are not systematically losing (or gaining) galaxies at high redshift because of photometric errors or uncertainties, then Loh and Spillar have succeeded in measuring the volume element as a function of redshift. It does not matter whether the mass in the Universe is luminous or dark. This measurement of the geometry of the Universe yields a determination of $\Omega = 0.9^{+0.7}_{-0.5}$ (95 per cent confidence limits) and is consistent with the inflationary prediction of an Einstein–de Sitter cosmology ($\Omega = 1$).

This result is especially noteworthy because it is the first, seemingly unambiguous astronomical determination of Ω that yields a value of unity on large (~ 1,000 megaparsecs) scales. Before accepting this as the new cosmological gospel, however, notes of caution have been urged. Redshift errors may be systematically underestimated because the 4,000-Å spectral indicator has only been calibrated against cluster galaxies, whereas this feature is known to be weaker for the field galaxies that dominate the Loh–Spillar sample (Richard Ellis, University of Durham, personal communication). There may be systematic errors in photometry that make the smaller, more distant galaxies appear brighter relative to the nearby galaxies, thereby mimicking the effect of high Ω (S.D. Jorgovski, personal communication). Dust in distant galaxies may also affect the photometric redshift determinations. These effects need to be critically evaluated, presumably by time-consuming spectroscopic studies, before the $\Omega = 1$ result can be considered to be definitive. For now, however, it remains a tantalizing possibility: perhaps observational cosmologists (and other astronomers) should pay more heed to theorists. □

Joseph Silk is in the Astronomy Department at the University of California, Berkeley, California 94720, USA.

Mathematics

The Waring experience

from Ian Stewart

In 1770 Joseph-Louis Lagrange proved a long-sought result in number theory, the four-squares theorem: every whole number is a sum of four perfect squares. Earlier that year Edward Waring, Lucasian Professor of Mathematics at Cambridge, had published his *Meditationes Algebraicae*. It contained a far-reaching generalization, stated without proof: every whole number is a sum of 9 cubes, 19 fourth powers "and so on". The implied conjecture, that for all n the number of perfect nth powers needed to represent any whole number is finite, became known as Waring's problem. The finiteness was proved by David Hilbert in 1909 using an identity in 25-fold multiple integrals, but his method did not yield the actual number of nth powers required. Other methods have been developed to estimate their number, and various workers have improved the upper bounds until they have become quite respectable. R.C. Vaughan has now obtained substantial improvements to these bounds when n lies between 5 and 9 (*Proc. Lond. Math. Soc.* **52**, 445; 1986 and *J. Lond. Math. Soc.* **33**, 227; 1986).

More precisely, let $g(n)$ be the smallest number such that every whole number is a sum of $g(n)$ nth powers. Then Lagrange's result shows that $g(2) \leq 4$. Because the number 7 actually requires all four squares ($7 = 1^2+1^2+1^2+2^2$ is the best we can do), $g(2)$ is exactly 4. Waring's conjecture is that $g(3) = 9$, $g(4) = 19$ and that $g(n)$ exists and is finite for all values of n.

He almost certainly hit on these numbers by trial and error. A search of all numbers up to, say, a few thousand will show that at most 9 cubes appear to be needed; and only 2 numbers, 23 and 239, require all 9. In fact only 15 numbers seem to need 8, the largest of these being 454. In *An Introduction to the Theory of Numbers* (4th edn; Oxford University Press, 1960), G.H. Hardy and E.M. Wright comment as follows: "It is plain, if this be so, that 9 is not the number which is really most significant in the problem. The facts, if they be facts, that just 2 numbers require 9 cubes, and just 15 require 8, are, so to say, arithmetical flukes... The most fundamental and most difficult problem is that of deciding, not how many cubes are required for the representation of all numbers, but how many are required for the representation of all large numbers, that is, of all numbers with some finite number of exceptions".

Thus there is a second, deeper quantity, called $G(n)$, defined to be the smallest number of nth powers needed to represent any whole number, with finitely many exceptions. The "facts, if they be facts," imply that $G(3)$ is 7 (or less) rather than 9. Because no number of the form $8k+7$ can be a sum of three squares, $G(2)$ is the same as $g(2)$, namely 4. But in general $G(n)$ will be smaller than $g(n)$.

Many authors have studied the functions $g(n)$ and $G(n)$, either for special values of n or for general ones. Thus, in 1942 Linnik showed that $G(3) \leq 7$ (so that the "facts, if they be facts" be facts). Between 1920 and 1928 Hardy and Littlewood developed an approach by way of complex function theory, and proved that $G(n)$ is never larger than $(n-2)2^{n-1}+5$. They improved this to yield $G(4) \leq 19$, $G(5) \leq 41$, $G(6) \leq 87$, $G(7) \leq 193$, $G(8) \leq 425$ and so on. By comparison the best reults known by 1930 for $g(n)$ were 37, 58, 478, 3,806 and 31,353, respectively.

Vinogradov introduced powerful methods based on estimating trigonometric sums (a standard approach in analytic number theory but not an easy one to carry through), and in 1936 Heilbronn simplified the method to yield

$$G(n) \leq 6n\log n + \{4+3\log(3+2/n)\}n+3$$

whence $G(7) \leq 137$, $G(8) \leq 163$. A list of various subsequent improvements is presented in the table (for references see the works cited above). Only for $G(4)$ is the current bound known to be best possible.

Vaughan's methods are related to those of Hardy and Littlewood and Davenport, and involve very careful and delicate analytical estimates. They have applications to other questions in number theory, and may also shed light on Waring's problem for tenth and higher powers.

n	4	5	6	7	8	9		
$G(n) \leq$	19	41	87	193	425		Hardy & Littlewood	(1928)
				137	163		Heilbronn	(1936)
	16*	23	36				Davenport	(1939, 1942)
				53	73		Narasimhamurti	(1941)
		22	34	50	68	87	Thanigasalam	(1982)
		21	31	45	62	82	Vaughan	(1986)

*$G(4) = 16$ exactly.

Although number theory may not be the most directly useful part of mathematics, it exerts an endless fascination because of the gulf between the simplicity of its raw materials and the complexity of its methods. There is some practical payoff too (see, for example, M. Schröder's *Number Theory in Science and Communications* 2nd edn; Springer, New York, 1986). It is remarkable that a problem posed more than two centuries ago, based on simple empirical observations, continues to exercise the ingenuity of number theorists and to generate new mathematics. The Waring experience is far from over. □

Ian Stewart is in the Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK.