Applied mathematics

Scattering by fractal objects

from Eric Jakeman

"A FRACTAL is a mathematical set or object whose form is extremely irregular and/or fragmented at all scales." So runs Mandelbrot's definition of the term he coined in an essay first published[1] in French in 1975. He points out that classical mathematics has on the whole been concerned with Euclidean geometry based on the notion of smooth shapes and with the continuous evolving dynamics of Newton, whereas most natural phenomena, objects and patterns are abruptly changing, with structure on many scales. Thus the mathematical description of a whole range of familiar objects ranging from coastlines to trees presents great difficulties. In his essay, Mendelbrot shows how these can be resolved by the concepts of fractal geometry[2].

To reduce the totality of fragmented objects to a class requiring minimal parameterization, it is necessary to introduce the notion of 'scaling', which refers to the similarity in appearance of an object when it is subject to arbitrary magnification. Regular scaling fractals can be generated by following simple iterative construction procedures. The best known object of this kind is the Von-Koch 'snowflake' curve[3] obtained by repeatedly replacing the middle third of each element of the construct by two vertices of the appropriate equilateral triangle (Fig. 1). This leads to an object of finite area but infinite perimeter. Its bounding curve is evidently continuous but does not have a locally well-defined direction (slope or derivative).

Although a number of these strange but regular geometrical constructs are known, the vast majority of natural objects are fragmented in an irregular way and it is in the description of these randomly varying structures that the concepts of fractal geometry are proving most valuable. An early investigation of one such structure is attributed to L.F. Richardson, who found that the length of land frontiers and coastlines appears to increase as the resolution of the measurement improves[2]. Thus, if the length of the coastline of Britain is determined using a 100-km measuring stick, the answer will be smaller than that obtained using a 10-km stick, which will resolve and therefore have to negotiate the longer distance round smaller scale structures such as river estuaries and inlets.

This observation leads quite naturally to the principle parameter characterizing a scaling fractal curve, for it is found that if the length of the measuring stick is l then the apparent length of such a curve is proportional to l^{1-D}, where D is a number greater than unity, but not necessarily integral, which uniquely characterizes the scaling behaviour of the curve. The value of D is in fact a measure of the density with which the curve fills the space (surface, volume) in which it is embedded and can be interpreted as an anomalous or fractal dimension.

Although Mandelbrot's ideas have diffused rapidly, the most striking applications of his work remain the numerical simulations of random fractal objects such as islands and landscapes which, in spite of being entirely artificially contrived, acquire an uncannily familiar appearance when supplemented by suitable shadowing effects[2]. The ability to recognize the familiar in this context is dependent, of course, on information being carried to the eye by light scattered from the object and there is indeed increasing interest in the properties not only of light but of other frequencies of the electromagnetic spectrum as well as of sound waves which have been scattered by fractal objects.

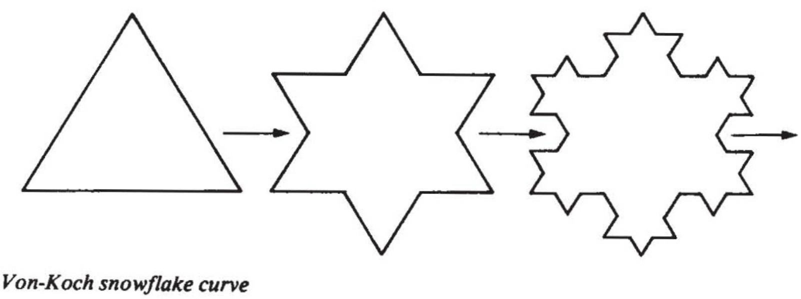
In fact there is a long history of interest in one multi-scale scattering system — the turbulent medium. Although not referred to as such, a fractal description involving the Kolmogorov spectrum has commonly been employed since the pioneering work of Tatarski in the early 1960s[4]. The full implications of using models of this type have only become clear recently, however, through the quantitative investigation of scattering by simple models of fractal surfaces[5]. Since the slope of a fractal surface is ill-defined, ray or geometrical optics do not apply and the scattering is of an entirely diffractive nature. This results in relatively low contrast intensity (brightness) patterns[6] that are very different from those generated by smoothly varying surfaces whose lens-like behaviour produces bright geometrical features of the kind which can often be seen on the floor of a swimming pool.

In the case of pulsed scattering, the fractal dimension of the surface is represented in the temporal decay of the tail of the return pulse, so that this kind of surface should in principle be easily recognizable and capable of characterization by remote sensing techniques[7]. In practice variations of reflectivity may confuse such measurements, while real surfaces will also have smallest (inner) and largest (outer) scale sizes with a hierarchical or fractal structure in between. To investigate the latter regime, the wavelength of the probing radiation has to be much greater than the height fluctuations over inner-scale sized elements of the surface (so that their scattering effect can be neglected by comparison with that of the larger scales) and the receiving optics must be designed to collect radiation from an area of the surface which is smaller than the outer scale size.

Although many solid surfaces of both microscopic and geophysical proportions appear to be fractal[8], at least over some limited range of scale sizes, there is evidence that certain fluid systems are better described as objects with fractal slope. The sea surface is one such system, for example. Ray or geometrical optics effects do play an important role in determining the characteristics of the scattered intensity pattern in this case, leading to strong variations in brightness. Since the local surface curvature remains ill-defined, however, lens-like behaviour does not occur and a new kind of geometrical optics or short-wave limit is found which is free from the catastrophes (caustics, focusing points) associated with smoothly varying surfaces. The statistical properties of the scattered intensity remain finite in this limit and can again be related to the fractal dimension[9].

The full implications of current work on fractal scattering cannot yet be properly assessed but will surely extend beyond the immediate practical objectives of the investigations. Certainly many new and sometimes surprising results are emerging by recognizing the role of fractals in well-established areas of science as well as in the more exotic frontier areas such as dynamical systems and chaos. □

Eric Jakeman is in the Royal Signals and Radar Establishment, St Andrews Road, Great Malvern, WR14 3PS.



The Von-Koch snowflake curve

1. Mandelbrot, B.B., *Les objects fractals: forme, hasard et dimension (Flammarion, Paris, 1975).*
2. Mandelbrot, B.B., *The fractal geometry of nature* (Freeman, San Francisco 1982).
3. Cundy, H.M. & Rollet, A.P., *Mathematical models* (Clarendon, Oxford 1954).
4. Tatarski, V.I., *Wave propagation in a turbulent medium* (McGraw-Hill, New York, 1961).
5. Berry, M.V., *J. Phys.* A *12* 781 (1979).
6. Jordan D.L., Hollins, R.C. & Jakeman, E. *Appl. Phys.* B*31* 179 (1983).
7. Berry, M.V. & Blackwell, T.M., *J. Phys.* A *14*, 3101 (1981).
8. Sayles, R.S. & Thomas, T.R. *Nature* **271** 431 (1978).
9. Jakeman, E. *J. Opt. Soc. Am.* **72**, 1034 (1982).

0028-0836/84/020110-01$01.00