holding. When waterlogged, however, ammonia volatilization was 2–4 times greater than for the aerobic moisture levels. The higher incubation temperature (40° C) gave slightly greater ammonia volatilization. The differences caused by temperature increases became much more pronounced with the longer incubation period. Among the six acid soils there was no correlation between the extent of volatilization of ammonia and the pH.

As expected, the Saline-Sodic and Saline Palmaseca soils lost more ammonia than did the acid soils. The loss of ammonia by volatilization even from acid soils is difficult to explain. The mineralogical analysis of these acid soils showed that they all contained "alkaline" feldspars and other "basic" minerals. In spite of the fact that the soils as a whole were acid, the presence of these alkaline-producing materials distributed throughout the soils may be responsible for ammonia volatilization. It is notable, for example, that the Cerritos soil, which gave the greatest volatilization of ammonia among the acid soils, also had the highest content of these "alkaline" and "basic" minerals.

This preliminary experiment has shown that loss of ammonia by volatilization can be important even from acid soils. Much further work is required on this problem, particularly to determine why the loss occurs. In general, losses in incubation tests may be expected to be greater than in the field, since in the latter case ammonium can diffuse downward through the soil, increasing the possibility of absorption by organic and inorganic colloids, or by plants.

The possibility that ammonia may be lost by volatilization from acid soils after the application of ammonium sulphate or ammonia-producing fertilizers such as urea must be considered. This applies particularly in the tropics, where the soils commonly reach temperatures of more than 30° C.

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[1] Allison, F. E., *Adv. in Agron.*, **7**, 213 (1955).
[2] Mortland, M. M., *Adv. in Agron.*, **10**, 325 (1958).
[3] Wahhab, A., *et al.*, *Soil Sci.*, **84**, 249 (1957).
[4] Cornfield, A. H., *Plant and Soil*, **14**, 90 (1961).
[5] Yuen, S. H., and Pollard, A. G., *J. Sci. Food Agric.*, **3**, 441 (1952).

MATHEMATICS

Square-root Law for Solving Two-ended Problems

SOME mathematical problems can be expressed in terms of threading mazes: we are given a non-oriented linear graph (notes and branches) and wish to find a path from a node A to a node B, where A corresponds to the premises and B to the conclusions of some theorem[1,2]. As is well known, it is likely to be more economical to "work from both ends", that is, from both A and B, rather than from A alone. For many years I have been remarking that the saving in work by this "two-ended" method is likely to be given approximately by a square root law. By this I mean that if the number of steps used when threading the maze from one end is S, then the number used when working from both ends is liable to be of the order of the square root of S. These estimates, S and $\sqrt{S}$, are both intended to be expected values, and the result is true only if the maze is very difficult to thread, so that the shortest path from A to B is long and consists of far fewer than S steps. The condition that the maze is difficult is to be interpreted in the sense that there is not at first much indication of whether we are going in the right direction, that is, there is at first no very useful estimate of "distance". We can define the distance from A to B as the number of steps required to get from A to B along the shortest path, but we have no quick and effective method of telling when we are making steps that decrease the distance, when A and B are far apart.

The most well known uneconomical method of threading a maze makes no use of the estimation of the distance from A to B. It consists in locating all nodes at distance 1 from A, then all at distance 2, and so on, thus generating a tree rooted at A which eventually reaches B if the graph is connected and finite, as we assume[3]. This method will, of course, find the shortest path from A to B, say, of length n. Let b be the average number of "children" of a node in this tree. (Clearly b is at least one less than the average number of branches at each node of the graph.) The expected number of steps required by this method is

$$b+b^2+b^3+\ \ldots\ +b^{n-1}+\tfrac{1}{2}b^n \approx \tfrac{1}{2}b^n(b+1)/(b-1)$$

If instead we work from both ends, the expected number of steps will be about $b^{\frac{1}{2}n}(b+1)/(b-1)$. If n is large, this is of the order of the square root of the number of steps required in one-ended working.

If we take into account some estimate of distance, it will begin to become of value when we are within some distance, d, of the goal, where $d<n$ in virtue of our hypothesis that the maze is difficult to thread. The tree, or the pair of trees, that are generated before the estimation of distance becomes of value may be described as the "groping" trees. For one-ended working, the groping tree will contain of the order of b^{n-d} steps, whereas, for two-ended working, the groping trees (the closest nodes of which will be at distance d from one another) will contain a number of steps of the order of the square root of this since the number of generations in each of them will be about $\frac{1}{2}(n-d)$. After the distance estimation begins to become useful, the number of steps that remain will be negligible compared with the sizes of the groping trees, provided that n is large enough. This completes the demonstration of the square root law for sufficiently difficult mazes.

This discussion can be made more formal by regarding the expected number of children of a node as a function of the shortest distance, m, to the goal, or, for two-ended working, to the other tree. Thus we replace b by a function b_m. Then the expected number of steps for one-ended working is $f(n)$ defined as

$$f(n) = \sum_{r=0}^{n-1} b_n b_{n-1} \ldots b_{n-r}$$

whereas, for two-ended working, it is $2f(\frac{1}{2}n)$. Perhaps the simplest form to assume for b_m is $(1-q^m)b$.

It will be interesting to see whether these formulae are consistent with the experimental results of the graph-traverser programme[2], when and if it is modified to cope with two-ended working.

For problems in which the goal is represented by a set of more than one node, the method of two-ended working generalizes in an obvious manner. But a generalization of the above formulae would be difficult as it must depend on how close the various goal-nodes are to one another, and also on their individual distances from the "origin", A. But the square root law remains valid for sufficiently difficult problems.

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[1] Newell, A., Shaw, J. C., and Simon, H. A., in *Self-Organizing Systems* (edit. by Yovits and Cameron), 153 (Pergamon, 1960).
[2] Doran, J. E., and Michie, D., *Proc. Roy. Soc.*, A, **294**, 235 (1966).
[3] See, for example, Berge, Claude, *Théorie des Graphes et ses Applications*, 65 (Dunod, Paris, 1958).