Mathematics

The battle of the biquadrates

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ADVANCES in mathematics take many forms. Entire new subjects can arise from a single brilliant idea. At the other extreme, long-standing problems can crumble under a sufficiently powerful attack. The metaphor is apt: research and war have much in common. Some problems are stormed by force of arms and superior generalship, some are devastated by new weaponry, some exist in a permanent state of siege. And some surrender only after a lengthy war of attrition.

Waring's problem (see my News and Views article in *Nature* **323**, 674; 1986) falls into this last category. In his *Meditationes Algebraicae* of 1770, Edward Waring stated without proof that every positive integer is a sum of at most 9 cubes, 19 biquadrates, "and so on". One case of Waring's problem, that for biquadrates — fourth powers — has finally been resolved by the collaborative work of Ramachandran Balasubramanian at the Institute of Mathematical Sciences, Madras and Jean-Marc Deshouillers and François Dress at the University of Bordeaux. In two short notes (*C.r. Acad. hebd. Séanc. Sci., Paris* **303**, 85–88; 1986 and **303**, 161–163; 1986) they show that, for fourth powers, Waring was right.

Waring was presumably led to his conjectures by numerical experiments. For example the number 79 can be written as $4\times 2^4 + 15\times 1^4$, involving 19 fourth powers in all. Because the only fourth powers less than 79 are 1 and 16 it is very easy to see that 79 cannot be written using 18 fourth powers or fewer. Thus the maximum must be at least 19 fourth powers, and it remains to show that no number requires more than 19. In 1974, H.E. Thomas (*Trans. Am. math. Soc.* **193**, 427–430; 1974) showed that at most 22 fourth powers are needed. Thus the correct number lies somewhere between 19 and 22. The new work reduces the upper bound from 22 to 19, closing the gap completely.

Proof

The proof falls into two distinct parts. In the first, it is shown that every sufficiently large integer is a sum of 19 fourth powers. In the second, the remaining cases are disposed of. This is often a sound strategy in number theory, for reasons that would be not unfamiliar to a sociologist: small numbers often behave in exceptional ways, whereas large numbers are usually more predictable. (Against this remark we must set the celebrated proof that all numbers are exceptional; if not, there would be a smallest unexceptional number, which, by its very definition, would be exceptional.)

Such is the strategy, but it is not always easy to carry it out successfully. In the case of Waring's problem for biquadrates, sufficiently large means "greater than 10^{367}". That is, Balasubramanian and his colleagues prove by one method that every number with 367 digits or more is a sum of 19 fourth powers, and then deploy new forces to mop up the stragglers with a mere 366 digits or less.

The first step may be considered classical: it is the circle method of G.H. Hardy, J.E. Littlewood and I.M. Vinogradov. The original number-theoretical question is turned into one in complex analysis, and solved by powerful analytical techniques. To understand how the analysis enters, it is convenient to start with an easier problem: Lagrange's theorem that every positive integer is a sum of four squares. This is just 'Waring's problem' for squares, but Lagrange got there first. Consider the generating function $(1+z+z^4+z^9+z^{16}+z^{25}+\ldots)^4$. Expand this as a power series in z. Then a little thought shows that the coefficient of z^N is precisely the number of different ways in which N can be written as a sum of four squares. If it were possible to prove, analytically, that every such coefficient is greater than zero, then we would have an analytical proof of Lagrange's four-squares theorem.

Waring's problem for fourth powers can be attacked similarly, using the generating function $(1+z+z^{16}+z^{81}+z^{625}+\ldots)^{19}$. The coefficient of z^N now gives the number of ways in which N can be written as a sum of 19 fourth powers, and again we have to show it is greater than zero. To pick out a single coefficient we effectively apply Fourier analysis. Multiply the generating function by z^{-N}; replace the variable z by $e^{i\alpha}$, a general point on the unit circle in the complex plane; and integrate round the circle with respect to α. All terms vanish except one — this is what makes Fourier analysis tick — and that term is the required coefficient. In brief, there is a rather complicated integral whose value is the number of ways in which N can be represented as a sum of 19 fourth powers, and we have to prove that it is greater than zero.

The Hardy–Littlewood–Vinogradov circle method tackles this by dividing the circle into two subsets, the major and minor arcs. On the major arcs, α can be well approximated by a rational number with small denominator; on the minor arcs it cannot. It is then shown that the integral over the major arcs is so large that the integral over the minor arcs cannot cancel it out. Thus the total integral is non-zero.

The method is not easy to use: the choice of major and minor arcs is crucial, and less than obvious, and it is not straightforward to estimate the sizes of the two integrals. In the earlier work of Thomas, the estimation of the integral over the major arcs took 40 pages and several computer calculations. Balasubramanian *et al.* use an idea from probability theory to obtain, in only half a page, an estimate that is almost as accurate — and good enough for the proof to be carried through. Their main effort is expended on the minor arcs, and while it follows the traditional strategy it contains several innovations. In particular they use an idea of R.C. Vaughan (*Camb. Tracts Math.* **80**, 1981), in which the traditional choice of major arcs is modified by making them slightly larger: this makes the difficult minor arcs smaller and simplifies some of the arguments.

Victory

This completes step one: every integer with more than 367 digits is a sum of 19 fourth powers. The large number of digits may seem surprising, but it is necessary to make the estimates valid. Large numbers like this are common in analytical number theory because it is often $\log(N)$ or even $\log(\log(N))$ that must be large rather than N itself.

Entirely different weaponry is now brought to bear to mop up the stragglers. In fact the authors give themselves some room to manoeuvre by proving that every integer with less than 378 digits is a sum of 19 fourth powers. Of course, in principle this could be achieved by trial and error on a computer, but the number of digits is so great that the battle of the biquadrates would be interrupted by Armageddon.

So something more subtle is needed. The main idea is to find a sufficiently big set of numbers which, save for rare exceptions, are sums of a mere five fourth powers. These provide a bridgehead from which an invasion can be mounted on the remaining numbers, by deploying only a further 14 fourth powers. Finding this set of numbers took 150 hours on a mainframe computer and needed special tricks to reduce the demands on memory capacity. The invasion on the remaining numbers uses a variation of the 'greedy algorithm': start by subtracting the largest fourth power available and work on what is left.

Thus the 216-year war of attrition against Waring's problem for fourth powers has ended in victory. The battle of the biquadrates is won, VB Day is declared, mathematics is triumphant. But what of cubes, fifth powers, sixth powers . . ., googolplexiquates? The greater conflict continues. Cry 'havoc!' and let slip the dogs of Waring. □

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