These results are in contradiction to the predictions of statistical detection theory and make it impossible to continue to explain Piper's law in this way.

C. I. HOWARTH
G. LOWE

Department of Psychology,
University of Nottingham.

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GENERAL

Asymptotic Dam Theory

IN a previous communication[1], an asymptotic theory was outlined for the probabilistic behaviour of a finite dam or reservoir subject to random input and output. A further problem raised by Herman Rubin—that of the amount of water lost by overflow—may be solved by similar methods. In the case of normal diffusion which is treated here, the distribution of the overflow may be found by evaluating its moment generating function. There is an alternative and more general method, which is useful when only the expected overflow is required. The alternative, suggested by R. Morton, is discussed in the succeeding communication.

As before, let the total capacity of the dam be C and the initial content c. Suppose the water lost before the dam first reaches emptiness be W, and let $H(\alpha|c) = E(e^{\alpha W}|c)$, where E denotes expectation. For the simple discrete case with probability p of one unit input and q of one unit output, we readily find

$$H(\alpha|c) = P_0(c) + P_C(c)H(\alpha|C) \quad (1)$$

$$H(\alpha|C-1) = P_0(C-1) + P_C(C-1)H(\alpha|C) \quad (2)$$

where $P_0(c)$ is the chance of emptiness before the dam becomes full, and $P_C(c)$ is the chance of fullness before the dam becomes empty, given the starting point c. As we have also

$$H(\alpha|C) = pe^{\alpha}H(\alpha|C) + qH(\alpha|C-1) \quad (3)$$

$H(\alpha|c)$ can be determined. In particular

$$H(\alpha|C) = \frac{qP_0(C-1)}{1 - pe^{\alpha} - qP_C(C-1)} \quad (4)$$

representing a geometric-type distribution.

The asymptotic case of normal diffusion can either be derived as a limiting case of the above, or from the following more direct argument. In place of equation (2) we have

$$H(\alpha|C-\varepsilon) = P_0(C-\varepsilon) + P_C(C-\varepsilon)H(\alpha|C)$$

whence (as $P_C(c)$ is differentiable at C, so is $H(\alpha|c)$)

$$\left[\frac{\partial H(\alpha|n)}{\partial n}\right]_C = [H(\alpha|C-1)]\left[\frac{\partial P_C(n)}{\partial n}\right]_C \quad (5)$$

In place of equation (3) we write

$$H(\alpha|C) = E(e^{\alpha\delta W}H(\alpha|C + \delta V - \delta W))$$

where δV is the net input and δW is the amount of overflow during a short time δt. Let m and σ^2 be the net mean and variance of the flow per unit time (m measured positively if mean output exceeds mean input—this corrects the wrong convention given above equation (5) of the earlier communication[1]).

The properties of normal diffusion imply that $E(\delta W)$ is of order $(\delta t)^{\frac{1}{2}}$, whereas δV contributes only terms of order δt.

Thus we obtain

$$\alpha H(\alpha|C) = \left[\frac{\partial H(\alpha|n)}{\delta n}\right]_C \quad (6)$$

whence $H(\alpha|c)$ may be determined from equations (1), (5) and (6).

In particular, as

$$P_C(c) = (e^{c\gamma} - 1)/(e^{C\gamma} - 1) \quad (7)$$

where $\gamma = 2m/\sigma^2$, we find

$$H(\alpha|C) = \frac{1}{1 - \alpha(1 - e^{-C\gamma})/\gamma}$$

representing an exponential distribution.

M. S. BARTLETT

Department of Statistics,
University College, London.

J. A. BATHER

Statistical Laboratory,
Department of Mathematics,
University of Manchester.

[1] Bartlett, M. S., *Nature*, **202**, 731 (1964).

Expected Overflow of a Dam

CONSIDER a dam with total capacity C and initial content c. If the dam becomes full the surplus water overflows and is lost. Let $W(t)$ be the total overflow up to time t, and T be the time at which the dam first becomes empty. For certain types of input and output Bartlett[1] derives the distribution of T, and with Bather in the preceding communication, the distribution of $W(T)$, the total overflow before the dam becomes empty.

For arbitrary input and output, the expectation of $W(T)$ can be expressed in terms of the distribution of T and the excess demand for output at emptiness.

Let $m(t)$ be the mean flow per unit time (output minus input) at time t, and suppose that this does not depend on $W(t)$ or $X(t)$, the current content of the dam. By convention, $X(T)$ is negative if there is excess demand when the dam is empty.

Denote $M(t) = \int_0^t m(s)\,ds$, then

$$S(t) = W(t) + X(t) + M(t)$$

can be regarded as a fair game in that for any $\tau < t$ $E[S(t)|S(\tau)] = S(\tau)$. In particular, if $\tau = 0$, $E[S(t)] = c$. The game is fair for any stopping rule, and in our case we stop at time T when first $X(T) \leqslant 0$. We have, therefore,

$$E[S(T)] = c = E[W(T)] + E[X(T)] + E[M(T)] \quad (1)$$

In cases where the output is continuous or the changes in content occur only in unit steps, there can be no excess demand and so $X(T) = 0$. Thus

$$E[W(T)] = c - E[M(T)] \quad (2)$$

In particular, if $m(t) \equiv m$ is constant, equation (2) becomes

$$E[W(T)] = c - m\,E[T] \quad (3)$$

If the output consists solely of discrete jumps with a negative exponential distribution, then $-X(T)$ is also negative exponential, but in other cases $E[X(T)]$ may be hard to evaluate.

Equation (1) can be explained intuitively by saying that the expected surplus $c - E[M(T)]$ must come out as overflow.

I would point out that because $W(T) \geqslant 0$, equation (3) implies the inequality $E(T) \leqslant c/m$ when $m > 0$, equality occurring only when no overflow is possible; for example if $C = \infty$, or if the content certainly decreases.

R. MORTON

Statistical Laboratory,
Department of Mathematics,
University of Manchester.

[1] Bartlett, M. S., *Nature*, **202**, 731 (1964).

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