matters arising

General relativistic incompressibility

COOPERSTOCK and Sarracino[1] have attempted to redefine the concept of incompressibility in general relativity. Their result, if correct, would lead to a higher allowable maximum redshift from the surface of a bound object than is usually said to be permitted[2]. We point out here a serious physical difficulty associated with their work. In calculating the mass of an equilibrium configuration in general relativity, the local mass density (a scalar quantity) can always be related to the pressure through an equation of state established in an inertial reference frame, without regard to the local gravitational potential. Nonetheless, Cooperstock and Sarracino choose to define a constant proper mass density $\rho_{\text{proper}} \equiv \rho(r)/g_{\text{rr}}^{1/2} = \text{constant} = a$ (say). They then compute a stellar mass from the relation

$$M = \int_0^{r_0} \rho_{\text{proper}} 4\pi r^2 g_{\text{rr}}^{1/2}\, \mathrm{d}r \qquad (1)$$

where g_{rr} is the radial component of the metric tensor. This definition of the mass is correct, being simply the general relativistic expression with $\rho_{\text{proper}} g_{\text{rr}}^{1/2}$ in place of $\rho(r)$.

As $\rho(r)$ is already a scalar, however, ρ_{proper} has no well defined physical significance. The division of the integral for mass into a product of a 'proper' volume and 'proper' density is arbitrary and misleading. Cooperstock and Sarracino, in asserting that the global contribution of gravitation to the overall mass-energy of a star will affect the local stress tensor, have missed the point of the equivalence principle, the very foundation of general relativity. The validity of this proposition guarantees that ρ itself (not $\rho/g_{\text{rr}}^{1/2}$) is what one would measure when applying an 'ergometer' to a small piece of matter, whether it was inside a neutron star or in empty space. In principle, the source stress tensor $T_{\mu\nu}$ appearing on the right hand side of the field equations $G_{\mu\nu} = kT_{\mu\nu}$ should contain all sources of mass-energy except gravitation. In practice, the gravitational binding energy of two neutrons, whether 10^{28} or 10^{-13} cm apart is negligible. Therefore ρ is the physical quantity relevant for local dynamical effects in neutron stars.

Nonetheless, following the procedure of Tolman[3], one can treat the relation $\rho(r) = ag_{\text{rr}}^{1/2}$ as a defining equation for the radial variation of $\rho(r)$ and then deduce the equation of state $p = p(\rho)$. In that case, the equation of hydrostatic equilibrium is still given by[4]

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{-GM(r)\rho(r)}{r^2}\left[1 + \frac{p(r)}{\rho(r)c^2}\right] \times \left[1 + \frac{4\pi r^3 p(r)}{M(r)c^2}\right]\left[1 - 2GM(r)rc^2\right]^{-1} \qquad (2)$$

Substituting $ag_{\text{rr}}^{1/2}$ for $\rho(r)$ and noting that in a physically allowable object p must be positive, all the terms on the right hand side of equation (2) must be positive, so that the pressure decreases with radius, that is $\mathrm{d}p/\mathrm{d}r < 0$. Now the numerical results of Cooperstock and Sarracino[1] show that $p(r)$ decreases with increasing $g_{\text{rr}}^{1/2}$, that is

$$\mathrm{d}p/\mathrm{d}(g_{\text{rr}}^{1/2}) < 0$$

But since $\rho(r) = ag_{\text{rr}}^{1/2}$ must be a positive quantity, $\mathrm{d}p/\mathrm{d}\rho < 0$. This is an unstable and unphysical situation. In order to have microscopic stability[4], one requires $\mathrm{d}p/\mathrm{d}\rho > 0$. Furthermore, dividing $\mathrm{d}p/\mathrm{d}r$ by $\mathrm{d}p/\mathrm{d}\rho$, one finds $\mathrm{d}\rho/\mathrm{d}r > 0$. It is hard to imagine that a physically realistic star with ρ increasing outward can be made. Though the resulting configuration is in equilibrium since it is a solution to the general relativistic hydrostatic equilibrium equation, it is unstable. Therefore, the resulting star constructed from Cooperstock and Sarracino's definition of incompressibility in general relativity is physically unrealisable. It seems that the redshift limit $z = 2$ set by Bondi[2] remains the largest allowable surface redshift consistent with both general relativity and microscopic stability.

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COOPERSTOCK AND SARRACINO REPLY— For Brecher and Wasserman, ρ_{proper} would have "well defined physical significance" only if it were completely invariant. In earlier correspondence, we had directed them to the equivalence principle, noting that this principle renders such invariance *a priori* unattainable. The equivalence principle implies that the gravitational field can be locally transformed away by free-fall. That is not the point. The point is that with respect to a frame at rest relative to a spherically-symmetric body, ρ_{proper} certainly is well defined. It assumes the form $\rho g_{\text{rr}}^{-1/2}$ in Schwarzschild coordinates. This form has been justified by Misner and Sharp[1,2] from dynamical considerations. We have justified the form from static considerations, and the extension has been made to charged fluid spheres (F.I.C. and R.S.S., in preparation, also, F.I.C. and V. de la Cruz, in preparation). In another coordinate system, say isotropic coordinates, it would not assume this form, but rather be determined by the integrand of the energy integral of the body over proper volume

$$M(r) = \int_0^r \rho_{\text{proper}}\, \mathrm{d}V_{\text{proper}}$$

All energy, including gravitational energy, contributes to the total mass of the body and hence ρ_{proper}, which includes all energy, is the physically relevant quantity in general relativity. The failure to recognise its central role has been perpetuated by Newtonian conditioning.

On the one hand, Brecher and Wasserman assert that the validity of the equivalence principle guarantees that ρ and not ρ_{proper} is what their "ergometer" will measure. On the other hand, they then say that the gravitational binding energy of two neutrons, even 10^{-13} cm apart, is very small at any rate. If the energy is not there in principle, why worry about it in practice?

We feel that Brecher and Wasserman miss the point again. Certainly the gravitational binding for two neutrons is negligible. But, we are not concerned here about two neutrons nor indeed, necessarily about neutrons. We are concerned about conditions where large amounts of matter are being compressed towards their limit and gravitational energy is very significant indeed. This significance is made quite evident in the distinction between the bodies which satisfy the $\rho = \text{constant}$ and $\rho_{\text{proper}} = \text{constant}$ equations of state. We are *not* "asserting that the global contribution of gravitation to the overall mass-energy of a star will affect the local stress tensor ($T_{\mu\nu}$) . . .". We are asserting that the local contribution of gravitational energy, in addition to $T_{\mu\nu}$, determines the physically relevant proper total energy density.

Without entering the controversy regard-

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