PSYCHOLOGY

Sources of Difficulty in Learning Arithmetical Facts

SEVERAL investigations have been carried out on the relative difficulties of learning the various 'arithmetical facts' or tautologies contained in the multiplication and addition tables. A variety of operational criteria of 'difficulty' have been used by different workers, one of whom[1] gives a comparative review of earlier work, with which his own results substantially agree. In particular it seems clear that the chief source of difficulty with a given combination (such as $2.7 = 14$ or $4 + 5 = 9$) is the size of the numbers concerned. However, on the question of which numbers, exactly, are responsible for the difficulty—the addends and factors, or the sums and products—both the evidence and the considered opinions of different investigators are nicely balanced between the two points of view.

In the case of addition, Knight and Behrens[2] concluded that the size of the sum determined the difficulty, defining the latter in terms of learning, mistakes and calculation-time. Yet they suspected that the size of the addends influenced the way in which the relative difficulties fell into groups. Wheeler[1] found similar results, but favoured the size of the addends as the primary source of difficulty. In Wheeler's experiment, after some preliminary teaching 125 children practised addition by playing an educational card-game for a fixed period each day. The different additive combinations were then ranked in order of increasing difficulty, according to the proportion of children answering correctly in a written test on the 20th day. Wheeler found correlation coefficients of 0·87 and 0·95 respectively, between difficulty-rank and size of sum, and difficulty-rank and size of addends. Since the combinations were differently grouped for the two tests it can only be inferred that 'difficulty' is closely related to both variables.

In the case of multiplication, opinion is still more narrowly divided. Wheeler[1] cites two investigations in which the size of the product, and three in which the size of the factors, were judged to be the primary source of difficulty. He himself, using a second educational card-game, found almost identical correlations; for the product, $r = 0{\cdot}92$, for the factors, $r = 0{\cdot}94$.

Much of the ambiguity throughout clearly stems from the fact that for both additive and multiplicative combinations (generally expressible in the forms $x_1 + x_2 = x_3$ and $x_1.x_2 = x_3$ respectively) x_3 is heavily dependent on x_1 and x_2 by definition. Consequently the two hypotheses, linking difficulty with x_3 on one hand or with a joint measure of (x_1, x_2) on the other, are to a large extent confounded and statistically indistinguishable. Even if only for this reason, a new unifying hypothesis involving x_1, x_2 and x_3 in combination, which made it unnecessary to choose between these two alternatives by giving at least equally high correlations, would be preferable to either of them.

Some recent work on timed calculations by adults, including simple one-stage additions and multiplications of the type considered here, suggests a formula which may be the appropriate collective measure of 'calculation-size' or 'complexity' for use in the present dilemma[3]. In some circumstances at least, the average time taken to perform a simple one-stage arithmetical operation R on a pair of one-digit numbers (x_1, x_2) to give a result x_3 is strictly proportional to $\log(x_1 + x_2 + x_3)$. Thus for multiplications, $x_3 = x_1.x_2$; for additions, $x_3 = x_1 + x_2$ and $\log(x_1 + x_2 + x_3) = \log 2(x_1 + x_2)$. The correlations between 'difficulty' and the appropriate values of $\log(x_1 + x_2 + x_3)$, according to Wheeler's data, have therefore been examined and compared with correlations obtained on the hypothesis that either (x_1, x_2) or x_3 alone is responsible for the difficulty.

The first test was applied to the ungrouped data on multiplications. 'Double' combinations for which $x_1 = x_2$ were excluded and, apart from these, a combination and its reverse were treated as distinct. All combinations with 0 or 1 as a factor were excluded. The remaining combinations were then ranked according to the percentage of children performing each one correctly in the final test. For these 64 combinations, ungrouped, the correlation coefficients (with respect to difficulty-rank number in each case) are as follows: *Product*, $(x_1.x_2)$: $r = 0{\cdot}79$; *mean factor-size*, $\frac{1}{2}(x_1 + x_2)$: $r = 0{\cdot}80$; and for $\log(x_1 + x_2 + x_1.x_2)$: $r = 0{\cdot}84$.

In the second test the multiplicative combinations were classed into 18 groups in the manner adopted by Wheeler, including 0 and 1 as factors but excluding 'doubles'; in each group either x_1 or x_2 is held constant. For each combination, $\frac{1}{2}(x_1 + x_2)$, $(x_1.x_2)$ and $\log(x_1 + x_2 + x_1.x_2)$ were separately computed and the difficulty-rank taken from Wheeler's complete list. Correlations were computed for the group arithmetic means of these quantities. Correlation coefficients with respect to difficulty-rank are: *Mean product:* $r = 0{\cdot}88$; *mean factor-size:* $r = 0{\cdot}93$; *mean* $\log(x_1 + x_2 + x_1.x_2)$: $r = 0{\cdot}95$.

Applying the second test to Wheeler's data for additions, excluding 0 as addend and excluding 'doubles', grouping and averaging as before, the correlations are: *Mean sum:* $r = 0{\cdot}97$ (mean addend would give the same r); *Mean* $\log 2(x_1 + x_2)$: $r = 0{\cdot}99$.

In every case, therefore, difficulty-rank correlates at least as well with $\log(x_1 + x_2 + x_3)$ as with the arithmetic mean of (x_1, x_2) or with x_3.

As already observed, the argument in favour of $\log(x_1 + x_2 + x_3)$ as the pertinent psychophysical variable only requires this condition to be satisfied; the value of the present hypothesis is, primarily, that it removes the need to decide between two statistically indistinguishable hypotheses, each of which only takes one part of the calculation into account. However, the hypothesis gains indirect support from the fact that, unlike x_3 or $\frac{1}{2}(x_1 + x_2)$, the function $\log(x_1 + x_2 + x_3)$—under suitable conditions—is strictly proportional to the time taken by adults to perform the corresponding calculation. It may be noted that although Wheeler ranked additions purely by 'learning-difficulty', whereas Knight and Behrens used calculation-time as one of their criteria of difficulty, the two rankings agree closely ($r = 0{\cdot}91$), a fact which again suggests that one and the same measure of calculation 'size' may be a major factor in both learning-difficulty and, subsequently, calculation-time.

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Influence of Stroboscopic Illumination on the After-effect of Seen Movement

THE after-effect of seen movement (sometimes known as the waterfall effect) is a well-known illusion in which steady viewing of a moving patterned surface is followed, on transfer of one's gaze to a stationary surface, by an apparent motion in the opposite direction. Pickersgill[1] gives an exhaustive review of the literature.

We have recently investigated the effect of lighting the stationary surface stroboscopically instead of constantly. The stimulus pattern first used (by R. L. G. and S. M. A.) was a rotating Archimedes's spiral which is seen as expand-