Divide, conquer and unify

Fluid dynamics: Ludwig Prandtl's ideas brought hydraulics and hydrodynamics together to found a new field.

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In August 1904 Ludwig Prandtl, a 29-year old professor of mechanics at the Technical University of Hanover, presented a remarkable paper at the Third International Mathematical Congress in Heidelberg. The paper was a scientific time bomb — it made no great impact at the congress, and was not translated into English until 1928. But by the 1920s and 1930s, the powerful idea in that paper and the reputation of its author had spread across the world, helping to create modern fluid dynamics out of ancient hydraulics and nineteenth-century hydrodynamics.

In retrospect, the paper, with twelve photographs of water flow past bodies, ten other diagrams and only eight equations, can be seen as full of understated daring. The heart of the paper is in one paragraph, sandwiched between some preliminary mathematics and a terse but wide-ranging exposition of the explanatory power of the idea. That exposition covered how vortices emerge in mixing layers, behind cylinders, at the edge of a plate moving normal to its plane and so on, and of when and why flows separate from the solid surface they are supposed to follow.

The tone of that key paragraph is deceptively casual. The formal problem it tackled is the flow past an aligned flat plate — a problem that is trivial in inviscid hydrodynamics, but was crucial to the new fluid dynamics that was being created. First, a small parameter, ε, is identified to represent viscosity, or more precisely a reciprocal of the Reynolds number. Classical perturbation methods would not work here, for the limit $\varepsilon \to 0$ is singular: the limit of the full solution (for example, at the surface) is not the solution of the limiting equation. Prandtl's method was to divide the flow into different regions in which the dynamics are different — an outer inviscid flow, and a thin inner viscous ('boundary') layer next to the surface. In each of these regions only the dominant terms in the Navier–Stokes equations were collected. The equations for each region were separately solved, but their boundary conditions were cleverly selected so that the solutions blended into each other to yield a 'unified' approximation.

In that same paragraph, Prandtl also showed how the partial differential equations that governed the boundary layer could be reduced to one ordinary differential equation using a combination of the independent variables, and thus solved the resulting nonlinear equation (which was not even explicitly written out) numerically, and provided a rough value for the drag of the plate and a sketch of the solution. And he did all this using only roughly 25 lines!

Extraordinary insights: Ludwig Prandtl in 1936.

V. M. GHATAGE, 1936; COURTESY CHAIRMAN DEPT AEROSPACE ENG., INDIAN INST. SCI.

This potentially precise calculation of laminar-flow flat-plate drag opened a door between the previously sealed chambers of hydraulics and hydrodynamics. Practitioners of these disciplines had long poured scorn on each other — hydraulics was often called a science of variable constants, and hydrodynamics the mathematics of dry water. Reality, Prandtl demonstrated, was not only within the scope of Navier–Stokes equations, but even accessible to mathematics.

The idea was subsequently developed and extended, mainly through the pupils who flocked to work with Prandtl in Göttingen, where the mathematician Felix Klein helped him to establish a renowned centre for research in fluid dynamics. And the theory showed excellent agreement with experiment. Yet, astonishingly, it was regarded as no more than a clever and successful approximation or 'model' for nearly five decades. It was eventually the work of Paco Lagerstrom and his colleagues at Caltech that distilled a systematic 'method of matched asymptotic expansions' (MAX for short) as the mathematical idea inherent in Prandtl's work. In the simplest case, MAX involves studying the limiting solutions as $\varepsilon \to 0$ separately in inner and outer regions, making in each of them appropriate asymptotic expansions in so-called inner and outer variables, matching the two expansions by equating (to appropriate order) the outer limit of the inner expansion with the inner limit of the outer expansion, and finally constructing a unified ('composite') solution that is everywhere asymptotic to the exact solution.

That formal mathematization of solving a whole class of singular perturbation problems had several important sequels. First it was discovered that many puzzles (from the mathematician Jean-le-Rond d'Alembert's famous paradox to others, including low Reynolds-number Stokes-type hydrodynamics) yielded to MAX. Then it turned out that the many other problems that Prandtl had brilliantly solved — such as finite wings in aerodynamics — had essentially been based on use of MAX too. When Milton van Dyke wrote the first book on the subject he crowned MAX with the adjective 'rational' — distinguishing it from the numerous non-rational, more or less ad hoc, approximations and theories that then abounded in fluid dynamics.

Werner Heisenberg said that Prandtl had "the ability to see the solution of equations without going through the calculations". Prandtl demurred, "No, I strive to form in my mind a thorough picture… the equations come only later when I believe I have understood… [and are] good means of proving my conclusions in a way that others can accept." His papers have a simplicity and directness born of supreme self-confidence. They do not trumpet their success or criticize others, but just get on with solving the central problems using all the tools available — observation (plenty of it), mathematics, calculation and modelling. Prandtl's methodological eclecticism set the style of fluid dynamics research in the twentieth century. No wonder G. I. Taylor called him 'our chief', and helped nominate Prandtl for a Nobel prize he never won. ■

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FURTHER READING

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