Can anyons be made more tractable?

Particles that are neither fermions nor bosons, but something in between, appear to exist only fitfully, in special circumstances, but may now have been made more easily calculable.

THERE are almost as many accounts of the difference between a boson and a fermion as there are people with an interest in the question. One solution is to list them; electrons are fermions and photons are bosons, for example. Others may prefer to refer to the intrinsic quantum spin of the particles concerned; particles with half-integral spin (such as electrons) are fermions, those with zero or integral spin are bosons. Still others will choose to refer to Pauli's exclusion principle; two electrons (or two other fermions) cannot simultaneously occupy the same physical state, but the exclusion does not apply to bosons.

The exclusion principle derives from a mathematical condition that is simply put. The many-particle wavefunction for a system of, say, N fermions, will change sign when any two are interchanged, but the same operation applied to a system of bosons will yield a wavefunction that is unchanged, both in functional form and algebraic sign. This is why theoretical chemists, when constructing wave-equations for many-electron systems, are driven to write them in the form of determinants. For then the interchange of two particles will automatically bring about a change of sign; swapping any two rows (or columns) of a determinant will change the sign of the algebraic form it represents.

The practical consequences of all this are considerable. Atoms would not have electronic shell structures if the exclusion principle did not apply, nor would the electrons in a metal at low temperature fill the successive energy levels in the so-called Fermi sea up to a certain maximum, ensuring that electrons make a much smaller contribution to the specific heat of a metal than would be expected from the equipartition of energy, or on the old Dulong and Petit prediction. But liquid ^{4}He would not be a superfluid at low temperatures if its atoms (which have zero spin) were not bosons.

As the textbooks explain, properties such as these are best described by the "statistics", or the statistical mechanics, characteristic of fermions and bosons: fermions follow Fermi statistics, but bosons follow Bose statistics. But then the question arises whether the natural world is as sharply divided into two categories of particles as this nomenclature implies. May there not be particles that behave neither as fermions nor as bosons, but as something in between?

The question would be pointless if it could not be answered in the affirmative. Indeed, there are circumstances in which particles behave as if their properties were in between those of fermions and bosons. The best-known are those in which electrons move in a conductor under the influence of a magnetic field, when what is known as the fractional quantum Hall effect comes into its own. (Electrons are effectively confined to two dimensions by sandwiching a microscopically thin conducting layer between two insulating layers by the techniques used in making electronic devices.)

Inevitably, there is now a name by which to call particles which are neither fermions nor bosons; *anyon* is the name used for the past decade. (The similarity with the English pronoun "anyone" is part of the joke.) But the prospect of being able to deal with them as if they were no more complicated than electrons is illusory. When electrons are confined in a narrow two-dimensional slice of conductor, for example, the properties they acquire are necessarily a function of the geometry, while the problem is a statistical problem only because there are several of them. The anyon problem is essentially a many-body problem.

What follows is not a detailed account of how the problem can now be tackled, but is rather a celebration of some work that appears to make anyons more manageable than they have been. It is relevant that people have recently become skilled at trapping small numbers of atoms in electromagnetic traps, and are robbing them of kinetic energy by suitably devised lasers. The search for direct observation of the cooperative Bose condensation expected of bosons is well under way. Using magnetic fields or other external influences to make atoms into anyons is not beyond the bounds of possibility.

The place to start is with the classical Hall effect, which is simply the observation that if a current is made to flow in some direction through a conductor, say a piece of metal foil, a magnetic field perpendicular to the plane of the conductor will induce a potential difference across the conductor in a direction transverse to the current.

The qualitative explanation is that the influence of the magnetic field is to tend to make the electrons travel in circles around its lines of flux. The quantum Hall effect, observable only at low temperatures, arises because low-energy electrons in a magnetic field should be confined to orbits in which their momentum is quantized. But observations (notably by the German physicist K von Klitzing, who won a Nobel Prize a decade ago for his pains) show that there are many more quantized Hall voltages than expected (whence the fractionality). These are anyons made manifest.

The impediments to the solution of this problem are easily imagined. The quantized motion of a single electron in an external magnetic field is not straightforward, the application of the exclusion principle is something else again. But three years ago, F. D. M. Haldane from Princeton University described a "generalization of the Pauli principle" that seemed to offer a way of dealing with anyons in a coherent fashion (*Phys. Rev. Lett.* **67**, 937; 1991). The trouble was that the argument involved counting the number of dimensions in a Hilbert space in which the state of the quantum system could be represented, hardly common practice.

But it makes sense. A Hilbert space is most simply thought of as one in which each axis is one of a set of orthogonal (and, in general, complex) functions, which may, for example, be the eigenfunctions of a Schrödinger equation, each corresponding to a quantum state. Then any point in the Hilbert space will represent a mixture or superposition of quantum states. In most applications, the Hilbert space has an infinite number of dimensions. But for electrons moving in a two-dimensional piece of conductor with finite dimensions (and ignoring the complication of excitation into states of higher energy), the Hilbert space describing the motion of a single particle will have finite dimensions.

To describe N particles, of course, extra dimensions will be necessary, in the extreme case of fermions, as many single-particle spaces as there are particles. Haldane's generalization of the exclusion principle was simply a statement of how the size of the single-particle Hilbert space would change as extra particles were added.

M. V. N. Murthy and R. Shankar from the University of Madras have now taken the argument an important step further (*Phys. Rev. Lett.* **72**, 3629–3633; 1994) by showing that the single-particle Hilbert space need not necessarily have a finite number of dimensions. Among other things, they are able to calculate the single-particle partition function of a system of anyons, from which all thermodynamic properties should flow. As people do, the authors have calculated a few simple systems to show that their argument holds water. But the prize that lies ahead is that it may be useable in tackling the outstanding problem of the interacting spins of ferromagnetic materials, perhaps even of high-T_c superconductors.

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