not subject to systematic bias. Although less important for the individual case, such issues are of much greater importance when dealing with aggregate data and determining statistical significance of treatment effects. These effects are of significance when dealing with high-powered lenses such as intraocular lenses or the cornea. Consideration of these effects need to be taken into account particularly for calculations of intraocular lens power, toric intraocular lenses, and cornea refractive surgery if outcomes are to be improved.

Conflict of interest

The author declares no conflict of interest.

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