LETTERS TO NATURE

PHYSICAL SCIENCES

The Non-existence of Stationary Infinite Newtonian Universes and a Multi-dimensional Model of Shot Noise

HOLTSMARK[1], in an electrostatic context, and Chandrasekhar[2], in the case of a Newtonian gravitational field, have given a simple model which describes the electrostatic or gravitational field at an arbitrary point in space. They take the point as the origin of coordinates and calculate the field there due to point charges or masses which are scattered randomly and homogeneously throughout the space. Specifically, the field is

$$\mathbf{X}_R(0) = K \sum_{j=1}^{N} \mathbf{t}_j/|\mathbf{t}_j|^3 \tag{1}$$

where K is a proportionality constant, and in a sphere of radius R it is supposed that there are N points scattered randomly and located at $\mathbf{t}_1, \ldots, \mathbf{t}_N$. By letting N and $R\to\infty$ in such a way that the density of points $3N/4\pi R^3 \to \lambda$, Holtsmark showed that the probability distribution of $\mathbf{X}_R(0)$ converges to a limit, being the distribution of $\mathbf{X}^P(0)$, say.

Now, if this model is to be at all realistic, it must have the property of stationarity. That is, if our point is taken not at the origin (which for equation (1) is the centre of the sphere R) but at $\mathbf{t}$, say, then by starting from

$$\mathbf{X}_R(\mathbf{t}) = K \sum_{j=1}^{N} (\mathbf{t}_j - \mathbf{t})/|\mathbf{t}_j - \mathbf{t}|^3 \tag{2}$$

and proceeding to the limit with R and $N\to\infty$ as before, then the resulting distribution, of $\mathbf{X}^P(\mathbf{t})$, say, should be the same as for $\mathbf{X}^P(0)$. A mathematical analysis to be given elsewhere[5] shows that this is not so; hence the non-existence statement in the title.

The point of this conclusion, which may at first sight seem to be a piece of mathematical pedantry, is more apparent when we consider simultaneously[3] the field at two points separated by a distance $2\mathbf{t}$. The limiting distributions obtained by taking the limit R and $N\to\infty$ as before in the two possible definitions (equations (3) and (4) below) are different, yet comparison with equations (1) and (2) gives no reason for preferring either definition.

$$(\mathbf{X}_R(0),\ \mathbf{X}_R(2\mathbf{t})) = K \sum_{j=1}^{N} (\mathbf{t}_j/|\mathbf{t}_j|^3,\ (\mathbf{t}_j - 2\mathbf{t})/|\mathbf{t}_j - 2\mathbf{t}|^3) \tag{3}$$

$$(\mathbf{X}_R(-\mathbf{t}), \mathbf{X}_R(\mathbf{t})) = K \sum_{j=1}^{N} ((\mathbf{t}_j + \mathbf{t})/|\mathbf{t}_j + \mathbf{t}|^3, (\mathbf{t}_j - \mathbf{t})/|\mathbf{t}_j - \mathbf{t}|^3) \tag{4}$$

The model here is a special case of a generalization of shot noise in which points are scattered homogeneously and randomly (as a Poisson process) in an n-dimensional Euclidean space with density λ, with any point $\mathbf{t}_j$ contributing the disturbance $\mathbf{f}(\mathbf{t}_j - \mathbf{t})$ to the total disturbance (summed over j)

$$\mathbf{Y}(\mathbf{t}) = \Sigma \mathbf{f}(\mathbf{t}_j - \mathbf{t}) \tag{5}$$

at any point $\mathbf{t}$. [In equations (1) and (2), $n=3$ and $\mathbf{f}(\mathbf{t}) = K\mathbf{t}/|\mathbf{t}|^3$. If instead the Yukawa[4] disturbance function $\mathbf{f}(\mathbf{t}) = Ke^{-\alpha t}(1+\alpha t)\mathbf{t}/t^3$, where $t = |\mathbf{t}|$, is used, then the distributions of $\mathbf{X}^P(\mathbf{t})$ and $\mathbf{X}^P(0)$ are the same.] The existence of the limiting distributions for $\mathbf{X}^P(\mathbf{t})$ and $\mathbf{Y}(\mathbf{t})$ depends on the behaviour of $\mathbf{f}(\mathbf{t})$ for large $|\mathbf{t}|$, and possibly also (as with equation (1)) on the order of summation of the points $\mathbf{t}_j$ in equation (5). Assuming that $\mathbf{f}(\mathbf{t})$ is radially symmetric so that $\mathbf{f}(\mathbf{t}) = g(t)\mathbf{t}/t$ for $t = |\mathbf{t}|$ and some function $g(t) \geq 0$ with $g(t) \downarrow 0$ for sufficiently large $t\to\infty$, the sum for equation (5) is well defined and finite for all $\mathbf{t}$ when, for sufficiently large a,

$$\int_a^\infty t^{n-1}\ g(t)\mathrm{d}t < \infty \tag{6}$$

Sums analogous to equations (1) and (2), that is,

$$\mathbf{Y}_R(\mathbf{t}) = \sum_{|\mathbf{t}_j|<R} \mathbf{f}(\mathbf{t}_j - \mathbf{t}) \tag{7}$$

have probability distributions with a limit for $\mathbf{t} = 0$ when

$$\int_a^\infty t^{n-1}\ g^2(t)\mathrm{d}t < \infty \tag{8}$$

and there is a limit for all $\mathbf{t}$ when, additionally,

$$\int_R^{R+a} t^{n-1}\ g(t)\mathrm{d}t \tag{9}$$

has a finite limit as $R\to\infty$. These limiting distributions coincide for all $\mathbf{t}$ when the limit of the integral (9) is zero.

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Giorgi System

THE letter by Stopes-Roe[1] on "Essential Features of the Giorgi System of Electromagnetism and a Basic Error by Sommerfeld" reveals that a bewildering number of formulations of the equations of electromagnetism are being advocated. Some related problems are discussed by Rosser[2].

In SI units the constant $\mu_0 = 4\pi\times10^{-7}$ is indispensable in many formulae, if the dimensions are to balance and the correct numerical values are to be obtained. The difficulty arises from attributing to this constant the physical properties of permeability. As shown by Stopes-Roe, different definitions of magnetic moment lead to different versions of magnetic formulae; these definitions with their associated formulae are self-consistent and all equally correct in free space. If, however, μ_0 in the free space formulae is replaced by $\mu_0\mu_r$ in a medium of relative permeability, μ_r, the different systems predict different results. In particular the Kennelly formulation predicts that the field H due to a magnetic dipole is inversely proportional to μ_r, while the Sommerfeld formulation predicts that it is independent of μ_r.

I have so far failed to find references to any experimental evidence that would distinguish between these predictions. Because the experiments would have to use a fluid with μ_r very close to unity the experiments would be difficult, although probably not impossible.

Many older textbooks, such as Starling[3], deduce from a dubious analogy with electrostatics that H is inversely proportional to μ_r. The electrostatic proof rests on the Gauss theorem in the form that the integral of the outward normal displacement D_m over a closed surface

$$\int_S D_n \mathrm{d}s = 4\pi q \tag{1}$$