Mathematics

Beyond reasonable doubt?

from Ian Stewart

WHEREAS the mathematical discipline of Number Theory deals with properties of whole numbers, that of Analysis is concerned with the continuous properties of real and complex numbers. Despite this difference, Bernhard Riemann, in 1859, discovered profound connections between the two, leading to what is now known as Analytic Number Theory. Riemann developed a theory of the zeta function, defined for real $s > 1$ by

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \ldots$$

and for complex s by analytical continuation, and showed how it could be used to study the distribution of prime numbers (*Monatsber. Akad. Berlin*, 671; 1859). In particular, the number of primes in a given interval could be estimated provided enough was known about the zeros of the zeta function: values s such that $\zeta(s) = 0$. There are some relatively obvious zeros at $s = -2, -4, -6, \ldots$. Riemann conjectured that all the remaining zeros lie on the complex line $s = \frac{1}{2} + iy$ where y is real. This, the Riemann Hypothesis, if true, has extensive repercussions — not all within Number Theory. It remains an intriguing and important unsolved problem.

Computer calculations, based on various known features of the zeta function, have shown that Riemann's hypothesis holds good for the first 320 million zeros (Edwards, H.M. *Riemann's Zeta Function*, Academic, New York; 1974). This would appear to be strong evidence in favour of the hypothesis — but is it? An analogous question is illuminating. In 1885 T.J. Stieltjes came up with a different conjecture which would, if true, imply the Riemann Hypothesis (*C. r. hebd. Seanc. Acad. Sci., Paris* **101**, 368). It was proposed again by F. Mertens in 1897, after whom it is now named (*Sber. preuss. Akad. Wiss.* **106**, 761). It can be stated as follows. For any integer n, let $M(n)$ be the difference between the number of integers less than n that are products of an even number of distinct primes and the number that are products of an odd number of primes. The Mertens Conjecture is that $M(n) < \sqrt{n}$ for $n > 1$. For example when $\sqrt{n} = 16$ there are five numbers in the 'even' case (1,6,10,14,15) and six 'odd' (2,3,5,7,11,13), so $M(16) = 1$ which is certainly less than 4. The Mertens Conjecture has been verified by computer calculations for the first 10 billion values of n.

Despite the apparently overwhelming evidence in its favour, it has recently been discovered that the Mertens Conjecture is false. Andrew Odlyzko (Bell Laboratories) and Herman te Riele (Centre for Mathematics and Computer Science, Amsterdam) have shown in unpublished work, based upon fast computer techniques, that there exist infinitely many values of n for which the conjecture fails. The first 'bad' n is no larger than 10 to the power 10 to the power 70 (or thereabouts), an order of magnitude totally inaccessible to direct computation. It is possible that an explicit calculation of the actual 'bad' values would not be an improvement on the demonstration that they must exist.

The Riemann Hypothesis is unaffected by this result: it is still unsolved. Experts have long felt that to prove it by way of the Mertens Conjecture would be 'too easy', and their suspicions are now vindicated. Meanwhile, if the first 10 billion numbers are conspiring to mislead us about the Mertens Conjecture, what faith can we place on a mere 320 million in favour of the Riemann Hypothesis? □

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Astronomy

IRAS circular 11

THE sources in this circular all have a firm detection at 60 μm, are at least 20° away from the galactic plane and have continua representative of 'warm' galaxies. Each search box has been inspected on SERC/ESO Schmidt survey plates or on National Geographic/Palomar Schmidt plates, and contains an obvious candidate, in most cases a galaxy. Selection of sources and the optical search were done by G.K. Miley and M.H.K. de Grijp, Sterrewacht Leiden, The Netherlands. Source names derived as before; see *Nature* **309**, 480; 1983. Position is given at equinox 1950.0. Measurements were made between epochs 1983.1 and 1983.9. In the table, a m denotes arc min.

Also now available, but not to be published in *Nature* is IRAS circular 10 which lists 179 sources selected from the IRAS minisurvey of an area of approximately 300 square degrees between 9 and 16 February 1983 (see Rowan-Robinson *et al. Astrophys. J. Lett.* **278**, L7; 1984). The sources satisfied the criteria of being more than 20° away from the galactic plane, reproducible on successive scans and with a signal-to-noise ratio of greater than 9:1 in at least one of the wavelength bands.

Source IRAS	RA h min s	Dec deg a m	Flux density (Jy) 12 μm	25 μm	60 μm	100 μm
0425−072P11	04 25 22.2	−07 15	<0.4	0.4	0.9	1.3
0425−046P11	04 25 57.1	−04 40	<0.2	1.6	4.5	4.3
0428−097P11	04 28 11.0	−09 44	<0.3	0.4	0.7	<1.4
0432−143P11	04 32 32.5	−14 19	1.2	3.8	7.9	11.3
0438−084P11	04 38 28.8	−08 28	0.5	1.8	3.4	2.8
0450−032P11	04 50 14.1	−03 17	<0.6	0.5	1.0	1.5
0450−184P11	04 50 40.8	−18 26	<0.2	<0.5	0.9	2.5
0509−024P11	05 09 03.8	−02 26	0.3	1.2	2.0	<1.5
0513−235P11	05 13 44.2	−23 31	<0.2	<0.5	0.9	3.1
0521−122P11	05 21 47.0	−12 12	<0.2	0.4	0.6	<2.8
0531−206P11	05 31 57.0	−20 36	<0.2	<0.4	1.2	2.0
0556−348P11	05 56 31.9	−34 53	<0.7	<0.3	0.5	<1.3
0611−326P11	06 11 30.1	−32 40	<0.2	0.4	0.9	1.5
0815+035P11	08 15 18.0	+03 31	<0.4	<0.3	0.7	<1.7
0818+033P11	08 18 49.8	+03 19	<0.2	<0.4	0.8	2.0
1036−190P11	10 36 39.5	−19 04	<0.4	<0.3	0.6	<1.4
1051−273P11	10 51 09.1	−27 22	<0.2	0.4	1.0	<1.2
1105−115P11	11 05 48.9	−11 31	<0.2	0.4	0.8	<1.4
1119+045P11	11 19 55.6	+04 31	<0.7	0.5	0.9	2.7
1121−281P11	11 21 33.3	−28 06	<0.4	0.4	0.7	<0.8
1246−111P11	12 46 53.3	−11 07	<0.2	0.8	1.7	2.1
1249−131P11	12 49 35.1	−13 08	<0.2	<0.5	1.3	2.8
1304−234P11	13 04 23.5	−23 24	0.4	1.3	2.6	4.1
1305−241P11	13 05 59.1	−24 07	<0.2	0.7	1.6	2.1
1315−098P11	13 15 31.4	−09 49	<0.2	<0.5	0.9	<1.6
1316−242P11	13 16 49.3	−24 13	<0.2	<0.4	0.9	<2.1
1319−164P11	13 19 42.3	−16 27	0.8	2.9	6.3	7.1
1320−342P11	13 20 44.8	−34 15	<0.4	<0.3	0.6	<1.7
1329+022P11	13 29 19.7	+02 16	<0.2	<0.6	1.1	1.2
1331−301P11	13 31 28.9	−30 07	<0.2	<0.3	0.8	<1.0
1331−234P11	13 31 51.2	−23 25	<0.2	<0.4	1.0	2.2
1331−231P11	13 31 56.4	−23 11	<0.9	<0.5	1.0	2.0
1333−340P11	13 33 01.8	−34 02	0.4	0.7	1.2	1.5
1354−203P11	13 54 33.1	−20 22	<0.6	<0.8	1.5	2.5
1356−188P11	13 56 16.2	−18 48	<0.3	<0.7	1.4	3.6
1402−316P11	14 02 09.7	−31 40	<0.5	<0.3	0.7	<1.5
1404+012P11	14 04 04.9	+01 17	<0.2	<0.5	1.1	<1.2
1423−116P11	14 23 27.8	−11 40	<0.4	<0.4	0.8	1.6
1428−030P11	14 28 51.4	−03 04	<0.2	<0.4	1.0	2.0
1431−326P11	14 31 42.8	−32 37	<0.3	0.4	1.0	<1.2
1444−219P11	14 44 35.4	−21 56	<0.6	<0.4	1.1	1.8
1458−222P11	14 58 56.7	−22 15	<0.3	<0.4	0.8	2.6
1509−211P11	15 09 06.6	−21 07	<0.4	0.7	1.8	2.0
1524+007P11	15 24 04.5	+00 46	<0.2	0.5	1.0	1.5
1548−037P11	15 48 03.4	−03 44	<0.4	0.8	1.3	<1.7
1618+068P11	16 18 30.1	+06 51	<0.3	<0.3	0.7	<1.2
1832−594P11	18 32 32.8	−59 26	0.6	1.5	3.6	5.6
1833−654P11	18 33 21.8	−65 28	0.8	2.5	2.6	<1.7
1840−624P11	18 40 07.9	−62 25	0.4	1.1	2.2	4.4
1844−532P11	18 44 14.7	−53 12	<0.2	<0.4	0.8	<1.7
1919−421P11	19 19 23.9	−42 06	<0.4	<0.5	1.1	<2.1
1955−140P11	19 55 49.9	−14 05	1.0	1.5	1.3	3.1
1958−183P11	19 58 02.7	−18 18	<0.5	<0.7	1.1	1.4
2037−383P11	20 37 58.7	−38 22	<0.4	<0.5	1.5	<2.2