Making quantum mechanics relativistic

A novel attempt to reconcile dull old Schrödinger's equation with special relativity would be more winsome if it had more explicitly acknowledged its purpose.

DIRAC's achievement in the late 1920s notwithstanding, people seem still to be brooding about the best way of making quantum mechanics relativistic. That, at least, is the burden of a long article by A. Kyprianidis, from the Poincaré Institute in Paris, now published (*Physics Reports* **155**, 1; 1987). It is only appropriate that the issue should have been raised from an address in Paris; the French physicist Louis de Broglie was always slightly alienated from the mainstream in the 1920s and 1930s by his insistence that there is something odd about the treatment of time, in the Schrödinger equation and elsewhere in elementary quantum mechanics.

The problem is simply put. The time-dependent form of Schrödinger's wave equation is a linear differential equation for the wave function ψ in which the first derivate of ψ with respect to time is equated with the effect on ψ of an operator that includes second-order differentials with respect the the space coordinates. For a single particle, the space-derivative side of the equation represents the Hamiltonian of the particle, H, the sum of its kinetic energy and potential energy and that the time-derivative side of the equation represents the energy, E, which amounts simply to $H\psi = E\psi$.

The simple rule for arriving at the differential equation is that the kinetic energy part of the Hamiltonian should be written in terms of momentum $\mathbf{p}$, or $\mathbf{p}^2/2m$, where m is the mass, whereupon each of the three components of momentum must be replaced by the differential operator $\hbar/i.d/dx$, where x is one of the three space coordinates, while E is replaced by $-\hbar/i$. (In all this, i is the square root of -1.)

Plainly, the time and space coordinates are not dealt with on an equal footing, and so Schrödinger's equation cannot be relativistic. De Broglie was well within his rights to protest that time is dealt with as if it were simply a parameter describing the evolution of the system the equation represents, although his assertion that Schrödinger's equation says nothing of the mutual uncertainty in measurements of time and energy is more disputable.

Historically, there was no obvious way to generalize Schrödinger's equation. The essential difficulty is that special relativity relates the energy of a free particle to its momentum in a more complicated fashion than in classical mechanics. Indeed, the energy of a free particle is $c\sqrt{(\mathbf{p}^2 + mc^2)}$, which makes for an awkward differential equation when appropriate partial differential operators are substituted for the components of the momenta.

Unfortunately, the obvious way round the difficulty, that of deriving a differential equation by writing $E^2 = c^2\mathbf{p}^2 + m^2c^4$ and substituting differential operators as above, leads to the Klein–Gordon equation, which suffers from several defects, not least that it will not allow of the simple interpretation that the square of the magnitude of ψ should represent the probability density, if only because this quantity may be negative.

Dirac's achievement was his demonstration that Schrödinger's equation can nevertheless be generalized by analogy into a form in which both the space and time derivatives, representing momenta and energy respectively, appear only to the first order. One penalty (but, of course, it is the prize) is that such an equation compels the existence of Pauli's electron spin as well as the electron states with negative energy which, in the sequel of Dirac's argument, led to the prediction that positrons should be holes in this limitless sea of negative electrons.

That seems part of the incentive for what Kyprianidis has attempted, although Dirac rates a mention only once (in the subtitle to de Broglie's book appearing among the references). The argument is nevertheless instructive, even in parts entertaining. The stated purpose is to stay with the ordinary quantum mechanics of the kind described by Schrödinger, but to do so relativistically, but to avoid using the language of quautum field theory.

In short, the objective is to be able to represent the states of a single particle by wave functions ψ, to represent stationary states of a single particle by particular wave functions (which must be function of the space and time coordinates) and to require, as in the familiar Schrödinger calculus, that these stationary states are all orthogonal to each other when integrated over all coordinates from $-\infty$ to $+\infty$.

The essential trick is the definition of an evolutionary parameter distinct both from the ordinary or 'fourth' coordinate and from the proper time of the particle. The progress of a particle through space and time is represented by a stochastic process (strictly, a Markov process) in which the state of affairs at one value of the evolutionary parameter is determined by the state of affairs at earlier and/or later values (allowing for such happenings as the creation and the annihilation of the particle).

The result is a Schrödinger equation in which the wave functions have five coordinates (three representing space, one time and one evolutionary direction). Loyal to his fellow city-dweller de Broglie, Kyprianidis claims that the objections of de Broglie to the wave equation are overcome. He says that the "advantages of this formalism [over what?] are immense", but also acknowledges that "explicit solution of open problems in the frame of this formalism will be the best proof of its validity".

There are some points in the argument, however, when a suspension of disbelief is necessary. For example, to arrange for the orthogonality of the wave equations representing stationary states, it is necessary to require that they should vanish at infinity in all direction, in time as well as space. Innocents will find themselves asking just what physical significance there can be in the concept of a free and isolated particle whose wave function gives zero probability of existance on all sufficiently distant space-like surfaces, let alone in the use of such a set of functions, however orthogonal they may be, to represent more general states of existence.

Sadly, what starts off as an appealing argument becomes, before its end, more than a little contrived and, in the process, introspective, as if the author is writing principally for the people whom he knows will understand what he intends. To say this is not to complain that the mathematics becomes more complicated as the pages turn, but that there is a hidden agenda — to find a way off the awkward conceptual hooks of quantum mechanics.

The sense one has of being let down by articles of this kind derives from an easily identified deficiency — Kyprianidis, while referring to the Klein–Gordon equation as an earlier (but unsatisfactory) landmark on the road to relativistic quantum mechanics, fails to say what he thinks is wrong with Dirac's equation or, for that matter, with the development of field theory in the past forty years (although Feynmann does win a reference in his own right). Instead, his problem is presented to innocent readers as one that is about to be tackled properly for the first time. It is forgiveable that authors should write like that, but should not editors ask that their readers should be given a little help?

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