TOPOLOGY

Just out of pigtails

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HOW, in a simple and forceful way, do we characterize the dynamics of systems with several moving components? When the components move in two dimensions, methods based on the theory of braids may provide the answer. That is why an experiment on the motion of beads drifting in a magnetized fluid[1] will be of general interest to those studying nonlinear dynamics. The space–time diagrams reveal a rich topological structure that would not be readily apparent in a motion picture of the beads.

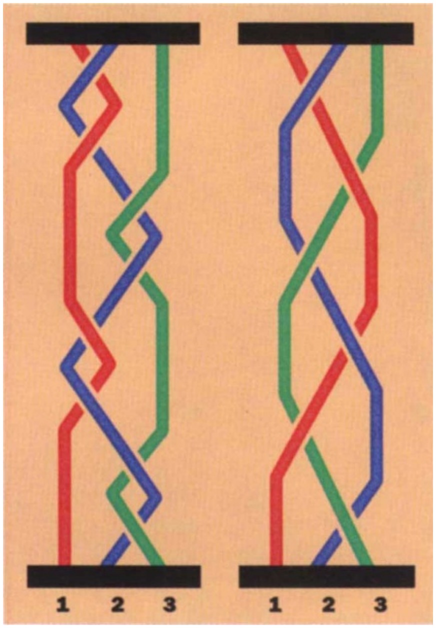
People generally visualize objects as they exist at one instant of time; in a drawing, a one-dimensional curve represents the position and shape of a filament. But just over a century ago in *The Time Machine*, H. G. Wells advocated a different way of looking at objects — as they exist in both space and time. This adds an extra dimension: in a diagram with one axis representing the time coordinate, particles generate one-dimensional curves, and a loop of string becomes a tube. A decade later, special relativity made this viewpoint fundamental to our understanding of nature. Even though most scientific work does not involve relativistic effects, time generally does play a role, and researchers may well wonder how their data would look as plotted in a space–time diagram.

In the new experiment by Pieranski *et al.*[1], non-magnetic spherical beads are placed in a thin layer of fluid containing iron particles. A rotating magnetic field sets the fluid in motion. The beads behave like magnetic holes, and interact with one another according to simple, but nonlinear, equations. The essentially two-dimensional motion of a bead can be represented as a curve in a three-dimensional space–time diagram, and so several beads in motion produce a set of braided curves.

The authors suggest that the topological description of this braid provides a simple and concise language for describing the dynamics of the system. Here one ignores all the little wiggles in the motions of the magnetic holes, concentrating instead on the overall pattern of movement. The holes perform a complicated dance as they move about one another, and the braid encodes the choreography of this dance.

To understand what is meant by the topology of a braid, think of a set of strings stretching between two parallel planes (see figure), crossing one another at several places. Emil Artin in 1925 gave a simple way of keeping track of the crossings, and hence describing the braid structure. Just below each crossing, label the strings 1, 2, 3... from left to right. If string 2 crosses over string 3, label the crossing σ_2; if it crosses under string 3, label the crossing σ_2^{-1} (the character σ is just something to attach subscripts and superscripts to). The entire braid can be coded as a sequence of these symbols.

Suppose we fix the ends of the strings at the two boundary planes, but deform the strings in between. Such an operation is said to preserve the topology of the braid,



Two braid diagrams. Although their sequences of crossings are different ($\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_2\sigma_1\sigma_1$ and $\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$; see text), they are topologically equivalent. The right-hand braid is the standard pattern for plaiting pigtails.

even though the sequence of Artin symbols may change. For example, a braid with the sequence $\sigma_2\sigma_1\sigma_2$ can readily be converted to one with the sequence $\sigma_1\sigma_2\sigma_1$. Simple algebraic algorithms can be used to check whether two sequences belong to the same braid topology[2].

Once a sequence of motions has been converted into braid notation, it can be further examined to obtain numbers characterizing the structure and complexity of the braid. For example, the number of positive crossings minus the number of negative crossings (those with the –1 superscript) is a topological invariant known as the 'writhe', which characterizes the net twist of the braid. And the minimum possible length of the braid sequence provides a measure of complexity. At present, simple algorithms exist only for minimizing braids with three strings[3].

The algebra of braids has applications in several areas of mathematics and theoretical physics, including statistical mechanics and field theory[4]. In 1983, Joan Birman and R. F. Williams developed mathematical tools for analysing the knottedness of trajectories[5]. Since then, braid theory, a subset of knot theory, has been a particularly rich source of insight[6]. For example, Alan McRobie and Mike Thompson used braid diagrams to look at the trajectories of nonlinear oscillators[7].

To illustrate this technique, consider a ship rolling in heavy seas. On the *x–y* plane, plot the angle of roll against the velocity of rolling, and then follow the evolution of these two coordinates with time. A set of a few different initial values will lead to trajectories that trace out a braid, and the braid topology will change as the parameters of the physical system change — dramatically so when there is a bifurcation or when qualitatively new behaviour emerges. In this case, "qualitatively new behaviour" can mean the ship capsizing.

Braid theory has applications in astrophysics as well[8]. X-ray pictures of the Sun often show clouds of gas as long loops or arches; these loops trace out the direction of the magnetic lines of force, and often display a complex braided and twisted structure that reflects the pattern of motion of magnetic flux at the surface of the Sun. Braid complexity correlates with magnetic energy storage, and violent changes in the braid pattern can release this energy, heating the solar atmosphere to millions of degrees and accelerating charged particles to high energy. We may be able to predict such magnetic storms by monitoring the complexity of the braids.

Theoretical work by Christopher Moore[9] and the experiment by Pieranski *et al.* point to new directions for research. Moore looked at two-dimensional dynamical systems in general, and proved that any braid type can be realized as a set of trajectories in some dynamical system. But linking braid structures to particular systems, especially those that occur in natural or experimental settings, remains to be done.

Braids generated by random mechanisms can also be usefully studied. The tangle of wires under your desk or behind your stereo system may prove to be a ripe source of investigation. □

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