itself: its information content is incompressible.

A sequence of digits 0 and 1 can be interpreted as the binary expansion of a real (non-integer) number, by inserting a 'decimal point' (or rather, a 'binary point') in front. Thus the infinite repeating sequence 010101... corresponds to the real number .010101... in binary; that is ⅓. A real number is computable if its binary expansion can be the output of a computer program. It is a theorem — at first sight surprising, but true, and even easy — that 'almost all' real numbers are not computable. For example, Turing used an argument that goes back to Georg Cantor to prove the existence of at least one non-computable real number.

The proof is based on the idea that all possible computer programs, each of which must be represented by a finite sequence of digits, can be listed in order. To do this, interpret the program's defining sequence as a whole number, expressed in binary, and arrange these numbers in increasing numerical order. Now assign to each program in this list its output data, a real number expressed in binary. Run down the diagonal of this table of numbers, changing the nth digit in the nth number. The new diagonal is a number that is not on the list, which therefore corresponds to the output of no computer program whatsoever.

Turing machine

Another of Turing's basic discoveries is that it is possible to construct a universal Turing machine — a computer program capable of simulating any other program. Consider the following 'halting' problem for a universal Turing machine: given a particular input, does the computation eventually stop or does it go on for ever? Using the above 'diagonal' argument, Turing showed that the halting problem is logically undecidable.

Correspondingly, Chaitin defines a real number Ω which can be thought of as the probability that a universal Turing machine, given a random program, will eventually halt. It is impossible to compute Ω because its digits solve the halting problem for the universal Turing machine. Chaitin shows that any formal system of axioms can yield only a finite number of (scattered) digits of Ω. Thus Ω is a random real number in the above sense: it has no compact computable description.

The final step is to recast this circle of ideas in terms of whole numbers, rather than arbitrary real numbers. Chaitin does this using 'exponential diophantine' equations. These are equations, to be solved in positive integers, that can be built up from the usual algebraic operations of sums, differences and products, together with exponentiation, x^n.

Over the past two decades, logicians have solved a longstanding problem, showing that any computation can be encoded in the set of solutions of some diophantine equation. There is a diophantine formula, for example, to generate the primes. An undecidable computation — such as the halting problem for a universal Turing machine — leads to an undecidable diophantine equation; that is, an equation in arithmetic for which there can be no proof that solutions exist, and no proof that they do not.

Finally, the randomness. Chaitin's main theorem asserts that it is possible to construct a specific exponential diophantine equation, of the form $L(n)=R(n)$, where L and R are functions that depend on a finite number of additional variables, with the following property. The equation has infinitely many whole-number solutions if and only if the nth digit of Ω is 1. Because Ω is random, the question 'does $L(n)=R(n)$ have infinitely many solutions?' has an answer which varies randomly as n takes the values 0,1,2,3,... in turn.

In the proof, Chaitin constructs an exponential diophantine equation in 17,000 variables, about 900,000 symbols long. Instead of Turing's formulation of computation, he uses a version of the standard computer language Lisp. The practical and theoretical aspects of computation become almost inseparable in this work.

For the foundations of mathematics, and even the philosophy of its application to science, this century has been one of shattered illusions. Cosy assumption after cosy assumption has exploded in mathematicians' faces. The assumption that the formal structure of arithmetic is precise and regular turns out to have been a time-bomb, and Chaitin has just pushed the detonator. His book ends with some speculations on a possible analogy with biological complexity. Resemblances to Deep Thought in Douglas Adams' *The Hitchhiker's Guide to the Galaxy* are strong, but presumably accidental. In the following quotation from Chaitin's book, the phrase 'oracle for the halting problem' boils down to 'complete specification of the number Ω': "We have seen that Ω is about as random, patternless, unpredictable and incomprehensible as possible ... However, with computations in the limit, which is equivalent to having an oracle for the halting problem, Ω seems quite understandable ... Biological evolution is the nearest thing to an infinite computation in the limit that we will ever see: it is a computation with molecular components that has proceeded for 10^9 years in parallel over the entire surface of the earth ... Perhaps biological structures are simple and easy to understand only if one has an oracle for the halting problem." □

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Daedalus

Warm wind

The temperature of the atmosphere drops with increasing altitude. Its adiabatic lapse-rate of 6.5 °C km^{-1} is maintained by its turbulence: rising air expands and cools while descending air is compressed and heated. Other gases have different adiabatic lapse-rates. The record is held by xenon, an atmosphere of which would establish a gradient of 63 °C km^{-1}.

Daedalus plans to exploit this in a new wind-powered heating system. Imagine, he says, a flexible plastic bag rising a kilometre into the atmosphere, and filled with xenon. The turbulence and buffeting of the wind would soon stir the xenon into full convective equilibrium, when the bottom of the bag would be 63 °C hotter than the top. If the top were maintained at local atmospheric temperature by a suitable heat-exchanger, domestically useful heat at about 70 °C could be extracted from the bottom of the bag.

This ingenious scheme extracts energy, not from the 'd.c.' component of the wind, but from its turbulent 'a.c.' component. At ground-level, especially near buildings or in regions of irregular topography, much of the wind's energy is gusty and fluctuating and unavailable to conventional wind-turbines. So the thermal wind-bag has great promise.

There are two main problems. First, a flabby bag a kilometre high is rather an awkward object. Second, such a tall column of a dense gas like xenon would exert greater than atmospheric pressure at its base. This would hold the plastic bag taut, reducing its ability to flap and flex and stir the gas within. Thermal wind-bags only a hundred metres high would be more practical; they could then be anchored to buildings or lattice-towers, or even steep hillsides. If excess base-pressure is then still troublesome, Daedalus will replace his xenon by a mixture of argon and neon with the same density as the atmosphere, but an adiabatic lapse-rate about twice as great: 14 °C km^{-1}. All this will reduce the temperature-rise at the bottom of the bag to only a few degrees, so Daedalus plans to multiply it up by regenerative heating. A counter-current heat-exchanger inside the bag will transfer most of the bottom heat straight up to the top again. The convective process will therefore increase the temperature at the bottom still more, and this too will be returned to the top. With a sufficiently high recycle-ratio, any desired temperature could be attained at the bottom of the bag, though with a corresponding reduction of extracted power per effective cycle. Even so, the more the cold wind buffeted your house, the more heat it would unwittingly supply! David Jones