

Mathematics

How bent is a knot?

from Ian Stewart

Anyone who has fiddled about with one of those lengths of plastic-coated spring from which curtains are sometimes hung will have noticed that it is harder to bend the spring into a knotted loop than it is into an unknotted one. Since elastic energy is concentrated on curves, this suggests that a knot is in general more bent than a simple, unknotted loop. The natural context in which to ask whether any mathematical sense can be made of this observation is differential topology, a hybrid creation in which topological notions, such as knottedness and connectedness, are blended with analytical and differential-geometric ideas such as curvature and smoothness. It is one of the deeper areas of today's mathematics, with important successes and applications to its credit. Recent research by N.H. Kuiper and W. Meeks (*Inventiones Mathematicae* 77, 25; 1984) has uncovered some fundamental relations between the topology of a multi-dimensional space (or manifold) and its metric properties, notably curvature.

The notion of curvature goes back to C.F. Gauss in 1828. He defined curvature for surfaces, proved it was an intrinsic property, not dependent on the specific embedding of the surface in 3-space, and applied it to geodesy. It was generalized by B. Riemann to n -dimensional manifolds, and used by A. Einstein as a basic concept in general relativity. The earliest result relating curvature and topology is the Gauss-Bonnet Theorem: the total Gaussian curvature of a surface is $2\pi C$, where C is its Euler characteristic. This is a topological invariant, equal to $2-2g$, where g is the number of 'holes' in the surface. Roughly speaking, the Gauss-Bonnet theorem says that to make holes in a surface you have to put more bend into it. Its importance may be gauged from a remark in M. Spivak's *Differential Geometry* (a classic 5-volume text, described by its author as "The Great American Differential Geometry Book", published by Publish or Perish Inc.). Spivak says: "A whole slew of consequences follows immediately from this spectacular theorem, which expressed a differential geometric quantity in terms of a number which has nothing at all to do with curvature, or even with a Riemannian metric."

Opening remarks about the curvature of a knot can be made precise in a similar way. In 1949, J. Milnor and I. Fary proved that the total curvature of a knot is greater than 4π . For a planar convex loop, the total curvature is exactly 2π . The ordinary trefoil, or overhand knot, can be embedded in 3-space with a total curvature as close to 4π as required, so 4π is the 'best possible' bound.

Instead of knotting a loop in 3-space, topologists are extremely interested in knotting n -dimensional manifolds in N -space. Again, the intuitive expectation is that a knotted surface will have more bend in it than an unknotted one. The amount of bend is once more measured by the total curvature, which is 4π for a sphere and 0 for a torus (in their natural embeddings). Kuiper and Meeks prove a general result which implies, for example, that a knotted torus in 3-space must have total curvature greater than 16π .

To be specific, they relate the total curvature of an n -dimensional manifold M embedded in N -space, to two topological invariants. One of these depends only on the manifold M : it is a relative of the Euler characteristic, and is the sum, β , of the Betti numbers of M . The other depends on the way M is embedded, and measures the extent to which M is knotted: it is σ_1 , the first homotopy excess. (Space limitations preclude a precise explanation of either invariant.) Kuiper and Meeks show that the total curvature must be at least $\frac{1}{2}V(n)(\beta + 4\sigma_1)$, where $V(n)$ is the 'surface area' (or total curvature) of the n -dimensional unit sphere. For example $V(1) = 2\pi$,

$V(2) = 4\pi$, $V(3) = 2\pi^2$, $V(4) = 8\pi^2/3$. There are, however, two exceptional pairs of dimensions. If $n = 1$ (M is a curve) then 4_1 must be replaced by 2_1 . And if $n = 3$, $N = 4$, then v_1 is replaced by the maximum of two similarly defined quantities, calculated for the interior and exterior of M . In these exceptional cases the results are due to R.H. Fox and P. Wintgen.

The proofs rely heavily on new techniques, including Morse Theory (the study of critical points such as maxima, minima, and saddles), the handle decomposition due to S. Smale, the notion of a tight embedding, and Meeks's work on surfaces known as multipants because of their resemblance to generalized trousers for multilegged creatures. The argument is delightfully geometric, as is most of the deeper work in differential topology.

Much of the power of mathematical reasoning derives from its ability to relate concepts that had not been seen to have any connection with each other. The curvature properties of knotted surfaces are an intriguing example of this process and part of a continuing drive to apply metric concepts, such as curvature, to the topology of manifolds. More detailed results along similar lines cannot be far away; meanwhile the view of Gauss and Riemann that curvature is a fundamental property of a space is being vindicated with a vengeance. □

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Molecular genetics

Diverged genetic codes in protozoans and a bacterium

from Thomas D. Fox

THE genetic code, which is the same in *Homo sapiens* as in *Escherichia coli*, is widely thought to have been fixed prior to the divergence of all life forms and so to be universal. Since a mutation changing a codon assignment would effectively cause thousands of mistakes in the proteins translated, it has seemed quite reasonable to assume that such mutations would be lethal. Although studies on mitochondrial genes have shown that the code is not strictly universal, mitochondria could be thought of as exceptions that prove the rule: their genetic systems produce only a very limited number of proteins and so might tolerate changes. Now, some 'real' exceptions have come to light in both eukaryotic and prokaryotic free-living organisms, and the notion of universality will have to be discarded.

In the erstwhile universal code, the codons UAA and UAG mark the termination of translation of a sequence into protein. But in three different ciliated protozoans there is now a strong indication that UAA and UAG encode the amino acid

glutamine. S. Horowitz and M.A. Gorovsky (*Proc. natn. Acad. Sci. U.S.A.* in the press) have determined partial sequences of two genes for histone H3 in the macronucleus of *Tetrahymena thermophila*. (Ciliated protozoans contain a 'somatic' macronucleus and a 'germ-line' micronucleus.) The gene sequences predict the known amino acids of the histone except that each sequence contains two UAA codons at positions that correspond to glutamine in the protein. Thus, UAA must be translated unless both cloned sequences are derived from pseudogenes, which are not translated. This seems unlikely since all 12 mismatches between the two partial H3 sequences produce silent changes. Southern blot experiments show that there are a maximum of four H3 genes in *T. thermophila*, at least two of which must be active since there are two variant forms of the protein in this organism. Thus, while it is possible that the two sequenced H3 genes are silent copies, it is much more likely that UAA actually encodes glutamine.

This conclusion is supported by com-