STATISTICS

Combining the Probabilities from Significance Tests

IN a letter¹ with the above title, Leigh van Valen has proposed a method which appears at variance with a well-known procedure described by R. A. Fisher² (and not mentioned by van Valen). Thus for the probabilities considered by Fisher:

$$0.145, 0.263, 0.087$$

— 2 times the sum of the natural logarithms of the probabilities gives a χ^2 with degrees of freedom twice the number of tests, that is in this case:

$$\chi^2 = 11.42 \text{ with 6 d.f.}$$

giving a resulting significance level of 0.076. Van Valen's procedure, the justification of which is not clear to me, appears to give:

$$0.145 \times 0.263 \times 0.087 \times 8/2 = 0.0133$$

greatly exaggerating the significance.

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¹ *Nature*, 201, 642 (1964).

² *Statistical Methods for Research Workers*, § 21.1.

THE statistic Q proposed by van Valen¹ is such that, under the null hypothesis, $2 \log 1/2Q$ is distributed as the modulus of the difference of two independent χ^2 variables each of $2N$ degrees of freedom (where $N = n + m$ in van Valen's notation). It follows that the use of Q as 'the combined probability desired' is not correct without a functional transformation: namely, using the result of David *et al.*² for the distribution of the difference of two χ^2 variables, the value:

$$G_N(Q) = 1 - 2S_{N-1} \left(\log \frac{1}{2Q} \right)$$

should be used, where:

$$S_m(x) = \int_0^x \frac{z^m K_m(z)}{2^m \sqrt{\pi} \Gamma(m + \frac{1}{2})} dz$$

($K_m(z)$ being a standard notation for the modified Bessel function of second kind with imaginary argument).

Since the Bessel function of half-odd-integer order is elementary, the directly computable form

$$G_N(Q) = \frac{8Q}{4N} p_N \left(\log \frac{1}{2Q} \right)$$

where:

$$p_N(x) = \sum_{r=0}^{N-1} \frac{(2x)^r}{r!} \sum_{s=0}^{N-r-1} \frac{2^{N-1-r} C_s}{s!}$$

may be more convenient than reference to the tables of S_m .

Thus, for $N = 3$ as in the case considered by Prof. Bartlett in his preceding communication and using David's tables² of S_m :

$$Q = 0.0133 \text{ and } G_3(0.0133) = 0.1307$$

that is, a figure of somewhat less significance. The general sense of van Valen's letter, as opposed to a literal reading, indicates that small values only, of R_a/R_b , should be taken to measure lack of confidence in the null hypothesis. This gives a value of 0.065 ($= \frac{1}{2}(0.1307)$), in the case quoted by Prof. Bartlett, which is very close to Fisher's value of 0.076.

Thus modified, van Valen's method is a proper alternative to Fisher's; both, equally, valid. The choice between the two must depend on the application of Neyman and Pearson's theory of the power of tests and

this in turn on the better specification of the problem, that is, of the alternatives to the null hypothesis envisaged.

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¹ *Nature*, 201, 642 (1964).

² David, F. N., Pearson, K., and Stouffer, S. A., *Biometrika*, 24, 293 (1932).

An Asymptotic Theory for Dams

IN my contribution to the discussion at a symposium¹ in 1957 on the theory of dams, I suggested that the distribution of the time to emptiness of a finite dam or reservoir might be investigated by the use of Wald's identity. This method has recently been used by R. M. Phatarfod² in the case of inputs with negative exponential distribution. It seems worth noting how the asymptotic case of normal diffusion, which should be useful as an approximation in all cases where individual (independent) increments are fairly small in relation to the capacity of the dam, may be dealt with. This asymptotic theory has, moreover, an immediate extension to the case of correlated increments by our determining the effective diffusion per unit time in such a case (compare with my concluding remarks, *loc. cit.*).

Suppose water is coming into the dam at a rate equivalent to an additive process $I(t)$ and is leaving independently equivalent to an additive process $J(t)$. Let the capacity of the dam be C and let the initial volume of water at $t = 0$ be c . If the dam becomes full, the surplus water overflows and is lost. Let the characteristic function of the time to emptiness (irrespective of becoming full in the meantime) be $\xi(\varphi|c)$; and let the components of the characteristic function $\eta(\varphi|c)$ of first time to either boundary (emptiness or fullness) be given by the equation:

$$\eta(\varphi|c) = \eta_0(\varphi|c) + \eta_c(\varphi|c) \quad (1)$$

Then we have the relations:

$$\begin{aligned} \xi(\varphi|c) &= \eta_0(\varphi|c) + \eta_c(\varphi|c) \xi(\varphi|C) \\ \xi(\varphi|C-\varepsilon) &= \eta_0(\varphi|C-\varepsilon) + \eta_c(\varphi|C-\varepsilon) \xi(\varphi|C) \end{aligned} \quad (2)$$

These relations, which are exact, will determine $\xi(\varphi|c)$ provided the functions in (1) are known and a further relation between $\xi(\varphi|C-\varepsilon)$ and $\xi(\varphi|C)$ is available. This is so, for example, in the simple discrete time and increment case, with probability p of one unit input (except that if the dam is full the excess water overflows) and probability q of one unit output. In this case we consider the situation relating $\varepsilon = 0$ and $\varepsilon = 1$, and easily arrive at the results obtained by B. Weesakul³ by more direct methods.

In the normal diffusion case we let $\varepsilon \rightarrow 0$; as in a small time δt diffusion from the boundary extends back a distance of order $\sqrt{\delta t}$, consistency is only possible if $\partial \xi(\varphi|x)/\partial x$ is zero at $x = C$ (this conclusion may also be checked if desired from the simple discrete case above as $p \rightarrow q \rightarrow \frac{1}{2}$, and the time and volume scale units are appropriately reduced). Hence, as in the normal diffusion case the functions $\eta_c(\varphi|x)$ and $\eta_0(\varphi|x)$ are differentiable with respect to x , the relations (2) give finally:

$$\xi(\varphi|c) = \eta_0(\varphi|c) - \eta_c(\varphi|c) \left[\frac{\partial \eta_0(\varphi|x)}{\partial x} / \frac{\partial \eta_c(\varphi|x)}{\partial x} \right]_C \quad (3)$$

In this formula, from Wald's identity:

$$\begin{aligned} \eta_0(\varphi|c) &= (e^{c\theta_1} - e^{c\theta_2}) / (e^{c\theta_1} + (C-c)\theta_2 - e^{c\theta_2} + (C-c)\theta_1) \\ \eta_c(\varphi|c) &= (e^{(C-c)\theta_1} - e^{(C-c)\theta_2}) / (e^{c\theta_1} + (C-c)\theta_2 - e^{c\theta_2} + (C-c)\theta_1) \end{aligned} \quad (4)$$

where:

$$\theta_1, \theta_2 = \{ -m \pm \sqrt{m^2 - 2i\sigma^2\varphi} \} / \sigma^2$$

m and σ^2 being the net mean and variance per unit time (m positive if mean input exceeds mean output). This gives:

$$\left[\frac{\partial \eta_0(\varphi|x)}{\partial x} / \frac{\partial \eta_c(\varphi|x)}{\partial x} \right]_C = \frac{\theta_2 - \theta_1}{\theta_2 e^{C\theta_1} - \theta_1 e^{C\theta_2}} \quad (5)$$