Storm in teacup about Poynting vector

An argument about the ambiguity of the standard expression for energy-flux in the electromagnetic field asks too much of physics.

REFORM is an uncertain business, as Professor C. S. Lai must by now have learned from his attempt two years ago to modernize, even improve upon, the standard expression for the transport of energy by an electromagnetic field. It is now not quite a century since the literature was first adorned with J.H. Poynting's succinct statement of the energy flux of an electromagnetic field

$$\mathbf{S} = \mathbf{E} \wedge \mathbf{H}$$

where **E** and **H** are the vector fields specifying the electric and magnetic field strengths and the symbol connecting them indicates the usual vector product, itself a vector field. For the best part of a century, the Poynting vector has seemed to enjoy a great many advantages, not the least of which is that, in the plane-wave solution of Maxwell's electromagnetic wave equation, **E** and **H** are plainly perpendicular to each other and to the direction in which the wave is travelling and, given the phase relationships between the electric and magnetic vibrations, the energy-flux density at any point works out as a constant vector in a fixed direction.

Reform

Lai's programme of reform for the Poynting vector, first put forward in the *American Journal of Physics* (**49**, 841; 1981) starts from the simple observation that the revered Poynting vector yields an awkward non-zero result when some pattern of entirely static electric and magnetic fields is slotted into the equations. To be sure, the quantities concerned, **E** and **H**, are at some level related by Maxwell's equations to each other and to the patterns of moving electric charges (currents) that there may be, but it seems on the face of things to be offensive that static field should yield non-zero energy fluxes. So why not find some other way of constructing an energy-flux vector that will tie in with Poynting for the obviously crucial plane-wave solution of a vibrating electromagnetic field but otherwise also yield a zero energy flux for a pattern of static fields? That was Lai's programme two years ago.

Formally, the solution is quite straightforward. Given the relationships between the electric and magnetic field strengths, **E** and **H**, and the electromagnetic vector potential **A** and the electrostatic potential ϕ, the definition of the electric current **J** and Maxwell's equations, it turns out that it is always possible to add to the vector field represented by the Poynting vector the expression **curl** ($\phi\mathbf{H}$) or $\nabla\wedge(\phi\,\mathbf{H})$. In many physical respects, the two vector fields are the equivalents of each other, for the train or argument that leads to Gauss's theorem also shows that the integral of the added vector field over the surface of a closed volume (with an appropriate definition of the scalar product of the vector field at the surface and the normal to the unit of surface area) is identically zero. So the modified Poynting vector yields the same description of the transfer of power within the field, but plainly the modified vector can also, by suitable manipulation, also be set equal to zero in those parts of the field in which the electric and magnetic vectors themselves are static.

So is that not a great advantage? The vector field describing the energy flux in an electromagnetic field can be given the desirable property that when the fields are static, unchanging in the course of time, no energy flux need be involved. This feature, among others, seems to have been one of Lai's boasts two years ago. In the most recent issue of the *American Journal Physics* (**50**, 1162, 1165 and 1166; 1982), however, at least three critics take him to task for having overlooked a simple point, the fact that the original form of the Poynting vector is at least unambiguously defined by the electric and magnetic field strengths but that the modified version of the vector, including as it does a term dependent on the vector and scalar potentials of the electromagnetic field, is not unambiguous but, rather, dependent on a suitable gauge.

To put the difficulty another way, just as a scalar potential may always without physical consequences be modified by the addition or subtraction of some constant, so a vector potential such as **A** may always be modified, without physical consequence, by the addition of a vector field defined everywhere as the gradient of some scalar function.

So, the problem with the Poynting vector is, on the face of things, a simple question of choosing between one set of ambiguities and another — an energy flux vector field that yields a non-zero energy flux even when the fields are static, and a modified vector field which avoids the second difficulty but at the cost of yielding ambiguous expressions for the energy flux within an electromagnetic field except when the gauge (the scalar function whose gradient is added arbitrarily to the vector potential) is chosen suitably.

In the long history of this subject, now almost a century old, issues like these have been debated frequently and often fiercely. So what is, or should be, the correct form of a vector such as the Poynting vector, describing the energy flux within an electromagnetic field? The simple answer is simply to insist, as common sense requires, that if the field is static, the transport of energy should be zero. If only the expectations of common sense more often had the force of law.

Identity

For the identification of the classical Poynting vector with the energy-flux within an electromagetic field is, in its essence, a generalization of some kind. Physically, the simplest thing to say is that the transport of energy across some closed surface enclosing a small volume will be related to the change of the amount of energy closed within that volume. In the electromagnetic field, the internal energy of the field is determined by the scalars, quantities that represent the squares of the field strengths, and however the scalar and vector gauges are chosen, the calculation of the changes of energy content are entirely unambiguous.

Ernst Mach would have a simple comment on this storm in a teacup — always relate arguments about the meaning of physical qualities to calculations of quantities that can be measured. For what it is worth, Lai's discussion of alternatives to the Poynting vector is an interesting way of getting at the ambiguity introduced by the concept of gauge in the electromagnetic field.

The important point is merely that the ambiguities to which Lai and his three critics have drawn attention are neither unexpected nor physically meaningful. Just as, in a world in which the only attributes of numbers are their magnitudes (strictly, moduli), it makes no difference whether all numbers are multiplied by some complex number of unit modulus, so it is right and proper that the functions of position and time from which electric and magnetic field strengths are derived by differentiation should be within limits arbitrary. If they were not, it would not be possible to satisfy Mach's positivist criterion that all supposedly physical quanties should be unambiguous. □