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Relativistic Behaviour of Circumnavigating Clocks

CONSIDERABLE interest has been aroused by Hafele's suggestion^{1,2}, subsequently supported by experiment³, that caesium beam clocks, borne by commercial jets circumnavigating the Earth, may provide supporting evidence for the famous prediction of Einstein's theory of relativity that if one member of a set of twins departs from the Earth in a spacecraft and later returns to reunite with his sibling, he will be the younger of the two. Strictly speaking, this statement is valid only if the frame of reference in which the home-bound sibling remains at rest is an inertial frame, which is not true of a frame of reference attached to the (rotating) Earth. This fact was ordinarily disregarded because the relative velocities involved were supposed to be much larger than the velocity difference caused by this circumstance. But in the case of clocks employed by Hafele and Keating³, this supposition is not true and the influence of Earth's rotation must be reckoned with.

In Hafele's calculation^{1,2}, a nonrotating frame of reference embedded in the Schwarzschild metric of the Earth played an intermediate role. Of course, the final result was expressed in terms of the ground speed of the flying clock. For flights confined to circular orbits in the equatorial plane, Hafele showed that

$$(d\tau/d\tau_0) \simeq 1 + gh/c^2 - (2\Omega Ru + u^2)/2c^2 \quad (1)$$

where $g = GM/R^2 - \Omega^2 R$ and M , R and Ω are the mass, the radius and the angular velocity of the Earth. Equation (1) contains a directional effect which arises from the term linear in u (u positive for eastward flights, negative for westward) and is related to the fact that the speed u is measured in a rotating frame of reference. This seems to have been overlooked by Schlegel⁴ when he suggested that the 'directional' effect appearing in equation (1) may be erroneous. Here, I present a calculation, made directly in the rotating frame of reference, which not only vindicates Hafele's result but also generalizes it to more practical situations.

In an arbitrary frame of reference, the space-time metric is

$$ds^2 = g_{ik} dx^i dx^k \quad (2)$$

where $x^i \equiv (x^\alpha, ct)$; $\alpha = 1, 2, 3$. The metric tensor g_{ik} includes the effects of gravitation as well as of the noninertial nature of the frame of reference. The proper time of a standard clock is then given by⁵

$$d\tau = dt [-g_{44} - 2g_{u4} (u^u/c) - g_{u\beta} (u^\beta u^u/c^2)]^{1/2} \quad (3)$$

where $u^\alpha = (dx^\alpha/dt)$. For a clock at rest, equation (3) reduces to

$$d\tau = dt (-g_{44})^{1/2} \simeq dt(1 + \Phi/c^2) \quad (3a)$$

where Φ is the Newtonian potential at the position of the clock. In a Lorentz frame, this becomes

$$d\tau = dt(1 - u^2/c^2)^{1/2} \quad (3b)$$

Now, if the underlying metric is of the Schwarzschild type (arising from the mass M of the Earth) and is viewed from a frame of reference K which rotates, with angular velocity Ω , with respect to the system of "fixed" stars, then the nonvanishing elements of the metric tensor, in the spherical coordinates (r, θ, ϕ) , are given by⁶

$$\left. \begin{aligned} g_{11} &= \left(1 - \frac{2GM}{c^2 r}\right)^{-1}, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \theta \\ g_{44} &= \left(1 - \frac{2GM}{c^2 r} - \frac{\Omega^2 r^2 \sin^2 \theta}{c^2}\right), \quad g_{34} = g_{43} = \frac{\Omega r^2 \sin^2 \theta}{c} \end{aligned} \right\} \quad (4)$$

Substituting (4) into (3), and for slow motions ($u \ll c$) and weak fields ($\Phi \ll c^2$),

$$d\tau \simeq dt \left[1 - \left(\frac{GM}{c^2 r} + \frac{\Omega^2 r^2 \sin^2 \theta}{2c^2} \right) - \left(\frac{2\Omega r^2 \sin^2 \theta (d\phi/dt) + u^2}{2c^2} \right) \right] \quad (5)$$

In comparing the rates of two given clocks, the coordinate differential dt is eliminated.

Equation (5) contains two effects: the effect of altitude which, for $r = R + h$ ($h \ll R$), takes the form gh/c^2 where $g \equiv GM/R^2 - \Omega^2 R \sin^2 \theta$ is the "measured" value of the acceleration due to gravity at a place of colatitude θ ; and the effect of motion of the clock which is directional, because of the occurrence of the product $\Omega(d\phi/dt)$, and for $\theta = 90^\circ$ reduces to the Hafele expression $-(2\Omega Ru + u^2)/2c^2$.

Equation (5) may be applied to a satellite-borne clock as well. I consider motion in a circular orbit, of radius R^* , inclined at an angle λ to the equatorial plane of the Earth. Measuring ϕ from the line of nodes, in frame K

$$\theta = \cos^{-1}(\sin \lambda \sin \alpha), \quad \phi = \tan^{-1}(\cos \lambda \tan \alpha) - \Omega t$$

where angle α locates the satellite in its own plane of rotation: $\dot{\alpha} \equiv \omega = \sqrt{GM/R^*}$. Equation (5) then gives

$$d\tau^* \simeq dt (1 - 3GM/2c^2 R^*) \quad (6)$$

which is independent of λ and Ω , and depends only on R^* . In K_0 one would have

$$d\tau^* \simeq dt (1 + \Phi/c^2 - u^2/2c^2)$$

which, for $\Phi = -GM/R^*$ and $u^2 = \omega^2 R^{*2} = GM/R^*$, would again lead to equation (6).

Finally, comparing the satellite-borne clock with an Earth-bound clock (for which $\Phi = -GM/R$ and $u = 0$),

$$(d\tau^*/d\tau_E) \simeq 1 - (GM/2c^2)(3/R^* - 2/R) \quad (7)$$

the Ω -term is now clearly negligible. Effects predicted by equation (7) are about 100 times larger than the ones encountered in the case of jet-borne clocks. Establishment of Skylab may subject these effects to a more decisive test.

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