Abstract models in search of problems

Most models come into being as aids in the solution of real problems, but there is no reason why the process should not be inverted by the creation of models for which there are no problems — yet.

JUST as paintings are artists' models of the visual world, so are physicists' models of the physical world imperfect representations of reality. Impressionism is permissible in both fields. To physicists (and in science generally) it may be more beneficial that a model should have heuristic virtues, or be suggestive, than that it should slavishly represent all the details of a system. But almost always, in physics as in art, a model is some kind of a representation of something out there. Painters usually give even their most abstract work a title, suggesting that they had something in mind when they made it. And in physics, it may be thought, a model that is a representation of nothing in particular would be of no value at all. By definition. For is not the purpose of a model to assist in the solution of some problem that has been specified?

That, it seems, is to reckon without people's ingenuity. Or, more correctly, there may be models in physics that are analogous to strictly abstract paintings in that they evoke both problems and the solutions of them. One such has just appeared. Cleverly, for an essay in abstract modelling, it combines two trains of speculation that have recently been famously successful: the use of the vertices of a regular lattice as an approximation to a continuum, and the use of geometrical mapping rules to represent the behaviour of chaotic systems.

This is how the argument goes. First, define a lattice, which for simplicity is merely a set of points alone a one-dimensional line and let each point be occupied by a dynamical variable of some kind. That might, for example, be the energy of some component of a whole system, or its orientation relative to some direction or its speed. That means that there is a separate dynamical variable for every point of the lattice, say $x(j)$, where j is any integer between 1 and N, the total number of vertices in the lattice.

The next step in the definition is to make the dynamical variables chaotic using some nonlinear mapping rule to relate each $x(j)$ at one time to the same quantity at some later time. If, for example, it is supposed that time as well as space is discrete (or executed in jumps of fixed amount), then the dynamics of each sub-system may be represented by a rule defining each $x_{n+1}(j)$ in terms of $x_n(j)$, the value of the same quantity one time-step earlier.

At this point, there is little interesting to be said about the system as a whole — the collection of the N lattice points, each with its own subsystem. It is simply a collection of nonlinear systems evolving independently of each other. How to tie them together into a single system? Obviously there must be some interaction between one subsystem and its neighbours, which obviously must depend on the value of the variables sited on them. And the principle must be that simplicity is best. So why not think of the variables on each lattice point as quantities of something, and suppose that if the amount of the something at any point exceeds some predetermined value, the excess simply spills over onto the next lattice-point in the chain?

That is the model constructed by Sudeshna Sinha and Debabrata Biswas from the Bhabha Atomic Research Centre at Bombay (*Phys. Rev. Lett.* **71**, 2010; 27 September 1993). The obviously missing element is the rule specifying the evolution of the chaotic sub-systems, which they take to be $x_{n+1}(j) = 1 - 2(x_n(j))^2$ for each lattice point in the system, with the convention that x lies in value between -1 and 1.

To establish the interaction between subsystems, there must also be a rule for spilling over. The authors offer several versions. First they define some critical value of x, say x_c, which they take to be the threshold over which excess material spills. Then there are two choices: spilling only in one direction, or equally in both. And what happens if spill-over carries a neighbouring variable above the same critical value x_c? Then there must be further spill-over, to more distant lattice sites, until all the lattice variables are less than or equal to the critical value. To make the system consistent, something spilled over from the ends of the lattice disappears from further consideration.

Inevitably, numerical simulation takes over at this point. Sinha and Biswas first populate their lattice with random values of x (within the allowed range), let the spilling over happen, the system settle down, and then execute a further iteration of the chaotic variables. And then carry on indefinitely, as computers allow. A further refinement (which helps to make some behaviour of the model interesting) is the possibility of adding random quantities at random lattice sites.

A few simple properties of the system stand out. If, for example, the threshold value x_c is less in magnitude than 0.5, the system will eventually reach a steady state in which each dynamical variable $x(j)$ has the same value as the threshold. The reason for that stems from the properties of the mapping relation, which implies that x_{n+1} is always greater than x_n whenever the latter is less than 0.5 in magnitude. That in turn implies that there is always spillover from each lattice point, and always enough of it to ensure that each is filled to the level of x_c. Interest then turns to the nature of the spillover, which turns out to be much like the problem of avalanches on the sides of an artificial sand-pile formed by the addition of grains to the surface in the phenomenon called "self-organizing criticality".

The fun begins when the critical parameter is greater. Then spillover is less common — the mapping relationship does less to magnify the separate variables, while the threshold is higher. Then the simulations show surprisingly rich behaviour at individual lattice sites. Often there will be a cyclical variation of the variable at this site which is then interrupted by apparently random (or chaotic) motion. The periodicity is more pronounced at the centre of the lattice, more chaotic at the outsides. The size of the lattice seems to matter a great deal; the larger, the more pronounced the periodic behaviour of the lattice elements.

The temptation to replace "lattice elements" with the word "neuron" is bound, at that point, to be strong. If Sinha and Biswas's construction is a model looking for a problem to which it may apply, might it be a way in which groups of uninstructed neurons could organize themselves into coherent function? The authors themselves refer to the possibility that the coordination of synapses might be in such a way.

Other connections readily suggest themselves. The model now described, for example, has much common with the simulation technique of the cellular automaton (conceptually due to J. von Neumann). There again there is an array of lattice elements evolving separately in time, but according to rules that imply interaction among them. And again the result often seems to be a delicate mixture of order and chaos. Indeed, it seems the difference between the two schemes is simply that of Sinha and Biswas allows more delicate tuning of the timing of the interaction among the lattice elements.

It is readily imagined that the publication of these simulation will send other practitioners rushing to their own machines. There is endless scope for tractable elaboration, which is a virtue. But what can be said about the analogy with the behaviour of neurons, for example? Sadly, there is a sense in which the modelling puts the cart before the horse. It would be different if somebody had begun with the problem and then built the model. The equivalent of abstract art may have no place in physics.

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