implantations have shown that the active area extends anteriorly by at least another 2 mm, but implantations more than 1.25 mm lateral to the midline elicited eating or drinking without ejaculation. Effective current strengths were 30–150 μ amp root mean square, 50 c/s. Further details of this work will appear elsewhere⁴.

I thank Prof. R. C. Oldfield for laboratory facilities and the Medical Research Council for providing a stereotaxic instrument.

L. J. HERBERG*

Institute of Experimental Psychology,
University of Oxford.

* Leverhulme Mental Health Research Fund Fellow. Present address: Psychology Laboratory, Institute of Neurology, The National Hospital, Queen Square, London, W.C.1.

¹ Beach, F. A., and Jordan, L., *Quart. J. Exp. Psychol.*, **8**, 121 (1956).

² Vaughan, E., and Fisher, A. E., *Science*, **137**, 758 (1962).

³ MacLean, P. D., and Ploog, D. W., *J. Neurophysiol.*, **25**, 29 (1962).

⁴ Herberg, L. J., *J. Comp. Physiol. Psychol.* (in the press).

STATISTICS

Approximate Solutions to the Pure Birth Process

IN the course of an investigation of polymerization kinetics, I developed some useful approximation formulae by a technique which, while familiar to statisticians, is less so to physicists and chemists. As an illustrative calculation, the method is applied below to the pure birth process.

In mathematical statistics, the pure birth process is described by the infinite set of differential-difference equations:

$$\begin{aligned} df_1/dt &= -\beta_1 f_1 \\ df_2/dt &= \beta_1 f_1 - \beta_2 f_2 \\ df_3/dt &= \beta_2 f_2 - \beta_3 f_3 \\ &\dots \end{aligned} \quad (1)$$

together with the condition that at time $t = 0$

$$\begin{aligned} f_1(0) &= 1 \\ f_g(0) &= 0; g > 1 \end{aligned}$$

It is known¹ that an exact solution of these equations may be obtained by an iteration, starting with the first; but the resulting expressions rapidly become very awkward and for g large useless for computing purposes unless the β_g happen to depend on g in some particularly regular way. To make the solution to the pure birth process accessible to rapid computation, it is convenient to search for alternative procedures which avoid the complexity of the iteration technique.

In the case that the β_g are time independent, one such procedure may be based on the determination of the temporal moments:

$$\mu_n(g) = \int_0^\infty t^n f_g(t) dt$$

of each f_g . By transforming equations (1) with the Laplace transform and using the transform of each $f_g(t)$ as a moment generating function, the temporal moments are easily derived. The result is a general property of convolution products² and is most compactly expressed in terms of the temporal cumulants:

$$\kappa_n(g) = (n-1)! \sum_{j=1}^g \beta_j^{-n}$$

whence in particular the temporal mean and variance prove to be:

$$\begin{aligned} \kappa_1(g) &= \tau_g = \mu_1(g)/\mu_0(g) = \sum_1^g \beta_j^{-1}; \\ \kappa_2(g) &= \sigma_g^2 = \mu_2(g)/\mu_0(g) - \tau_g^2 = \sum_1^g \beta_j^{-2} \end{aligned} \quad (2)$$

To apply these results to the pure birth process, we note that once the $\mu_n(g)$ are known, the integral over t of the product of $f_g(t)$ with any polynomial $P(t)$ may be written down as a function of the β_g and that this fact permits us to calculate the coefficients of $f_g(t)$ in an expansion in orthogonal polynomials. Two such expansions prove to be particularly useful and will be demonstrated here. If $f_g(t)$ is expanded in a series of associated Laguerre polynomials, we have correct to three terms in the expansion:

$$f_g(t) = \beta_g^{-1} \frac{a}{\Gamma(m+1)} (at)^m e^{-at} \quad (3)$$

in which $a = \tau_g/\sigma_g^2$; $m = \tau_g^2/\sigma_g^2 - 1$; and $\Gamma(m+1)$ is the Γ function. If $f_g(t)$ is expanded in a series of Hermite polynomials, then correct to three terms there results:

$$f_g(t) = \beta_g^{-1} (2\pi\sigma_g^2)^{-1/2} \exp[-(t - \tau_g)^2/2\sigma_g^2] \quad (4)$$

Equation (4) is only asymptotically correct in the region t and g both large, the basic reason being the unsuitability of the Hermite polynomials in representing functions defined only over the positive, real t axis. The reader will recognize in (3) and (4) the leading terms of expansions of the Poisson-Charlier and Gram-Charlier³ type.

Fig. 1. The pure birth process for $\beta_g = g$. —, Exact; ---, Poisson-Charlier approximation (equation 3); - · -, Gram-Charlier approximation (equation 4).

To demonstrate the worth of these approximations, there is sketched in Fig. 1 the time dependence of $f_g(t)$ in the particular case $\beta_g = g$. The exact solution to the pure birth process is here known to be⁴:

$$f_g(t) = e^{-t}(1 - e^{-t})^{g-1}$$

and this curve is compared in the figure with those calculated from (3) and (4). The Poisson-Charlier approximation (3) is seen to be superior, as was to be expected from the fundamental properties of the associated Laguerre polynomials; but the Gram-Charlier formula (4) is computationally more convenient, particularly when both g and t are large.

These results can be extended to the birth and death process and in a forthcoming publication will be applied to some problems of particle growth in colloidal systems.

F. C. GOODRICH

California Research Corporation,
Richmond, California.

¹ Bharucha-Reid, A. T., *Elements of the Theory of Markov Processes and Their Applications* (McGraw-Hill Book Co., Inc., New York, 1960).

² Hirschman, I. I., and Widder, D. V., *The Convolution Transform*, 25 (Princeton, 1955).

³ Kendall, M. G., *The Advanced Theory of Statistics*, 1, 34, 147 (C. Griffin and Co., London, 1948).

⁴ Bharucha-Reid, A. T., *op. cit.*, 77.