Mathematics

The beat of a fractal drum

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EVERYTHING in the Universe vibrates. Light from the most distant galaxy and the sound of a nightingale are brought to us by vibrations — of the space-time continuum and the atmosphere, respectively. The tone of a Stradivarius violin, and the stability of a car wheel, depend on how they vibrate. And each object has not just one, but a whole range of characteristic resonant frequencies of vibration. Ancient Chinese bells were designed to have two especially loud natural frequencies: which one sounds depends upon where the bell is struck. Michel Lapidus (University of Georgia, Athens) and Jacqueline Fleckinger-Pellé (Université Paul-Sabatier, Toulouse) now prove in *Comptes Rendus de l'Académie de Science, Paris* (**306**, 171–175; 1988) an important conjecture about the vibrational frequencies of objects that have fractal boundaries.

Harmonics

The fundamental frequency of a violin is determined by the tension in the string. But it can also generate harmonics of the fundamental frequency — vibrations that occur twice as fast, or three times, or four times. . . . Here the pattern is easily described: the possible frequencies follow the series of whole numbers 1,2,3. . . . But more complicated shapes produce more complicated series of frequencies. Even the analysis of the vibrations of a circular disk, modelling the head of a drum, leads to sophisticated questions about Bessel functions.

Despite these complexities, Hermann Weyl discovered that all vibrating shapes have features in common — at least, if they have smooth edges. In particular, the statistical distribution of the characteristic frequencies obeys regular laws, to be described in more detail in a moment. In 1980 Michael Berry *(Proc. Symp. Pure Math.* **36**, 13–29; 1980) conjectured, on physical grounds, that a more accurate version of Weyl's results should be true, and that it should apply not just to the smooth shapes envisaged by Weyl, but to shapes with fractal boundaries. Now this Weyl–Berry conjecture has been proved — in slightly modified form — by Lapidus and Fleckinger-Pellé.

Mathematically, small-amplitude vibrations of any medium are described by the wave equation. This was originally devised in the eighteenth century by Leonhard Euler in a study of musical instruments, but soon Joseph-Louis Lagrange extended it to sound waves, and the floodgates burst. The wave equation became perhaps the most important of all the equations of mathematical physics. Its mathematical form, $d^2f/dt^2 = \nabla^2 f$ involves the laplacian differential operator ∇^2, defined by $\nabla^2 f = \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2$.

The characteristic frequencies of a vibrating shape correspond to the eigenvalues λ of the laplacian; that is, the solutions of the equation $\nabla^2 f + \lambda f = 0$. In this equation you should think of the boundary as being held fixed while the rest of the shape vibrates, just as in a drum or a violin string.

Weyl's formula describes the asymptotic properties of the vibrations at high frequency. Specifically, let $N(\lambda)$ be the number of characteristic frequencies less than a given value λ. How does $N(\lambda)$ behave as λ becomes large? The result is that $N(\lambda)$ is approximately proportional to $\lambda^{k/2}$, where k is the dimension of the vibrating object. The constant of proportionality depends on the volume of the vibrating shape.

Here 'approximately' means that the ratio of the true answer to that given by Weyl's formula tends to 1 as λ tends to infinity. But that gives only very coarse information. To improve Weyl's formula we must ask how large the error can become. Berry's idea is that the error depends on boundary effects, and should be of the same order of magnitude as $\lambda^{d/2}$, where d is the dimension of the boundary. For example, a drum has a circular head, for which $k = 2$ and $d = 1$. So $N(\lambda)$ is approximately proportional to λ, with an error no worse than $\lambda^{1/2}$.

Weyl's proof of his original formula assumed that the boundary of the shape is mathematically nice: smooth, or at least smooth except at finitely many corners. But Berry's physical reasoning appears to apply equally well when the boundary is irregular. Indeed Berry suggested it should be true when the boundary is fractal, in the sense of Benoit Mandelbrot *(The Fractal Geometry of Nature,* Freeman, San Francisco, 1982).

Fractal snowflake

The archetypal fractal is the snowflake curve. Begin with an equilateral triangle. To each edge add an equilateral triangle one-third the size. Now repeat indefinitely, adding ever-smaller triangles. How does a drum shaped like a snowflake vibrate? According to Berry, much like a drum with a smooth rim, until you look at the fine detail, the high-frequency vibrations that penetrate into tiny crevices of the boundary. Here, more vibrations should be possible, because fractal objects have lots of ever-smaller crevices. So the number $N(\lambda)$ should be bigger. Instead of missing Weyl's formula by about $\lambda^{1/2}$ (as for a circular drum), a snowflake should miss it by more. Berry argues that the error should be about $\lambda^{d/2}$ where d is the dimension of the boundary of the snowflake.

What do we mean by the dimension of the boundary, when the boundary is an irregular fractal? Berry's conjecture is that it should be the fractal dimension, or Hausdorff–Besicovitch dimension. This concept, which is fundamental to the analysis of fractals, measures the

Discovery of a new species of lemur

THESE lemurs were first observed in 1985 by Bernard Meier and Yves Rumpler in the Hanomafana region of Madagascar. Last month it was confirmed that they belong to a new species, *Hapalemur aureus*. Since their discovery, these monkeys have been investigated by a team supported by the World Wildlife Fund and Duke University in North Carolina. The team captured two specimens and chromosome analysis confirms that the animals are members of a new species. *H. aureus* weighs about 2 kilograms and is monogamous, each couple living in a territory of 15–20 hectares of forest. The offspring are born in December and live with the parents for 2 years before leaving to begin an independent existence. Adult monkeys live off freshly-grown leaves. Because these lemurs are so rare — only 500 individuals of *H. aureus* are believed to exist — the Madagascan authorities plan to turn this area of forest into a wildlife reserve. Photograph: Jean-Christophe Peyre/Gamma. □

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