approximation for these values $N(p)$ turns out to be $\frac{16}{691}(p^{11}+1)$. The error term on this approximation, defined as

$$\text{Error}(p) := N(p) - \tfrac{16}{691}(p^{11}+1),$$

has been proven to be square-root small[7]. As a matter of fact, Error(p) is no larger in absolute value than $\frac{66304}{691}\sqrt{p^{11}}$.

This type of precision in obtaining 'good approximations' to difficult mathematical problems and estimating their error has led mathematicians to consider the next, but significantly deeper, tier of questions: determining the probability distribution of those error terms whose magnitudes have been shown to be square-root small.

For example, in the problem described above, let us rescale our error terms by their proven maximum order of magnitude, and ask for the probability distribution of the numbers

$$\frac{\text{Error}(p)}{66304/691\sqrt{p^{11}}} = \frac{N(p) - 16/691(p^{11}+1)}{66304/691\sqrt{p^{11}}}$$

as p ranges through the primes. These numbers all lie in the interval between -1 and $+1$. In 1960, Mikio Sato (by studying numerical data) and John Tate (following a theoretical investigation)[8] predicted that the absolute values of the scaled error terms for data in many problems of current interest conform to a specific probability distribution; Sato and Tate shared the Wolf prize in 2003 (ref. 9). In this particular instance, this is the simple distribution curve $\frac{2}{\pi}\sqrt{1-x^2}$, where x ranges through the interval between -1 and $+1$ (Fig. 1). Although the Sato–Tate prediction remains unproved for this specific example, the agreement with numerical data for the first million prime numbers gives cause for optimism.

In March this year, extraordinary strides were made towards demonstrating the truth of the Sato–Tate conjecture for one class of problems[1–3] related to elliptic curves. The error terms in such problems hold the key to counting solutions of algebraic equations that have a wide range of applications, including cryptography and analyses of the speed of computer algorithms.

The proof came by combining some wonderful pieces of mathematics, and the key to it is all is so-called representation theory. This branch of mathematics, in its various guises, studies abstract groups by representing them as groups of linear transformations of vector spaces. By understanding the profound number-theoretic structure behind enough of the symmetric tensor powers of a certain representation of a certain group, one can compute the probability distribution of the corresponding scaled error terms, and so confirm the Sato–Tate conjecture.

The first article of the three, by Laurent Clozel, Michael Harris and Richard Taylor ('Automorphy for some l-adic lifts of automorphic mod l representations')[1], deals with the relationship between number theory and representation theory related to these symmetric tensor powers. Harris, Nicholas Shepherd-Barron and Taylor (in 'Ihara's lemma and potential automorphy')[2] then show that the necessary 'profound number-theoretic understanding' of the nth symmetric tensor power is, for even values of n, intimately connected to the algebraic geometry of a certain beautiful family of $(n-1)$-dimensional vector spaces. This family, in the homogeneous variables X_0, $X_1, X_2, \ldots X_n$, is defined by the equation

$$X_0^{n+1} + X_1^{n+1} + X_2^{n+1} + \ldots + X_n^{n+1} = tX_0X_1X_2\ldots X_n,$$

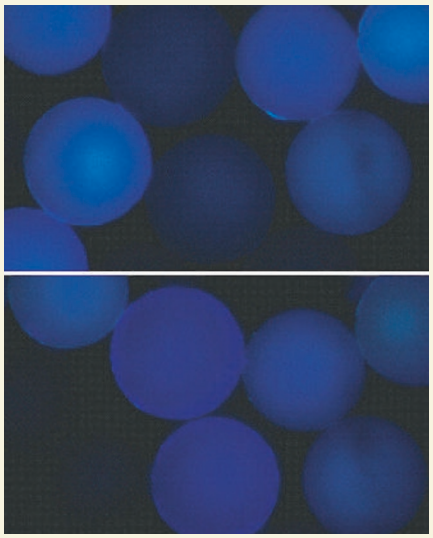
where t is a parametrization variable. When $n=2$, this family — known as the Hessian — already played an important role in nineteenth-century geometry. The higher examples ($n>2$) are well known to theoretical physicists because of their relevance to mirror symmetry and conformal field theory, theories that are important in string theory and statistical mechanics. Finally, Taylor (in 'Automorphy for some l-adic lifts of automorphic mod l representations II')[3] overcame the last obstacle, providing a striking argument that links this 'intimate connection'[2] to the truth of the Sato–Tate conjecture in the class of problems mentioned above.

This is a magnificent achievement for at least two reasons. First, the method brings synthetic unity to deep results in quite distinct mathematical fields. This coming together is as startling as the theory of continental drift that connects the shape of disparate continents.

Second, the work answers a question of delicate nature. Number theorists have long held the opinion that the 'error terms', despite the pejorative name, have a mesmerizingly rich structure (they are the Fourier coefficients of fascinating mathematical objects known as cusp forms) and that the keys to some of the deepest issues in their subject lie hidden in that structure. ■

ANALYTICAL CHEMISTRY

Playing molecular tag



With security tightened and concern about identity heightened, the computerized tagging of molecules with other molecules is an idea whose time has come. A. Prasanna de Silva and colleagues present an appropriate strategy, which bears the grand title of molecular computational identification, or MCID (*Nature Mater.* doi:10.1038/nmat1733; 2006).

In essence, the technique involves using a series of single-input 'logic gates' — in this case, fluorescent dyes — that respond differently to various chemical stimuli, and whose combined response is unique to a particular molecule. That might be useful to those in the pharmaceutical industry, for example, who need to be able to keep close tabs on libraries of different, but structurally related, compounds.

The authors set about finding fluorescent molecules that reproduce the effects of five different logic gates when exposed to the H^+ ion, which is responsible for acidity in water. Two of these gates, PASS 1 and PASS 0, produce a positive or a negative response regardless of the input and are trivial: they are, respectively, a fluorescent dye immobilized on a polymer bead, and no tag molecule at all.

For the YES gate, a molecule was chosen in which a process known as photoinduced electron transfer hinders fluorescence. When H^+ ions are present, they interact with electrons that would otherwise be transferred, so the molecule fluoresces. By using tag molecules with subtly different structures, the H^+ concentration at which this effect kicks in can be altered, and different colours of fluorescence can be created.

Similar principles apply to the authors' NOT gate, which is activated when the concentration of H^+ is low. Pictured is an array of tag beads that has been immersed in hydrochloric acid (where the H^+ concentration is high; top image) or the alkali sodium hydroxide (with a lower level of H^+ ions; bottom image). The disappearance of two YES beads (bottom left) and the appearance of two NOT beads (middle) is clearly visible.

The technique can be easily extended to multiple-input gates, as the authors demonstrate with a tag molecule that acts as an AND gate. It fluoresces only in the presence of both H^+ and the sodium ion Na^+. Other distinguishable input stimuli are calcium, caesium and zinc ions, glucose molecules, and even heat and light.

The various available inputs, gates, switching thresholds and output colours could make MCID a sophisticated cataloguing tool, say the authors. Molecules and other nanoscale objects, like so much in today's world, may soon become familiar with computerized ID.

Richard Webb

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