MATHEMATICS

Gravitational Collapse

IN a recent letter[1] Prof. H. Bondi states that it is impossible to determine the number of particles contained in a static distribution of matter and radiation by means of an integral over that configuration. In so doing he denies to a local observer the capability of measuring the number of particles contained in a small but finite volume, and thereby denies the possibility of assigning a physically meaningful particle density at every point in the configuration.

If, on the other hand, one asserts that an observer can determine a local number density, it is easy to show that the total particle content of any configuration is a well-defined invariant quantity.

The infinitesimal integral:

$$\int_{\Delta} \sqrt{-g}\, dx^1\, dx^2\, dx^3\, dx^4$$

is a known invariant[2]. It is always possible to choose the co-ordinate system in such a way that $g_{\mu 4} = g_{4\mu} = \delta_{4\mu}$. Hence:

$$\int_{\Delta'} A^4 \sqrt{-g}\, dx^1\, dx^2\, dx^3$$

where A^4 is the fourth component of an arbitrary vector, is also an invariant. In a local Cartesian frame of reference, $g_{11} = g_{22} = g_{33} = -1$, $g_{44} = 1$ and $dx^1\, dx^2\, dx^3$ represents an infinitesimal proper volume element. If A^4 is chosen as the local proper particle density, q, then:

$$dN = \int_{\Delta'} q\, dx^1\, dx^2\, dx^3$$

represents the invariant number of particles contained in the chosen volume element. Transforming to a co-ordinate system commonly used for spherically symmetric systems:[3]

$$(A^4)' = \frac{\partial (x^4)'}{\partial x^\nu} A^\nu = \frac{\partial (x^4)'}{\partial x^4} A^4$$

$$= e^{-\nu/2} q$$

$$\sqrt{-g} = e^{\nu/2} e^{\lambda/2} r^2 \sin\theta$$

and, finally:

$$N = \int_{\text{entire system}} q\, e^{\lambda/2} r^2 \sin\theta\, d\theta\, d\varphi\, dr$$

gives the total invariant particle content of the system.

In this same co-ordinate system:

$$M = \int_{\text{entire system}} \rho\, r^2\, dr \sin\theta\, d\theta\, d\varphi$$

where ρ is the proper energy density, represents the total gravitational mass of the system. It is clear that N and M are linearly independent unless very stringent and non-physical limitations are placed on the relationship between the particle density, energy density, and pressure distributions.

The particular choice of q (as well as of ρ and pressure), which has physical meaning, must, of course, involve further considerations. If the static configurations are to represent stages in the evolution of a real body losing energy, then considerations of energy flow within the body place restrictions on the relation between the pressure and temperature distributions which, in addition to a physically reasonable choice of equation of state, serve to specify uniquely the relation between the ρ, q, and pressure distributions at different stages in the evolution of the system.

Bondi appears to confuse the meaning of indistinguishability owing perhaps to his failure to recognize the distinction between energy density and particle density. Two static configurations with identical energy density and pressure distributions may certainly contain different numbers of particles, as Bondi himself states. But if we can attribute different nucleonic content to the two configurations, we have clearly distinguished them from one another, despite the fact that they may possess the same gravitational mass (that is, total energy content).

In the example which Bondi cites, the two systems initially have total mass $M_1(0)$ and $M_2(0)$ and particle number $N_1(0)$ and $N_2(0)$. Let m_1 and m_2 represent the rest masses of particles contained in 1 and 2. Since, in the chosen initial states, energy density and (particle-mass $\times$ particle-density) are identical, $m_1 N_1(0) = M_1(0)$ and $m_2 N_2(0) = M_2(0)$. Bondi demands that, after a time t, $M_1(t) = M_2(t)$, $M_1(t) = M_1(0)$, and $M_2(t) < M_2(0)$. In so doing he has simply assigned different physical properties to the matter-radiation field in the two cases. His particular choice of the relationship between energy density, pressure and particle density is such that, since $M_2(0) > M_1(0)$, $N_2(t) m_2 = N_2(0) m_2 > N_1(t) m_1 = N_1(0) m_1$. This conclusion, to which Bondi also subscribes, does not demonstrate the impossibility of determining particle number from a relativistic integral over a static configuration; on the contrary, it is reached only by considerations, implicit or otherwise, based on the existence of such an integral.

ICKO IBEN, JUN.

California Institute of Technology,
Pasadena.

[1] Bondi, H., *Nature*, **202**, 275 (1964).
[2] Einstein, A., *Ann. d. Phys.*, **49** (1916).
[3] Iben, I., *Astrophys. J.*, **138**, 1090 (1963).

I AM afraid Dr. Iben and I have been misunderstanding each other. In my communication I pointed out that an integral involving only the density and pressure of the material could not reveal the particle number. I correctly referred to a contrary statement by Eddington; but I mistakenly also referred to Dr. Iben's paper, which I had misunderstood. I now appreciate that no such error does in fact occur in his paper. On the other hand, Dr. Iben has misunderstood my communication. I stressed at the beginning that I was discussing integrals involving only the density and pressure of the material, and used the words 'relativistic integral' further on merely as a shorthand for this. In particular, I was thinking of Eddington's identification of the energy invariant as particle number density.

I trust that Dr. Iben agrees fully with me that no term or combination of terms of the energy tensor has any connexion with the particle number. The particle number may be put into the theory as is done in Dr. Iben's own work, but in no way can it be deduced from the metric.

H. BONDI

Department of Mathematics,
King's College
(University of London),
Strand, W.C.2.

I AGREE with Prof. Bondi that only energy density and pressure are included explicitly in the energy tensor. However, both energy density and pressure are related to particle density and any application of relativistic theory to a physically realistic description of a massive body in a highly condensed state requires a choice as to this relationship.

ICKO IBEN, JUN.

California Institute of Technology,
Pasadena.