at least partially non-autonomous in genetic mosaics.

Although large-scale non-autonomy has not been observed in the analysis of segmentation phenotypes, the sample of loci tested so far is relatively small[15]. Also, it is not clear whether the techniques presently available for marking cells in *Drosophila* are sufficiently accurate to exclude the limited non-autonomy needed to explain the discrepancy between the phenotypes and transcript distributions. However, genetic mosaics ultimately provide the most direct technique for relating phenotypes to individual cell requirements, and the variety of mosaic techniques available in *Drosophila* should provide a better understanding of the relationship between the gene products and the patterning process. □

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Mathematics

The class number problem

from Ian Stewart

SOME of the deepest mathematical theories have been built from very humble raw materials, and a classic case is number theory, which studies properties of ordinary whole numbers 1,2,3,... . It has long been remarked that, although it is easy to observe patterns in the behaviour of whole numbers, it can be astonishingly hard to prove that these patterns are generally valid. Number theorists have long been interested in the arithmetic of 'imaginary quadratic fields', systems of numbers in the form $x+y\sqrt{-D}$, where D is a whole number and x,y are rational.

Any whole number can be factorized into primes, and there is an analogous concept for such fields. For whole numbers, the prime factorization can be achieved in only one way. This is not always true for imaginary quadratic fields; and in his *Disquisitiones Arithmeticæ* of 1801, the fount of all modern number theory, Carl Friedrich Gauss in effect conjectured that unique factorization holds precisely when D = 1, 2, 3, 11, 19, 43, 67 and 163. This conjecture was proved more than a hundred years later by A. Baker (*Mathematika* **13,** 201; 1966) and independently by H.M. Stark (*Michigan Math. J.* **14,** 1; 1967). But Gauss also stated a more general conjecture, the class number problem, the solution of which by D.M. Goldfeld, B. Gross and D. Zagier has recently been described in a masterly survey by Goldfeld (*Bull. Am. Math. Soc.* **13,** 23; 1985).

The classic example of non-unique factorization occurs in the case $D = 5$. Here $6 = 2.3 = (1+\sqrt{-5})(1-\sqrt{-5})$ exhibits two distinct prime factorizations. (Each of the numbers 2, 3, $1+\sqrt{-5}$ and $1-\sqrt{-5}$ is prime in this particular number field.) The extent to which factorization is not unique can be measured by a quantity called the class number, h: when $h=1$, factorization is unique, and for larger values of h it is not unique. In some sense the larger h becomes the less unique factorization is. For a given D it is relatively easy to calculate the class number (for example, if $D = 5$ then $h = 2$) but the list of values so obtained varies in a very irregular way with D. This makes it hard to solve the inverse problem: given a value for h, find which Ds have that class number. And this is what Gauss's conjectures are about. The conjecture on unique factorization asks for a determination of all D such that the class number is 1, with the guess that they are precisely the 9 values listed above. The general class number problem is to prove that for any given class number h, the list of D having that value for h is finite.

Gauss actually phrased his conjectures in a slightly different setting, the theory of quadratic forms. This goes back to the time of Pierre de Fermat, who stated theorems of the following kind: "Every prime of the form $6n+1$ can be written as x^2+3y^2 for integers x and y." For example, 31 is such a prime, and $31 = 1^2+3.3^2$. The connection with quadratic fields is that $x^2+3y^2 = (x+y\sqrt{-3})(x-y\sqrt{-3})$, a factorization in the field corresponding to $D = 3$. So theorems about quadratic fields carry implications for quadratic forms, and conversely.

Gauss, and Joseph-Louis Lagrange before him, realized that it is not necessary to study all possible quadratic forms, because changes of variable can be used to turn one form into another. For example, if x and y are replaced by $x+2y$ and $x+y$, respectively, then the form x^2+3y^2 becomes $4x^2+10xy+7y^2$. Therefore, the numbers that can be represented in the form x^2+3y^2 are precisely those that can be represented in the form $4x^2+10xy+7y^2$, even though at first sight these are different questions. This led Gauss to the idea of 'equivalent' forms (transformable into each other by such changes of variable). He proved that there is only a finite number of distinct classes of equivalent forms, which is the class number of the associated quadratic field.

H. Heilbronn and E.H. Linfoot (*Q. Jl Math.* **5,** 293; 1934) showed that, apart from the nine known cases, at most one further field has class number 1. In 1952 Kurt Heegner claimed a proof that no such tenth field exists, which would solve the class number 1 problem. But as Goldfeld remarks: "Heegner's paper contained some mistakes and was generally discounted at the time. He died before anyone really understood what he had done."

Baker's proof uses new results from the theory of transcendental numbers (numbers that do not satisfy any polynomial equation with rational coefficients), whereas Stark's proof is much closer in spirit to Heegner's attempt. Then M. Deuring (*Inventiones Mathematicæ* **5,** 169; 1968) showed that the 'gap' in Heegner's attempted proof can be filled relatively painlessly. In 1971 Baker (*Ann. Math.* **94,** 139) and Stark (*Ann. Math.* **94,** 153), again independently, solved the class number 2 problem, finding exactly 18 values of D for which $h = 2$.

Although these results represent enormous progress compared with what was previously known, it is clearly not feasible to tackle the general problem one class number at a time, because the process will never end. The interest of such partial answers is that they suggest new methods. Goldfeld (*Astérisque* **41–42,** 219; 1976) exploited a connection between the class number problem and an elliptic curve, a type of cubic equation. He showed that if just one elliptic curve can be found with a particular property, then the class number problem is solved. Unfortunately, no such elliptic curve was then known. But later, Gross and Zagier (*C.r. hebd. Séanc. Acad. Sci., Paris* **297,** 85; 1983) established the necessary property for the elliptic curve $-139y^2 = x^3+4x^2-48x+80$. By such roundabout methods was Gauss's conjecture on the finiteness of the list of imaginary quadratic fields of given class number demonstrated.

Soon afterwards, J. Oesterlé (*Séminaire Nicolas Bourbaki* 631; 1984) obtained a specific estimate for the size of the largest D with given class number h, making is possible in principle to select values of h and find all possible Ds. As a result, the class number 3 problem (precisely which D gives $h = 3$?) has been solved, and many long-standing problems in number theory begin to appear tractable. For example, a solution of the class number 4 problem would answer a famous question: which integers can be expressed as a sum of three squares in exactly one way? The powerful and beautiful ideas laid bare by a 180-year assault on Gauss's conjecture strikingly attest to the quality of his insight into the deep properties of ordinary whole numbers. But the sad story of Kurt Heegner shows that genius is still not always appreciated during its lifetime. □

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